# Bifurcation theory

In general, it refers to a qualitative change in the behavior of a dynamical system as some parameter on which the system depends varies continuously.

Consider a scalar ODE

$$x_t = f(x;\mu) \tag{2.4}$$

depending on a single parameter $\mu \in R$ where $f$ is a smooth function. The qualitative dynamical behavior of a one-dimensional continuous dynamical system is determined by its equilibria and their stability, so all bifurcations are associated with bifurcations of equilibria. One possible definition (which does not refer directly to the stability of the equilibria) is as follows.

Definition 2.5. A point $(x_0,\mu_0)$ is a bifurcation point of equilibria for (2.4) if the number of solutions of the equation $f(x;\mu) = 0$ for $x$ in every neighborhood of $(x_0,\mu_0)$ is not a constant independent of $\mu$.

The three most important one-dimensional equilibrium bifurcations are described locally by the following ODEs:

$$\begin{aligned} x_t &= \mu - x^2, && \text{saddle-node;} \\ x_t &= \mu x - x^2, && \text{transcritical;} \\ x_t &= \mu x - x^3, && \text{pitchfork.} \end{aligned} \tag{2.5}$$

## Saddle-node bifurcation

Consider the ODE

$$x_t = \mu + x^2. \tag{2.6}$$

1 Draw a bifurcation diagram for $\dot{x} = \mu + x^2$.

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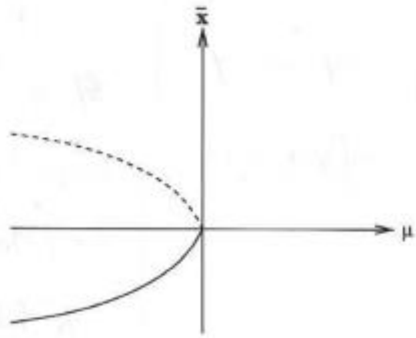
We have $f_\mu(x) = \mu + x^2, \quad f'_\mu(x) = 2x.$

Equilibrium points: $f_\mu(x) = 0 \Leftrightarrow x^2 = -\mu \Leftrightarrow x = \pm\sqrt{-\mu}.$

$\mu < 0$ Two equilibrium points at $-\sqrt{-\mu}$, $\sqrt{-\mu}$
$f'_\mu(-\sqrt{-\mu}) = -2\sqrt{-\mu} < 0$ (asymptotically stable)
$f'_\mu(\sqrt{-\mu}) = 2\sqrt{-\mu} > 0$ (unstable).

$\mu = 0$ One equilibrium point at 0
$f'(0) = 0$ so this test is inconclusive.
$xf(x) = x^3$ so $xf(x) < 0$ if $x < 0$ (stable) and $xf(x) > 0$ if $x > 0$ (unstable)
0 is unstable (or asymptotically stable from below).

$\mu > 0$ No equilibrium points.

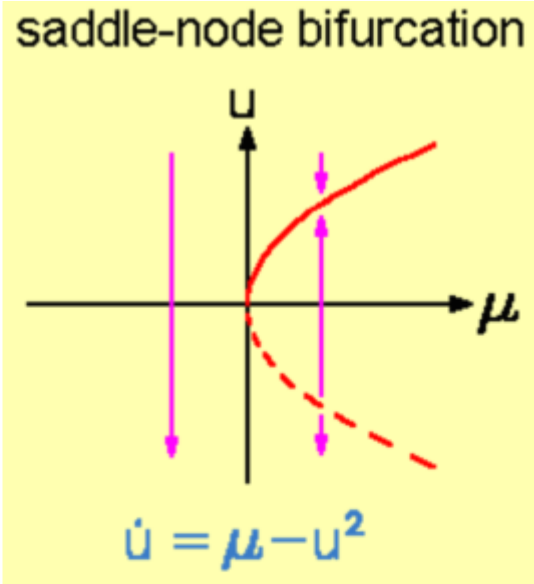



(- - - unstable, —— stable)
There is a fold (saddle-node) bifurcation at $\mu = 0$, with a subcritical bifurcation point at (0,0).

This bifurcation is called a saddle-node bifurcation. In it, a pair of hyperbolic equilibria, one stable and one unstable, coalesce at the bifurcation point, annihilate each other and disappear.[1] We refer to this bifurcation as a subcritical saddle-node bifurcation, since the equilibria exist for values of $\mu$ below the bifurcation value 0. With the opposite sign $x_t = \mu - x^2$, the equilibria appear at the bifurcation point $(x, \mu) = (0, 0)$ as $\mu$ increases through zero, and we get a supercritical saddle-node bifurcation. Saddle-node bifurcations are the generic way that the number of equilibrium solutions of a dynamical system changes as some parameter is varied.

The name "saddle-node" comes from the corresponding two-dimensional bifurcation in the phase plane, in which a saddle point and a node coalesce and disappear, but the other dimension plays no essential role in that case and this bifurcation is one-dimensional in nature.

[1] If we were to allow complex equilibria, the equilibria would remain but become imaginary.




## Transcritical bifurcation

Consider the ODE

$$\dot{x} = \mu x + x^2.$$

**2** Draw a bifurcation diagram for $\dot{x} = \mu x + x^2$.

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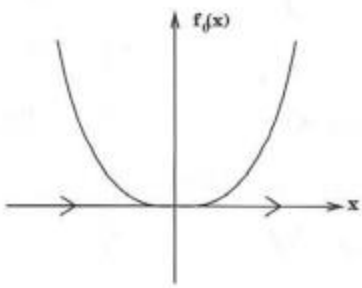
We have $f_\mu(x) = \mu x + x^2, \quad f'_\mu(x) = \mu + 2x.$

Equilibrium points: $f_\mu(x) = 0 \Leftrightarrow \mu x + x^2 = 0 \Leftrightarrow x = 0, -\mu.$

$\boldsymbol{\mu = 0}$ One equilibrium point at $x = 0$.
$f'_0(0) = 0$ so 0 is a non-hyperbolic EP. Graph of flow $f_0(x) = x^2$:

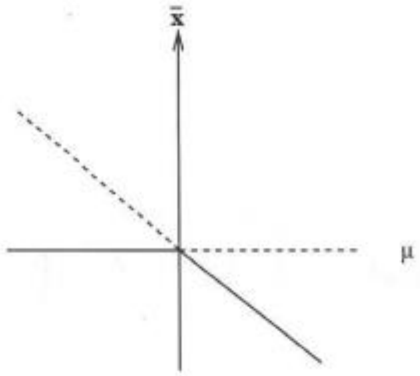


EP is asymptotically stable from below.

$\boldsymbol{\mu \neq 0}$ Two equilibrium points: $0, -\mu$

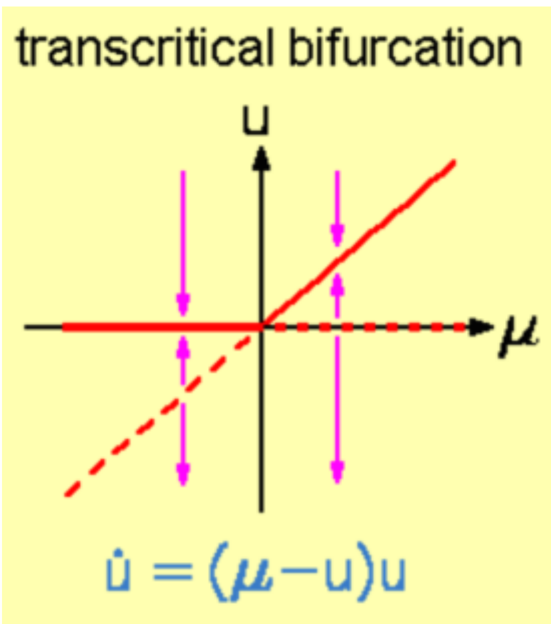
$f'_\mu(0) = \mu$ so $f'_0(0) < 0$ if $\mu < 0$ (stable) and $f'_0(0) > 0$ if $\mu > 0$ (unstable).

$f'_\mu(-\mu) = -\mu$ so $f'_\mu(-\mu) < 0$ if $\mu > 0$ (stable) and $f'_\mu(-\mu) > 0$ if $\mu < 0$ (unstable).




There is a transcritical bifurcation at (0,0).

This transcritical bifurcation arises in systems where there is some basic "trivial" solution branch, corresponding here to $x = 0$, that exists for all values of the parameter $\mu$. (This differs from the case of a saddle-node bifurcation, where the solution branches exist locally on only one side of the bifurcation point.). There is a second solution branch $x = \mu$ that crosses the first one at the bifurcation point $(x, \mu) = (0, 0)$. When the branches cross one solution goes from stable to unstable while the other goes from stable to unstable. This phenomenon is referred to as an "exchange of stability."



## Pitchfork Bifurcation

**3** Draw a bifurcation diagram for $\dot{x} = \mu x - x^3$.

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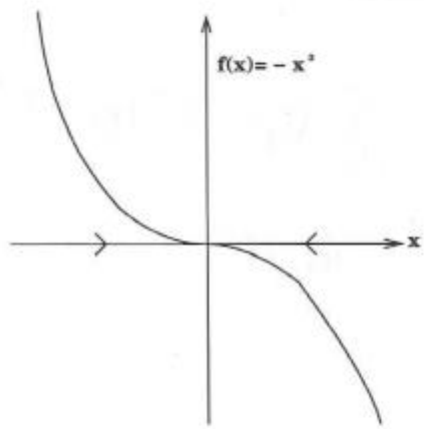
We have $f_\mu(x) = \mu x - x^3, \quad f'_\mu(x) = \mu - 3x^2$.

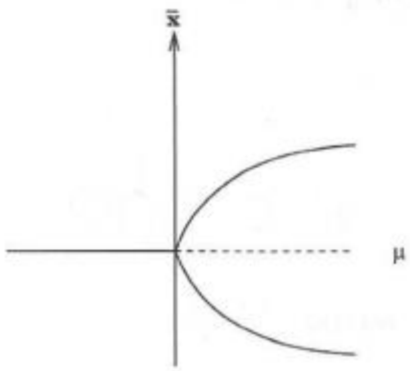
Equilibrium points: $f_\mu(x) = 0 \Leftrightarrow x(\mu - x^2) = 0 \Leftrightarrow x = 0, \; x = \pm\sqrt{\mu} \;\; (\mu \geq 0)$.

$\boldsymbol{\mu \leq 0}$ One equilibrium point at $x = 0$.
$f'_\mu(0) = \mu < 0$ if $\mu < 0$ so EP is stable if $\mu < 0$. When $\mu = 0$, EP is non-hyperbolic. Draw graph of flow $f_0(x) = -x^3$:




so EP $x = 0$ is stable for $\mu \leq 0$.

$\boldsymbol{\mu > 0}$ Three equilibrium points at $x = 0, \pm\mu$.
$f'_\mu(0) = \mu > 0$ if $\mu > 0$ so 0 is unstable.
$f'_\mu(\pm\sqrt{\mu}) = -2\mu < 0$ so both EPs are stable.




There is a supercritical pitchfork bifurcation at (0,0).

The system has one globally asymptotically stable equilibrium $x = 0$ if $\mu \leq 0$, and three equilibria $x = 0$, $x = \pm\sqrt{\mu}$ if $\mu$ is positive. The equilibria $\pm\sqrt{\mu}$ are stable and the equilibrium $x = 0$ is unstable for $\mu > 0$. Thus the stable equilibrium 0 loses stability at the bifurcation point, and two new stable equilibria appear. The resulting pitchfork-shape bifurcation diagram gives this bifurcation its name.

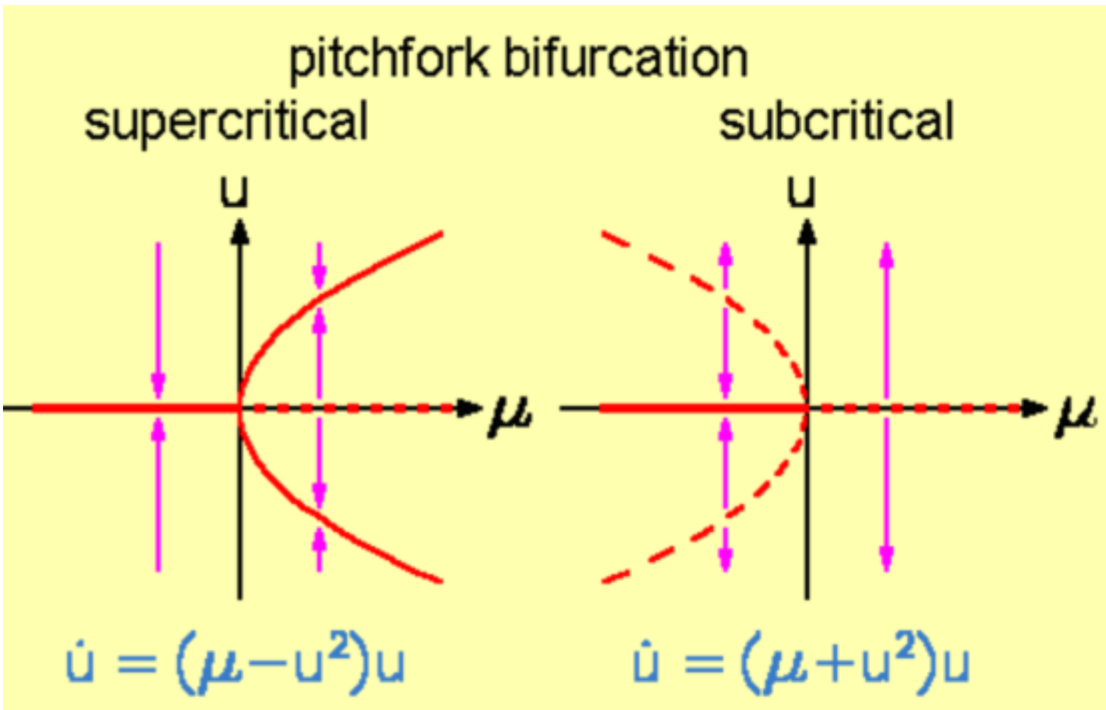
This pitchfork bifurcation, in which a stable solution branch bifurcates into two new stable branches as the parameter $\mu$ is increased, is called a supercritical bifurcation. Because the ODE is symmetric under $x \mapsto -x$, we cannot normalize all the signs in the ODE without changing the sign of $t$, which reverses the stability of equilibria.

Up to changes in the signs of $x$ and $\mu$, the other distinct possibility is the subcritical pitchfork bifurcation, described by

$$x_t = \mu x + x^3.$$

In this case, we have three equilibria $x = 0$ (stable), $x = \pm\sqrt{-\mu}$ (unstable) for $\mu < 0$, and one unstable equilibrium $x = 0$ for $\mu > 0$.

A supercritical pitchfork bifurcation leads to a “soft” loss of stability, in which the system can go to nearby stable equilibria $x = \pm\sqrt{\mu}$ when the equilibrium $x = 0$ loses stability as $\mu$ passes through zero. On the other hand, a subcritical pitchfork bifurcation leads to a “hard” lose of stability, in which there are no nearby equilibria and the system goes to some far-off dynamics (or perhaps to infinity) when the equilibrium $x = 0$ loses stability.



# Bifurcations in 2d

Like in 1d, in 2d existence and stability of fixed points depend on the parameters of the system. In contrast to 1d, however, now also oscillations can be switched on and off. As an example, look at the substrate-depletionoscillator.

There are three types of bifurcations in 2d:
1. 1d-like bifurcations (4 types)
2. Hopf bifurcation (local switch on/off of oscillations)
3. global bifurcations of cycles (3 types)

**Hopf Bifurcation**

**In the mathematical theory of bifurcations, a Hopf bifurcation is a critical point where a system's stability switches and a periodic solution arises.[1] More accurately, it is a local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues (of the linearization around the fixed point) cross the complex plane imaginary axis.**

(The **Hopf bifurcation** is a catastrophe in which as one gradually changes the parameters in an ordinary differential equation, a fixed point suddenly changes to a limit cycle)

It is easiest to understand the idea by considering an example. There are various differential equations that exhibit a Hopf bifurcation, but here's the simplest:

$$\frac{dx}{dt} = -y + \beta x - x(x^2 + y^2)$$

$$\frac{dy}{dt} = x + \beta y - y(x^2 + y^2)$$

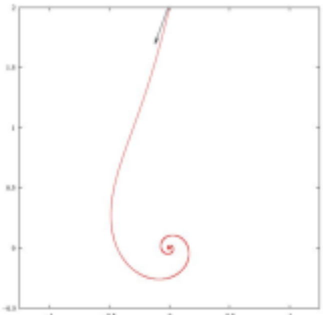
Here $x$ and $y$ are function in time, $t$ so these equations describe a point moving around on the plane. It's easier to see what's going on in polar coordinates:

$$\frac{dr}{dt} = \beta r - r^3$$

$$\frac{d\theta}{dt} = 1$$

The angle $\theta$ goes around at a constant rate while the radius $r$ does something more interesting. When $\beta \leq 0$, you can see that any solution spirals in towards the origin. Or, if it starts at the origin, it stays there. So, we call the origin a **stable equilibrium.**

Here's a typical solution for $\beta = -1/4$, drawn as a curve in the $xy$ plane. As time passes, the solution spirals in towards the origin:

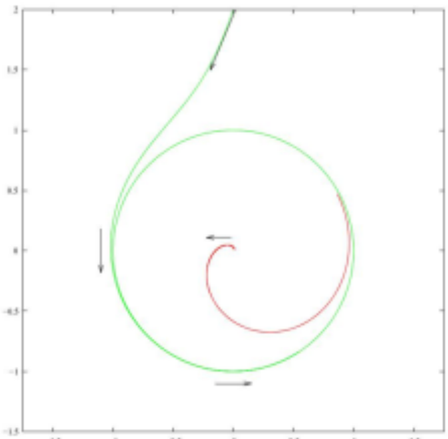


The equations are more interesting for $\beta > 0$. Then $\mathrm{dr}/\mathrm{dt} = 0$ whenever

$$\beta r - r^3 = 0 \tag{5}$$

This has two solutions, $r = 0$ and $r = \sqrt{\beta}$. Since $r = 0$ is a solution, the origin is still an equilibrium. But now it's not stable: if $r$ is between $0$ and $\sqrt{\beta}$, we'll have $\beta r - r^3 > 0$, so our solution will spiral *out*, away from the origin and towards the circle $r = \sqrt{\beta}$. So, we say the origin is an **unstable equilibrium**. On the other hand, if $r$ starts out bigger than $\sqrt{\beta}$, our solution will spiral in towards that circle.

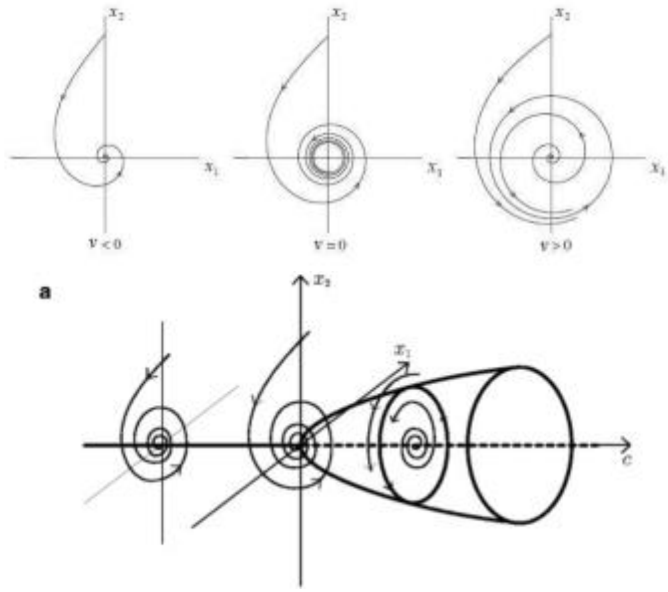
Here's a picture of two solutions for $\beta = 1$:



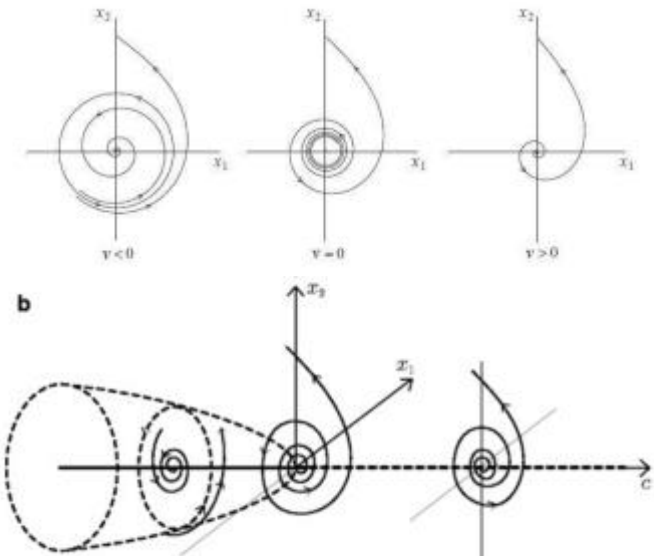
The red solution starts near the origin and spirals out towards the circle $r = \sqrt{\beta}$. The green solution starts outside this circle and spirals in towards it, soon becoming indistinguishable from the circle itself. So, this equation describes a system where $x$ and $y$ quickly settle down to a periodic oscillating behavior.

Since solutions that start anywhere near the circle $r = \sqrt{\beta}$ will keep going round and round getting closer to this circle, it's called a "stable limit cycle".

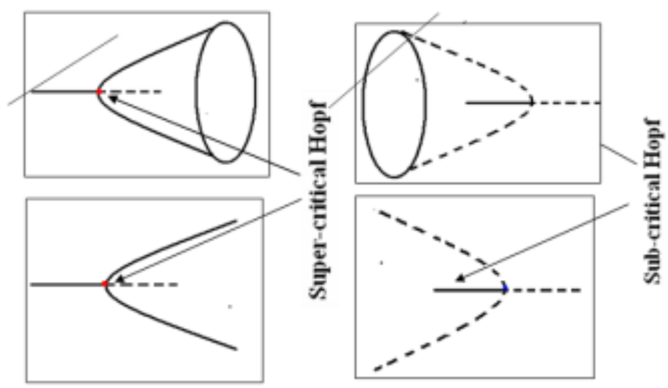
This is what the Hopf bifurcation is all about: we've got a dynamical system that depends on a parameter, and as we change this parameter, a stable fixed point become unstable, and a stable limit cycle forms around it.




**Supercritical Hopf Bifurcation**



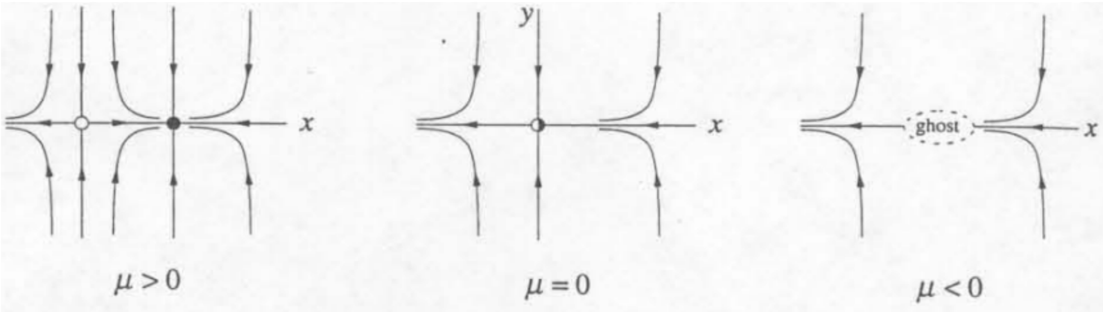

**Subcritical Hopf Bifurcation**



2D Bifurcation

Saddle-Node Bifurcation

$$\dot{x} = \mu - x^2 \qquad \dot{y} = -y$$




Transcritical Bifurcation

$$\dot{x} = \mu x - x^2, \qquad \dot{y} = -y.$$

Pitchfork Bifurcation

$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y.$$