DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITI TEKNOLOGI MALAYSIA

SSH 1033 MATHEMATICAL METHODS 2 **TUTORIAL 6**

1. By using the definition, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the following functions.

   (a) $f(x, y) = \dfrac{1}{y - x}$. (b) $f(x, y) = \dfrac{y}{3x - 2y}$.

2. By using the definition, find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial z}$ for the following functions.

   (a) $u(x, y, z) = \dfrac{z}{x - z}$. (b) $u(x, y, z) = \dfrac{x}{2x - y + z}$.

3. Use the definition to find $\dfrac{\partial z}{\partial x}$, given that

   (a) $z(x, y) = xy + \sin(x - y)$. (b) $z(x, y) = 2x^2y + e^{-y}$.

4. Given $z = \dfrac{x - y}{\sqrt{x^2 + y^2}}$.

   (a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ when $x = 8$ and $y = 6$. (b) Show that $\dfrac{\partial z/\partial x}{\partial z/\partial y} = -\dfrac{y}{x}$.

5. If $u = x \ln\left(1 + \dfrac{x}{y}\right) + y \ln\left(1 + \dfrac{y}{x}\right)$ for $x > 0$, $y > 0$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

6. If $f(x, y) = x \sin(y/x)$, find f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} when $x = 2$ and $y = \frac{1}{2}\pi$.

7. If $f(x, y) = y \cos(x - 2y)$, find f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} when $x = \frac{1}{4}\pi$ and $y = \pi$.

8. If $z = \dfrac{x^2 - y^2}{x^2 + y^2}$, find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$.

9. If $z = x^2 \tan^{-1}\left(\dfrac{y}{x}\right)$, find $\dfrac{\partial^2 z}{\partial x \partial y}$ when $x = y = 1$.

10. Find f_{xyz} when

    (a) $f(x, y, z) = e^{xyz}$, (b) $f(x, y, z) = \dfrac{xy}{2x + z}$,

    and verify in each case that $f_{xyz} = f_{yzx} = f_{zxy}$.

11. Use implicit partial differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

    (a) $x^2 - 2y^2 + 3z^2 - yz + y = 0$. (b) $xz + ye^{3y-2z} = x^2$.

    (c) $\ln(1 + x^2) + \ln(x + yz) = xy^2z^3$. (d) $\cos(x + y + z) = xyz$.

12. If $z^2 = x^2 + y^2$, show that $\dfrac{\partial z}{\partial x} = \dfrac{x}{z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{y}{z}$. Hence, deduce that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{z}.$$

13. If z is a function of x and y and is defined implicitly by $x^2 + y^2 + z^2 = 1$, show that

$$x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = z - \frac{1}{z}.$$

14. Assume that $F(x,\,y) = 0$ defines y implicitly as a differentiable function of x. Show that

$$\frac{dy}{dx} = -\,\frac{\partial F/\partial x}{\partial F/\partial y},$$

provided $\partial F/\partial y \neq 0$. Hence, use this formula to find $\dfrac{dy}{dx}$.

(a) $\dfrac{4y}{x} + \dfrac{2x}{y} = 3.$ (b) $x \tan y = y \sin x.$

(c) $x^2 + y^2 = e^{x/y}.$ (d) $y \ln \cos x = x \ln \sin y.$

15. Assume that $F(x,\,y,\,z) = 0$ defines z implicitly as a function of x and y. Show that

$$\frac{\partial z}{\partial x} = -\,\frac{\partial F/\partial x}{\partial F/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\,\frac{\partial F/\partial y}{\partial F/\partial z},$$

provided $\partial F/\partial z \neq 0$. Hence, use these formulas to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(a) $x^2 - 2y^2 + 3z^2 - yz + y = 0.$ (b) $xz + y \ln x - x^2 + 4 = 0.$

(c) $\ln\left(1 + x^2\right) + \ln\,(x + yz) = xy^2z^3.$ (d) $\cos\,(x + y + z) = xyz.$

16. The equation $x = r\cos\theta$ and $y = r\sin\theta$, which relate Cartesian and polar coordinates, define r and θ implicitly as functions of x and y.

(a) Use implicit differentiation with respect to the x on both equations to show that

$$\frac{\partial r}{\partial x} = \cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\,\frac{\sin\theta}{r}.$$

Similarly show that, implicit differentiation with respect to the y on both equations to show that

$$\frac{\partial r}{\partial y} = \sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

(b) Let $z = f(r,\,\theta)$, where r and θ are defined implicitly as functions of x and y by the equations $x = r\cos\theta$ and $y = r\sin\theta$. Use the results in (a), to show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\,\frac{\partial z}{\partial \theta}\sin\theta$$

and also

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\,\frac{\partial z}{\partial \theta}\cos\theta.$$

(c) Finally use the results in (b) to show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

17. If $f(x,y)$ is a function of x, y and $z = xy + f(x^2+y^2)$, show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2.$$

18. Use chain rule to find $\dfrac{dw}{dt}$.

(a) $w = 3x^2y^3;\quad x = t^4,\ y = t^2.$

(b) $w = \ln\left(2u^2 + v\right);\quad u = \sqrt{t},\ v = t^{3/2}.$

(c) $w = r^2 - s\tan v;\quad r = \sin^2 t,\ s = \cos t,\ v = 4t.$

(d) $w = e^{1-xy};\quad x = 2t^{1/3},\ y = t^3.$

(e) $w = \sqrt{1 + x - 2xy^2z^3};\quad x = e^{-t},\ y = 3t^2 + 2,\ z = \ln t.$

19. Use the chain rule to find the value of $\dfrac{dw}{ds}$ if $w = r^2 - r\tan\theta,\ r = \sqrt{s},\ \theta = \pi s$ when $s = \frac{1}{4}$.

20. Let $z = f(x,\,y)$, where $x = t - \cos t$, and $y = e^t$. Find $\dfrac{dz}{dt}$ at $t = 0$ given that $f_x(-1,\,1) = 4$ and $f_y(-1,\,1) = -3$.

21. Find $\dfrac{d^2z}{d\theta^2}$ for the following functions using the chain rule.

(a) $z = x^2 - y^2;\ x = \cos\theta,\ y = \sin\theta.$ (b) $z = \ln\left(x^2y\right);\ x = e^{\theta^2},\ y = \theta^2.$

22. Find the rate of change in the volume of a cylinder with radius 8 cm and height 12 cm if the radius increases at the rate of 0.2 cm/s while the height decreases at the rate of 0.5 cm/s.

23. The length, width and height of a rectangular box increases at the rate of 1 cm/s, 2 cm/s and 3 cm/s respectively. Calculate the rate of increase in the diagonal of the box when the length is 2 cm, width is 3 cm and height 6 cm.

24. Find the first partial derivatives for the following functions using the chain rule.

a. $z = x^2y - xy^3 + 2;\quad x = r\cos\theta,\ y = r\sin\theta.$

b. $z = \cos x \sin y;\quad x = u - v,\ y = u^2 + v^2.$

c. $z = x/y;\quad x = 2\cos u,\ y = 3\sin v.$

d. $z = r^3 + s + v;\quad r = xe^y,\ s = ye^x,\ v = x^2y.$

e. $z = pq + qw;\quad p = 2x - y,\ q = x - 2y,\ w = -2x + 2y.$

25. Given $w = \cos(uv);\quad u = xyz,\ v = \dfrac{\pi}{4(x^2+y^2)}$, find $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial z}$ when $x = y = z = 1$.

26. Use the chain rule to find the value of $\dfrac{\partial u}{\partial r}$ at point $(\sqrt{\pi}, \sqrt{\pi}, 1)$, given that $u = z\sin(xy)$, $x = r + s,\ \ y = r - s,$ and $z = r^2 + s^2$.

35. If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\frac{\partial z}{\partial t}.$$

36. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right].$$

37. Let $z = f(x, y)$, where $x = u + v$ and $y = u - v$. Show that

$$\frac{\partial^2 z}{\partial v \partial u} = \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}.$$

38. Given $u = \frac{1}{r}\{\,f(ct + r) + g(ct - r)\,\}$, where f and g are arbitrary functions, show that

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right).$$

39. If $u = x^n f(y/x)$, where f is an arbitrary function, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Hence deduce that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)u.$$

40. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right].$$

41. If $z = f(x, y)$, where $x = r^2 + s^2$, $y = 2rs$, find $\frac{\partial^2 z}{\partial r \partial s}$.

42. If $z = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, find

(a) $\frac{\partial z}{\partial r}$. (b) $\frac{\partial z}{\partial \theta}$. (c) $\frac{\partial^2 z}{\partial r \partial \theta}$.

43. If $u = f(x, y)$, where $x = 2r - s$ and $y = r + 2s$, use the chain rule to show that

$$\frac{\partial u}{\partial x} = \frac{2}{5}\frac{\partial u}{\partial r} - \frac{1}{5}\frac{\partial u}{\partial s}.$$

Hence, find $\frac{\partial^2 u}{\partial y \partial x}$ in terms of derivatives with respect to r and s.

44. If $u = x + y$, $v = xy$, and f is a function of x and y, express $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ in terms of $\dfrac{\partial f}{\partial u}$, $\dfrac{\partial f}{\partial v}$ and prove that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial u^2} + u\,\frac{\partial^2 f}{\partial u \partial v} + v\,\frac{\partial^2 f}{\partial v^2} + \frac{\partial f}{\partial v}.$$

45. Show that if $x = \rho\cos\phi$, $y = \rho\sin\phi$, the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{become} \qquad \frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \phi},\ \frac{\partial v}{\partial \rho} = -\frac{1}{\rho}\frac{\partial u}{\partial \phi}.$$

Hence, show that under the transformation $x = \rho\cos\phi$, $y = \rho\sin\phi$, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{becomes} \qquad \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = 0.$$

46. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (*)$$

for $u(x, y)$ by making a change of variables as follows. Define new variables

$$\xi = x - y, \quad \eta = x,$$

and evaluate the partial derivatives of x and y with respect to ξ and η.
Writing $v(\xi, \eta) = u(x, y)$, use these derivatives and the chain rule to show that

$$\frac{\partial v}{\partial \eta} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y},$$

and that the equation

$$\frac{\partial^2 v}{\partial \eta^2} = 0$$

is equivalent to equation (*).
Deduce that the most general solution of (*) is

$$u(x, y) = f(x - y) + xg(x - y),$$

where f and g are arbitrary functions.
Solve (*) completely given that $u(0, y) = 0$ for all y, whilst $u(x, 1) = x^2$ for all x.