Tangent Planes and Normal Lines

Previously, we learned about level curves and level surfaces. In general, if $f(x, y) = C$ and we assign a particular constant value to C, we obtain a level curve at C. Likewise, if $f(x, y, z) = C$ and we assign a particular constant value to C, we obtain a level surface at C.

Example:

If $f(x, y) = x^2 + y^2 = C$ and we let $C = 9$, we obtain the level curve $x^2 + y^2 = 9$ (a circle of radius 3). Similarly, if $f(x, y, z) = z^2 - 2x^2 - 2y^2 = C$ and we let $C = 12$, we obtain the level surface $z^2 - 2x^2 - 2y^2 = 12$ (a hyperboloid).

Up till now, we have represented surfaces as $z = f(x, y)$. For the concepts to follow, it is convenient to use the more general representation $F(x, y, z) = 0$, where $F(x, y, z) = f(x, y) - z$.

Normal Lines

Let S be a surface given by $F(x, y, z) = 0$, and let $P(x_0, y_0, z_0)$ be a point on S. Let a curve on S through P be defined by the vector-valued function $\vec{r}(t) = x(t)\vec{\mathrm{i}} + y(t)\vec{\mathrm{j}} + z(t)\vec{\mathrm{k}}$. Then, $F\big(x(t), y(t), z(t)\big) = 0$ for all t. If F, $x(t)$, $y(t)$, and $z(t)$ are all differentiable, then by the Chain Rule,

$$0 = F'(t) = F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t)$$

At the point (x_0, y_0, z_0), the vector form of the above equation becomes $0 = \nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0)$, where $\vec{r}\,'(t_0)$ is the tangent vector. This shows that the gradient at P is orthogonal to the tangent vector at every curve on S through P. That is, all tangent lines on S lie in a plane normal to $\nabla F(x_0, y_0, z_0)$.

So, the **normal line** through P has the direction vector $\nabla F(x_0, y_0, z_0)$ and is defined by the equations

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t$$

Example: Find the equations of the normal line to the surface $z = x^2 + 3y^2$ at the point (1, 2, 13).

$$F_x = 2x, \quad F_y = 6y, \quad F_z = -1$$

$$x = 1 + 2t, \quad y = 2 + 12t, \quad z = 13 - t$$

Example: Find the equations of the normal line to the cone $z^2 = x^2 + y^2$ at the point (-3, -4, 5).

$$F(x, y, z) = x^2 + y^2 - z^2, \quad \nabla F(x, y, z) = \langle 2x, 2y, -2z \rangle \quad \rightarrow \quad \nabla F(-3, -4, 5) = \langle -6, -8, -10 \rangle$$

$$x = -3 - 6t, \quad y = -4 - 8t, \quad z = 5 - 10t$$

Tangent Plane

To find an equation for the tangent plane to *S* at the point (x_0, y_0, z_0), choose an arbitrary point (x, y, z) in the tangent plane. Then, the vector $\vec{\mathrm{v}} = (x - x_0)\vec{\mathrm{i}} + (y - y_0)\vec{\mathrm{j}} + (z - z_0)\vec{\mathrm{k}}$ lies in the tangent plane. Since $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at the point (x_0, y_0, z_0), it must be orthogonal to every vector in the tangent plane, that is $\nabla F(x_0, y_0, z_0) \cdot \vec{\mathrm{v}} = 0$.

Thus, the equation of the **tangent plane** is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Example: Give the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point (1, 5, 26).

Let $F(x, y, z) = x^2 + y^2 - z$. Then $\nabla F(x, y, z) = \langle 2x, 2y, -1 \rangle$. At the point (1, 5, 26),

$$\nabla F(1,5,26) = \langle 2, 10, -1 \rangle$$

The equation of the tangent plane is

$$2(x - 1) + 10(y - 5) - 1(z - 26) = 0 \quad \rightarrow \quad 2x + 10y - z = 26$$

Example: Find the point on the surface $x^2 + y^2 - 2x + 4y + z + 1 = 0$ where the tangent plane is parallel to the *xy*-plane.

The tangent plane will be parallel to the *xy*-plane only if its gradient is parallel to the *z*-axis, that is, it must be parallel to the unit vector $k = \langle 0,0,1 \rangle$.

$$F_x = 2x - 2, \quad F_y = 2y + 4, \quad F_z = 1$$

So,

$$\nabla F(x, y, z) = \langle 2x - 2, 2y + 4, 1 \rangle = \langle 0,0,1 \rangle$$

implies that $x = 1$ and $y = -2$. Hence, there is a horizontal tangent plane at (1, -2, -4).

Example: At what point on the surface $x^2 + y^2 - 2x + 4y + z + 1 = 0$ is the tangent plane parallel to the vector $\langle 3, -1, 2 \rangle$?

If the tangent plane is parallel to the vector, then its normal vector must be perpendicular to the vector. If two vectors are perpendicular, their dot product must be equal to zero. So,

$$\nabla F(x, y, z) = \langle 2x - 2, 2y + 4, 1 \rangle$$

$$\langle 2x - 2, 2y + 4, 1 \rangle \cdot \langle 3, -1, 2 \rangle = 0 \quad \rightarrow \quad 3x - y = 4 \quad \rightarrow \quad \langle 2,8,1 \rangle \cdot \langle 3, -1, 2 \rangle = 0$$

If $x = 2$, $y = 2$, then from the original equation, $z = -13$. At (2, 2, -13) the tangent plane is parallel to the vector $\langle 3, -1, 2 \rangle$.

Angle Between Two Planes

When two planes intersect, the angle between the two planes is the same as the angle between their normal vectors. Let θ be the angle between the normal vectors N_1 and N_2. Then,

$$\cos \theta = \frac{N_1 \cdot N_2}{\|N_1\| \|N_2\|}$$

Example: Find the angle of intersection of the planes $3x + 6y - z - 5 = 0$ and $9x + y + 5z - 18 = 0$.

$$N_1 = \langle 3,6,-1\rangle, \quad N_2 = \langle 9,1,5\rangle$$

$$N_1 \cdot N_2 = 28, \quad \|N_1\| = \sqrt{46}, \quad \|N_2\| = \sqrt{107}$$

$$\theta = \cos^{-1}\frac{N_1 \cdot N_2}{\|N_1\|\|N_2\|} = \cos^{-1}\frac{28}{\sqrt{46}\sqrt{107}} \cong 66.5°$$

Exercises:

1. Find the equations for the normal line and tangent plane to the following surfaces at the point given.
   a. $z = 5x^2 + 3y^2 \quad \text{at} \quad (1,2,17)$
   b. $2x^2 + y^2 + z^2 = 2 \quad \text{at} \quad (1,0,0)$
2. Find the point(s) at which the tangent plane to the surface is horizontal.
   a. $z = 2x^2 + y^2$
   b. $4x^2 + 9y^2 - z^2 = 36 = 0$
3. Find the angle between the two surfaces at the given point.
   a. $z = x^2 + y^2, \quad x^2 + y^2 + z^2 = 20 \quad \text{at} \quad (0,2,4)$
   b. $z = x^2 - y^2, \quad z = 4 \quad \text{at} \quad (2,0,4)$

Answers:

1.
   a. $x = 1 + 10t, \quad y = 2 + 12t, \quad z = 17 - t; \quad 10x + 12y - z = 17$
   b. $x = 1 + 4t; \quad x = 1$
2.
   a. (0, 0, 0)
   b. (0, 0, 6)
3.
   a. $77.47°$
   b. $75.96°$