Chapter 4

One-dimensional maps

In this chapter we will concentrate on the difference equation

$$x_{n+1} = f(x_n), \quad x_0 \in I$$

where $I \subseteq \mathbb{R}$ and $f : I \to I$, i.e. we restrict ourselves to the case where the metric space X is an interval on the real line. When f is continuous on I we write $f \in C(I)$. Similarly $f \in C^n(I)$ denotes the fact that the first n derivatives of f (with respect to x) are all continuous on I.

4.1 Stability of Fixed Points and Cycles

Our aim in this section is to develop techniques that can be used to determine whether a fixed point or a cycle is stable or not. We begin by introducing the notion of **hyperbolic** fixed points and cycles.

Definition 10 *Let $f \in C^1(I)$.*

1. *$p \in \mathsf{Fix}(f)$ is a* **hyperbolic** *fixed point if $|f'(p)| \neq 1$; when $|f'(p)| = 1$, p is an* **indifferent** *(or* **non-hyperbolic***) fixed point.*
2. *$p \in \mathsf{Per}_n(f)$ is a* **hyperbolic period n point** *if $|f'(p)| \neq 1$; otherwise p is an* **indifferent** *period n point.*

Theorem 4 *Let $p_1, p_2, \cdots, p_n$ form an n-cycle for the map $f : I \to I$ where $f \in C^1(I)$. Then*

$$|(f^n)'(p_1)| = |(f^n)'(p_2)| = \cdots = |(f^n)'(p_n)|$$

and so if one point in the cycle is hyperbolic, so are all of the other points in the cycle.

Proof Consider $(f^n)'(p_1)$. By the chain rule,

$$\begin{aligned}
(f^n)'(p_1) &= f'(f^{n-1}(p_1))(f^{n-1})'(p_1) \\
&= f'(f^{n-1}(p_1))f'(f^{n-2}(p_1))(f^{n-2})'(p_1) \\
&= \cdots \\
&= f'(f^{n-1}(p_1))f'(f^{n-2}(p_1)) \cdots f'(f(p_1))f'(p_1) \\
&= f'(p_n)f'(p_{n-1})f'(p_{n-2}) \cdots f'(p_2)f'(p_1)
\end{aligned}$$

Applying the same argument to each p_k gives the stated result since we obtain

$$(f^n)'(p_k) = \prod_{i=1}^{n} f'(p_i), \quad k = 1, 2, \cdots, n.$$

For a fixed point p, we have $x_{n+1} = f(x_n) = f(p) = p$ so the orbit remains at p for all future iterations. To examine stability of a fixed point p, we consider a nearby orbit $x_n = p + \xi_n$, and ask whether this orbit is attracted to or repelled by p, that is, does the deviation ξ_n grow or decay as n increases? Mathematically, this means considering the **linearization** of the function f about a fixed point p.

Suppose f is differentiable on an open interval containing a fixed point p. If x_0 is sufficiently close to p then

$$f(x_0) \approx p + f'(p)(x_0 - p) \Rightarrow x_1 - p \approx f'(p)(x_0 - p) \quad \text{or } \xi_1 \approx f'(p)\xi_0.$$

Similarly,

$$f(x_1) - p \approx f'(p)(x_1 - p) \Rightarrow x_2 - p \approx f'(p)(x_1 - p) \quad \text{or } \xi_2 \approx f'(p)\xi_1,$$

and, in general

$$\xi_{k+1} \approx f'(p)\xi_k, \quad k = 0, 1, 2, \cdots$$

Thus, in terms of the small perturbation ξ_n, we can make the linear approximation

$$\xi_{n+1} = a\xi_n$$

where $\xi_n = x_n - p$ and $a = f'(p)$.

Theorem 5 *Let $p \in \text{Fix}(f)$ where $f : I \to I$ is such that f is continuous on I and continuously differentiable on an open bounded interval $J \subseteq I$ containing p.*

1. *If $|f'(p)| < 1$ then p is an* **attracting** *fixed point (or* **sink***).*
2. *If $|f'(p)| > 1$ then p is a* **repelling** *fixed point (or* **source***).*

proof

1. We have $f \in C^1(J)$ and $|f'(p)| < 1$ so $\exists \epsilon > 0 : |f'(x)| \leq \alpha < 1 \; \forall x \in [p-\epsilon, p+\epsilon] \in J$. Let $U = (p - \epsilon, p + \epsilon)$. Then

$$\begin{aligned} x \in U \Rightarrow |f(x) - p| &= |f(x) - f(p)| \\ &= |f'(c)||x - p| \quad (\text{ for some point } c \text{ between } p \text{ and } x) \\ &\leq \alpha|x - p| < \epsilon. \end{aligned}$$

Hence $f(U) \subseteq U$. Similarly

$$\begin{aligned} |f^k(x) - p| &= |f(f^{k-1}(x) - f(p)| = |f'(c)||f^{k-1}(x) - p| \\ &\leq \alpha |f^{k-1}(x) - p| \\ &\leq \cdots \leq \alpha^k |x - p| \to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence $f^k(x) \to p$ as $k \to \infty \quad \forall x \in U$ and so p is an attracting fixed point.

Alternatively: $x \in |p - \epsilon, p + \epsilon| \Rightarrow |f(x) - p| \leq \epsilon$ as above so $f(|p - \epsilon, p + \epsilon|) \subseteq |p - \epsilon, p + \epsilon|$. Therefore it follows that f is a contraction on $|p - \epsilon, p + \epsilon|$ and hence $f^k(x) \to p$ as $k \to \infty$ (by the contraction mapping principle).

2. *Omitted*

Notes:

1. When $f'(p) < 1$,the analysis used in the proof of the previous theorem shows that there exists an open interval U containing p satisfying

   (a) $f(U) \subseteq U$,

   (b) $U \subseteq W^s(p)$.

   The largest such U is called the **local stable set** of p, and is denoted by $W^s_{\text{loc}}(p)$. The corresponding set associated with a repelling fixed point p is called the **local unstable set** of p, and is denoted by $W^u_{\text{loc}}(p)$.

2. Similar stability results hold for cycles. If $\{p_1, p_2, \cdots, p_n\}$ is an n-cycle for f then the cycle is **attracting** (or **sink**) if $|f'(p_1)f'(p_2)\cdots f'(p_n)| < 1$, and **repelling** (or **source**) if $|f'(p_1)f'(p_2)\cdots f'(p_n)| > 1$.

Theorem 5 shows that the local behaviour at a hyperbolic fixed point p is determined by $f'(p)$. This is not the case for indifferent (non-hyperbolic) fixed points; to analyse these we use Taylor series expansions involving higher derivatives. We will examine the cases $f'(p) = 1$ and $f'(p) = -1$ separately. For simplicity we shall assume that $f : \mathbb{R} \to \mathbb{R}$ and also that f can be differentiated as often as required at each $x \in \mathbb{R}$.

Examples 4A

1. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \dfrac{x^3 + x}{2}$.
   Establish the stability of the fixed points of f.

   Solution:

$$f(x) = x \Rightarrow \frac{x^3 + x}{2} = x \Rightarrow \frac{x(x^2 - 1)}{2} = 0 \Longleftrightarrow x = 0, 1, -1.$$

$\text{Fix}(f) = \{0, 1, -1\}$. Also $f'(x) = \dfrac{3x^2 + 1}{2}$.
$|f'(0)| = \frac{1}{2} < 1$ so 0 is an attracting fixed point.
$|f'(\pm 1)| = 2 > 1$ so $1, -1$ are repelling fixed points.
Show the cobweb diagram

$$\begin{aligned} |x| < 1 &\Rightarrow f^n(x) \to 0 \\ x > 1 &\Rightarrow f^n(x) \to \infty \text{ as } n \to \infty \\ x < -1 &\Rightarrow f^n(x) \to -\infty \end{aligned}$$

so $W^s(0) = \{x : |x| < 1\}$ and $\text{Per}(f) = \text{Fix}(f)$

2. Let $g : \mathbb{R} \to \mathbb{R}$ with $g = -\dfrac{(x^3 + x)}{2}$. Establish the stability of the fixed and period 2 points of g.

Solution: Fixed points

$$g(x) = x \Rightarrow -\frac{(x^3 + x)}{2} = x \Rightarrow -\frac{x(x^2 + 3)}{2} = 0 \Longleftrightarrow x = 0$$

$\text{Fix}(g) = \{0\}$. As $g'(x) = \dfrac{-3x^2 - 1}{2}$, $|g'(0)| = \dfrac{1}{2} < 1$, hence 0 is an attractor.

Period 2 points:

$$g^2(x) = x \Rightarrow g\left(-\frac{x^3 + x}{2}\right) = -\frac{(-\frac{(x^3+x)}{2})^3 + (-\frac{(x^3+x)}{2})}{2} = x$$

$$\frac{(x^3 + x)^3}{16} + \frac{(x^3 + x)}{4} - x = 0 \Rightarrow \frac{x^9}{16} + \frac{3x^7}{16} + \frac{3x^5}{16} + \frac{5x^3}{16} - \frac{3x}{4} = 0$$
$$\frac{1}{16}x(x^2 + 3)(x - 1)(x + 1)(x^4 + x^2 + 4) = 0$$

so $\text{Per}_2(g) = \{0, -1, 1\}$.
From theorem 4, $|(g^2)'(1)| = |(g^2)'(-1)| = |g'(1)g'(-1)| = |(-2)(-2)| = 4$. Hence the period 2 are repelling.

show cobweb diagram

Case (i): $f'(p) = 1$

Theorem 6 *Let $f \in C^2(\mathbb{R})$ and let $p \in \text{Fix}(f)$ be such that $f'(p) = 1$.*

1. *If $f''(p) > 0$ then p is (***weakly***)* **attracting from the left***, (***weakly***)* **repelling from the right***. In this case, we say that p is* **semistable from below***.*

2. *If $f''(p) < 0$ then p is (***weakly***)* **attracting from the right***, (***weakly***)* **repelling from the left***. In this case, we say that p is* **semistable from above***.*

Proof: Ommited

Theorem 7 *Let $f \in C^3(\mathbb{R})$ and let $p \in \mathrm{Fix}(f)$ be such that $f'(p) = 1$ and $f''(p) = 0$.*

1. *If $f^{(3)}(p) < 0$ then p is a* **weakly attracting** *fixed point.*

2. *If $f^{(3)}(p) > 0$ then p is a* **weakly repelling** *fixed point.*

Proof: Ommited

Case (ii): $f'(p) = -1$
To determine the stability of a fixed point p for which $f'(p) = -1$, we apply the previous results to the function $g = f^2 = f \circ f$. Routine calculations using the chain rule show that $g'(p) = 1$, $g''(p) = 0$, $g^{(3)}(p) = -2f^{(3)}(p) - 3\left(f''(p)\right)^2$. This leads to the following result.

Theorem 8 *Let $f \in C^3(\mathbb{R})$ and let $p \in \mathrm{Fix}(f)$ be such that $f'(p) = -1$.*

1. *If $-2f^{(3)}(p) - 3\left(f''(p)\right)^2 < 0$ then p is (***weakly***)* **attracting***.*

2. *If $-2f^{(3)}(p) - 3\left(f''(p)\right)^2 > 0$ then p is (***weakly***)* **repelling***.*

Summary of Key Points so Far
Let $f \in C^3(\mathbb{R})$ and $p \in \mathrm{Fix}(f)$. Then

1. $|f'(p)| < 1$: p is an **attractor**

2. $|f'(p)| > 1$: p is a **repeller**

3. $f'(p) = 1$:

   - $f''(p) > 0 \Rightarrow p$ is **semistable from below**
   - $f''(p) < 0 \Rightarrow p$ is **semistable from above**

4. $f''(p) = 0$:

   (a) $f'''(p) < 0$: p is a **weak attractor**

   (b) $f'''(p) > 0$: p is a **weak repeller**

5. $f'(p) = -1$:
   - $-2f'''(p) - 3\left(f''(p)\right)^2 < 0$:p is a **weak attractor**
   - $-2f'''(p) - 3\left(f''(p)\right)^2 > 0$:$p$ is a **weak repeller**

Example 4B
Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x - x^3$. Establish the stability of the fixed point of f.

Solution: Fixed points

$$f(x) = x \Rightarrow x - x^3 = x \Rightarrow x^3 = 0 \iff x = 0.$$

$\mathrm{Fix}(f) = \{0\}$.

$$f'(x) = 1 - 3x^2 \text{ and } |f'(0)| = 1 \Rightarrow 0 \text{ is non-hyperbolic.}$$

$$f''(x) = -6x, \text{ and } f''(0) = 0, \text{ and } f'''(x) = -6 \Rightarrow 0 \text{ is a weak attractor.}$$

4.2 The Logistic Map

We now return to the logistic map $f_\mu(x) = \mu x(1 - x)$, $x \in I$, $\mu > 0$. Recall what we have established so far from example 3B and 3D:

- $\mathrm{Fix}(f_\mu) = \{0, p_\mu\}$ where $p_\mu = \dfrac{\mu - 1}{\mu}$
- for $\mu > 3$, f_μ has a 2-cycle $\left\{q_\mu^-, qu_\mu^+\right\}$ where $q_\mu^-, qu_\mu^+ = \dfrac{\mu + 1 \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$
- when $0 < \mu < 1$, 0 is an attracting fixed point with $f_\mu^n(x) \to 0$ as $n \to \infty$ for each $x \in [0, 1]$. (Note that $|f_\mu'(0)| = \mu$ and so the fact that 0 is an attracting fixed point for $0 < \mu < 1$ can be deduced immediately from Theorem 5).

We will now move on to consider the case when $\mu \geq 1$. We begin by proving the following theorem which shows that any interesting dynamical behaviour of the map f_μ must occur when x lies in the interval $I = [0, 1]$.

Theorem 9 *Let $\mu \geq 1$ and let $x < 0$ or $x > 1$. Then $f_\mu^n(x) \to -\infty$ as $n \to \infty$.*

Proof: Let $x_n = f_\mu^n(x)$ and suppose initially that $x < 0$. Then it follows that $x_n < 0$ $\forall n = 0, 1, 2, \cdots$. Moreover

$$x_{n+1} - x_n = f_\mu(x_n) - x_n = x_n(\mu - 1 - \mu x_n) < 0 \quad (\text{ since } x_n < 0 \text{ and } \mu \geq 1).$$

and so the sequence $\{x_n\}$ is bounded below (by a standard result on monotonic sequences)

$$x_n = f_\mu^n(x) \to l \text{ as } n \to \infty.$$

where l is finite and negative. Since f_μ is continuous, we must have

$$l = \lim_{n\to\infty} f_\mu^{n+1}(x) = \lim_{n\to\infty} f_\mu\left(f_\mu^n(x)\right) = f_\mu(l),$$

and therefore the only possible values of l are 0 and $\dfrac{\mu-1}{\mu}$. Clearly neither is possible. Therefore $\{x_n\}$ cannot be bounded below and we conclude that $x_n \to -\infty$ as $n \to \infty$. If $x > 1$, after one iteration we arrive at $f_\mu(x) < 0$ and it follows from the previous analysis that

$$f_\mu^n(x) = f_\mu^{n-1}(f_\mu(x)) \to -\infty \text{ for } n \geq 1 \text{ as } n \to \infty.$$

example 4C
Consider the logistic map $f_\mu = \mu x(1-x)$, $\mu > 0$. Establish the stability of the fixed and period 2 points when $u \geq 1$.

Solution: $f_\mu'(x) = \mu - 2\mu x$

Suppose that $\mu = 1$: $\text{Fix}(f_1) = \{0\}$. As $|f_1'(0)| = 1$, 0 is non-hyperbolic.

$$f_1''(x) = -2, \text{ so } f_1''(0) = -2 < 0 \quad \Rightarrow 0 \text{ is stable from above.}$$

Suppose $\mu \neq 1$: $\text{Fix}(f_\mu) = \{0, p_\mu\}$.

$$|f_\mu'(0)| = \mu \text{ and } |f_\mu'(p_\mu)| = \mu - 2\mu(\frac{\mu-1}{\mu}) = 2 - \mu$$

$$\text{So 0 is } \begin{cases} \text{attracting} & \text{if} \quad 0 < \mu < 1 \\ \text{repelling} & \text{if} \quad \mu > 1 \end{cases}$$

$$\text{So } p_\mu \text{ is } \begin{cases} \text{attracting} & \text{if} \quad |2-\mu| < 1 \quad (\text{ i.e if } 1 < \mu < 3) \\ \text{repelling} & \text{if} \quad |2-\mu| > 1 \quad (\text{ i.e if } \mu < 1 \text{ or } \mu > 3) \end{cases}$$

$\mu = 3$.

$$\begin{aligned} \text{Fix}(f_3(x)) &= \left\{0, \frac{2}{3}\right\} \\ f_3'\left(\frac{2}{3}\right) &= -1, \quad \text{so } \frac{2}{3} \text{ is nonhyperbolic} \\ -2f'''\left(\frac{2}{3}\right) - 3\left(f''\left(\frac{2}{3}\right)\right)^2 &< 0 \quad \Rightarrow \quad \frac{2}{3} \text{ is a weak attractor.} \end{aligned}$$

For $\mu > 3$: 2-cycle with $q_\mu^\pm = \dfrac{\mu + 1 \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$.
We have to check

$$\begin{aligned}
|(f_\mu^2)'(q_\mu^-)| &= |(f_\mu^2)'(q_\mu^+)| \\
&= |f_\mu'(q_\mu^-) f_\mu'(q_\mu^+)| \\
f_\mu'(q_\mu^-) &= \mu - (\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}) \\
&= -1 + \sqrt{\mu^2 - 2\mu - 3} \\
f_\mu'(q_\mu^+) &= \mu - (\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}) \\
&= -1 - \sqrt{\mu^2 - 2\mu - 3} \\
\text{so } |f_\mu'(q_\mu^-) f_\mu'(q_\mu^+)| &= |1 - (\mu^2 - 2\mu - 3)^2| \\
&= |4 + 2\mu - \mu^2| \\
&= |5 - (\mu - 1)^2| \\
|f_\mu'(q_\mu^-) f_\mu'(q_\mu^+)| < 1 &\iff -1 < 5 - (\mu - 1)^2 < 1 \\
&\iff 4 < (\mu - 1)^2 < 6 \\
&\iff 2 < \mu - 1 < \sqrt{6} \\
&\iff 3 < \mu < 1 + \sqrt{6} \approx 3.45
\end{aligned}$$

Hence $\left\{q_\mu^+, q_\mu^-\right\}$ is an attracting 2-cycle for $3 < \mu < 1 + \sqrt{6}$ and a repelling 2-cycle for $\mu > 1 + \sqrt{6}$.

Theorem 10 *Let $1 < \mu < 3$ and let $f_\mu(x) = \mu x(1 - x)$. Then $\mathrm{Per}(f_\mu) = \left\{0, \frac{\mu-1}{\mu}\right\}$, and $W^s(\frac{\mu-1}{\mu}) = (0, 1)$ (which means that $\omega(x) = \left\{\frac{\mu-1}{\mu}\right\}$ $\forall x \in (0, 1)$)*

4.3 Parameter Space Analysis: Bifurcation

We have seen how the dynamic behaviour of the logistic map changes as the parameter μ changes. In this section, we consider any general **one-parameter family** of maps $\{f_\mu : \mu \in I\}$, where I is an interval in $\mathbb{R}$. For each $\mu \in I$, we assume that $f_\mu : I_\mu \to I_\mu$ for some interval $I_\mu \subseteq \mathbb{R}$. If we define the function F (of two variables) by

$$F(\mu, x) = f_\mu(x), \quad \mu \in I, x \in I_\mu,$$

then we shall also assume that F is continuously differentiable with respect to both μ and x. Our aims are to

1. Examine what happens to the dynamical properties of f_μ as μ varies.
2. Identify the values of μ at which changes to the dynamical properties occur.

Definition 11 *Let $J \subseteq I$ be an interval and let $p : J \to \mathbb{R}$ be a continuous function such that*

$$f_\mu(p(\mu)) = p(\mu) \quad \forall \mu \in J.$$

(i.e $p(\mu)$ is a fixed point of f_μ for each $\mu \in J$) Then the function p is called a **branch** *of fixed point of the one-parameter family. Branches of periodic points of prime period n can be defined in a similar manner.*

Example 4D
Let $f_\mu = \mu x(1-x)$ for $x \in \mathbb{R}$, $\mu > 0$ i.e. $I_\mu = \mathbb{R}$, $\forall \mu \in I$ where $I = (0, \infty)$. Plot the branches of fixed points and points of prime period 2.

Solution:

Definition 12 *A value $\mu_0 \in I$ is a* **bifurcation value** *(or* **bifurcation point***) of the one-parameter family $\{f_\mu : \mu \in I\}$ if the following conditions hold:*

1. *there are two different functions $p_i : J_i \to \mathbb{R} (i = 1, 2)$ whose graphs are branches of fixed points or periodic points of the family, and $p_1(\mu_0) = p_2(\mu_0)$;*
2. *$J_1 \cap J_2 \neq \{\mu_0\}$ and $p_1(\mu) \neq p_2(\mu) \quad \forall \mu \in J_1 \cap J_2 : \mu \neq \mu_0$.*

Bifurcations can be represented graphically on a **bifurcation diagram** in which the location of fixed points (or periodic points) are plotted for each value of μ. Stable branches of fixed points (or periodic points) are usually represented by solid lines, with unstable branches represented by dashed lines. Below are some common types of bifurcation.

1. **Saddle node** (or **tangent** or **fold**) **bifurcation**. In this case, there are two branches p_1 and p_2 of fixed point (or periodic points) which are defined on a common interval J which begins (or ends) at the bifurcation value μ_0. In addition,

   $$p_1(\mu_0) = p_2(\mu_0), \quad p_1(\mu) \neq p_2(\mu) \text{ for } \mu \in J, \mu \neq \mu_0,$$

   and no other branch p has the property $p(\mu_0) = p_1(\mu_0)$.

   Show saddle node bifurcation diagram here

   The diagram shows a typical fold bifurcation. In the vicinity of μ_0, the fixed points $p_1(\mu)$ and $p_2(\mu)$ lie on a parabolic shaped curve with 'vertex' occurring at (μ_0, p_0) where $p_0 = p_1(\mu_0) = p_2(\mu_0)$.

2. **Transcritical bifurcation**. Two branches p_1 and p_2 of fixed points (or periodic points) intersects at $\mu = \mu_0$ and neither 'double back' at μ_0.

   Show transcritical bifurcation diagram here

3. **Pitchfork bifurcation**. This is a combination of saddle node and transcritical bifurcations. In the vicinity of $\mu = \mu_0$, the branches of fixed points p_1 and p_2 lie on a parabolic shaped curve with vertex (μ_0, p_0), with another fixed point branch, p_3 also passing through this vertex. Pitchfork bifurcation for periodic points of the same prime period is defined in a similar manner.

   Show pitchfork bifurcation diagram here

4. **Period doubling bifurcation**. Let f_μ^2 (the second iterate of f_μ) undergo pitchfork bifurcation at $\mu = \mu_0$ such that $p_1(\mu)$ and $p_2(\mu)$ form a 2-cycle for f_μ but $p_3(\mu)$ is a fixed point of f_μ. Then we say that period doubling bifurcation has occurred at μ_0. Typically this involves

   (a) a change from an attracting fixed point to a repelling fixed point;

   (b) the birth of an attracting 2-cycle.

   Period doubling bifurcation of period points of prime period n is also possible since the nth iterate f_μ^n can display the type of behaviour described above for f_μ. In fact, we shall see in the next section that one way in which a one-parameter family of maps ends up behaving in a 'chaotic' manner is by means of a cascade of period doubling bifurcations.

   show period doubling bifurcation diagram here

When describing bifurcation, the word **subcritical** indicates that branches **disappear** as μ increases through bifurcation point, and the word **supercritical** indicates that branches **emerges** as μ increases through bifurcation point.

Example 4E
Draw a bifurcation diagram representing the dynamic behaviour of the one-parameter family of maps $f_\mu : \mathbb{R} \to \mathbb{R}$ where

$$F(\mu, x) = f_\mu(x) = 1.8x - 0.8x^2 - \mu$$

solution:
Fixed points satisfy

$$\begin{aligned}
f_\mu(x) &= x \\
\frac{9}{5}x - \frac{4}{5}x^2 - \mu &= x \\
\frac{4}{5}x^2 - \frac{9}{5}x + \mu + x &= 0 \\
\frac{4}{5}x^2 - \frac{4}{5}x + \mu &= 0 \\
x &= \frac{1 \pm \sqrt{1 - 5\mu}}{2}
\end{aligned}$$

Hence the fixed points are $p_{1,2}(\mu) = \dfrac{1 \pm \sqrt{1-5\mu}}{2}$.

Note that $f'_\mu(x) = \dfrac{9}{5} - \dfrac{8x}{5}$ and $f''_\mu(x) = -\dfrac{8}{5}$.

- $\mu = \dfrac{1}{5}$
  There is only one fixed point $x = \dfrac{1}{2}$.
  $f'_{0.2}\left(\dfrac{1}{2}\right) = 1$, so $x = \dfrac{1}{2}$ is a non-hyperbolic fixed point.
  However, $f''_{0.2}(\frac{1}{2}) = -\frac{8}{5}$, so the fixed point is semi-stable from above.

- $\mu > \dfrac{1}{5}$
  There are no fixed points.

- $\mu < \dfrac{1}{5}$
  There are two fixed points $p_1(\mu)$ and $p_2(\mu)$.

$$\begin{aligned}
\left|f'_\mu(p_1(\mu))\right| &= \left|1 - \frac{4\sqrt{1-5\mu}}{5}\right| \\
\left|1 - \frac{4\sqrt{1-5\mu}}{5}\right| < 1 &\iff -2 < -\frac{4\sqrt{1-5\mu}}{5} < 0 \\
&\iff 0 < \sqrt{1-5\mu} < \frac{5}{2} \\
&\iff 0 < 1-5\mu < \frac{25}{4} \\
&\iff -\frac{21}{4} < 5\mu < 1 \\
&\iff -\frac{21}{20} < \mu < \frac{1}{5}.
\end{aligned}$$

  Thus $p_1(\mu)$ is an attracting fixed point for $-\dfrac{21}{20} < \mu < \dfrac{1}{5}$, and a repelling fixed point for $\mu < -\dfrac{21}{20}$.
  $f'_\mu(p_2(\mu)) = \dfrac{9}{5} - \dfrac{4(1-\sqrt{1-5\mu})}{5} = 1 + \dfrac{4\sqrt{1-5\mu}}{5} > 1$.
  Hence for $\mu < \dfrac{1}{5}$, $p_2(\mu)$ is a repelling fixed point.

- $\mu = -\dfrac{21}{20} = -1.05$
  For the fixed point at $p_1(-1.05) = 1.75$, we have $f'_{-1.05}(1.75) = -1$. So $x = 1.75$ is a non-hyperbolic fixed point and $-2f'''(1.75) - 3(f''(1.75))^2 < 0$, hence $x = 1.75$ is a weak attractor.

Period 2 points satisfy

$$
\begin{aligned}
f_\mu^2(x) &= x \\
\frac{9}{5}\left(\frac{9}{5}x - \frac{4}{5}x^2 - \mu\right) - \frac{4}{5}\left(\frac{9}{5}x - \frac{4}{5}x^2 - \mu\right)^2 - \mu &= x \\
f_\mu^2(x) - x &= 0 \\
-\frac{2}{125}\left(4x^2 - 4x + 5\mu\right)\left(8x^2 - 28x + 35 + 10\mu\right) &= 0
\end{aligned}
$$

$\left(4x^2 - 4x + 5\mu\right)$ is a factor from period 1, where we obtain $p_1(\mu)$ and $p_2(\mu)$. To obtain period 2-cycle, we need to use the second factor of the above equation.

$$
\begin{aligned}
8x^2 - 28x + 35 + 10\mu &= 0 \\
x &= \frac{7 \pm \sqrt{-21 - 20\mu}}{4} \\
p_{3,4}(\mu) &= \frac{7 \pm \sqrt{-21 - 20\mu}}{4}
\end{aligned}
$$

f_μ has a 2-cycle $p_3(\mu)$ and $p_4(\mu)$ for $-21 - 20\mu > 0 \Rightarrow \mu < -\dfrac{21}{20}$.

$$
\begin{aligned}
\left|f_\mu'(p_3(\mu)) f_\mu'(p_4(\mu))\right| &= |4.36 + 3.2\mu| < 1 \\
&\Rightarrow -1.675 < \mu < -1.05
\end{aligned}
$$

f_μ has an attracting 2-cycle for $-1.675 < \mu < -1.05$. Combine all the information to produce a bifurcation diagram.

Theorem 11 (The Implicit Function Theorem.) *Suppose that $G : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with respect to each variable, and let (x_0, y_0) be such that*

$$G(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial G}{\partial y}(x_0, y_0) \neq 0.$$

Then there exist open interval U (about x_0) and V (about y_0) and a continuously differentiable function $f : U \to V$ such that

1. $y_0 = f(x_0)$
2. $G(x, f(x)) = 0 \ \forall x \in U$
3. $G(x, y) \neq 0$ *for* $(x, y) \in U \times V$ *unless* $y = f(x)$

Moreover

$$f'(x) = -\frac{G_x(x, f(x))}{G_y(x, f(x))} \quad \forall x \in U.$$

Proof: omitted