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SSH 1033 MATHEMATICAL METHODS 2 **TUTORIAL 9**

1. Evaluate the following iterated integrals.

(a) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx.$ (b) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz.$

(c) $\displaystyle\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\, dy\, dx.$ (d) $\displaystyle\int_1^3 \int_x^{x^2} \int_0^{\ln z} xe^y\, dy\, dz\, dx.$

(e) $\displaystyle\int_0^{\pi} \int_0^{4\cos z} \int_0^{\sqrt{16-y^2}} y\, dx\, dy\, dz.$ (f) $\displaystyle\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\, dz\, dy\, dx.$

2. Evaluate the following integrals.

(a) $\displaystyle\iiint_G xy\sin(yz)\, dV$, where G is the rectangular box bounded by $0 \le x \le \pi$, $0 \le y \le 1$ and $0 \le z \le \pi/6$.

(b) $\displaystyle\iiint_G z\, dV$, where G is the tetrahedron in the first octant bounded by $x + y + z = 4$.

(c) $\displaystyle\iiint_G xyz\, dV$, where G is the solid in the first octant bounded by the parabolic cylinder $z = 2 - x^2$, planes $y = x$, $z = 0$ and $y = 0$.

3. Given $\displaystyle\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\, dy\, dx.$

Rewrite the integral as an equivalent iterated integral in the order

(a) $dy\, dz\, dx$. (b) $dy\, dx\, dz$. (c) $dx\, dy\, dz$.
(d) $dx\, dz\, dy$. (e) $dz\, dx\, dy$.

4. Find the volumes of the following solids using triple integrals.

(a) The tetrahedron in the first octant bounded by the coordinates planes and the plane passing through (1, 0, 0), (0, 2, 0) and (0, 0, 3).

(b) The solid bounded by the parabolic cylinder $z = y^2$ and rectangular planes $x = 0$, $x = 1$, $y = -1$ and $y = 1$.

(c) The solid bounded on its sides by the surface $y = x^2$, above by the plane $y + z = 1$ and below by the plane $z = 0$.

(d) The solid in the first octant bounded by the coordinate planes, the plane $y + z = 2$ and the parabolic cylinder $x = 4 - y^2$.

(e) The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$.

(f) The solid cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.

(g) The solid in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$.

5. Evaluate triple integrals

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy.$$

Give a physical interpretation of this integral.

6. Show that the volume of a common solid of the intersection between two paraboloids $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$ is $8\sqrt{2}\,\pi$.

7. Evaluate the following cylindrical coordinate integrals.

(a) $\displaystyle\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta.$

(b) $\displaystyle\int_{1}^{2\pi}\int_{1}^{3}\int_{r^2/3}^{\sqrt{18-r^2}} dz\,r\,dr\,d\theta.$

(c) $\displaystyle\int_{0}^{\pi}\int_{0}^{\theta/\pi}\int_{-\sqrt{4-r}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta.$

(d) $\displaystyle\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta.$

8. Convert the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy.$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

9. Use cylindrical coordinates to evaluate the following integrals.

(a) $\displaystyle\iiint_G \sqrt{x^2+y^2}\,dV,$ where G is the solid bounded by surfaces $z = 2$ and $z = x^2 + y^2$.

(b) $\displaystyle\iiint_G z\,dV,$ where G is the cylinder $y^2 + z^2 = 1$ which intersects with the planes $y = x$, $x = 0$ and $z = 0$.

(c) $\displaystyle\iiint_G dV,$ where G is the solid bounded by paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$.

10. Find the volumes of the following solids using triple integrals.

(a) The solid bounded by the paraboloid $z = r^2$ and the plane $z = 9$.

(b) The solid bounded below by the plane $z = 0$, on its side by the cylinder $r = 2\sin\theta$ and above by $z = r^2$.

(c) The solid bounded by the cylinder $x^2 + y^2 = 9$, and the planes $x + z = 5$ and $z = 1$.

(d) The solid is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the $xy-$plane and whose top lies in the plane $z = 4 - y$.

11. Evaluate the following integrals using cylindrical coordinates.

(a) $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{(x^2+y^2)^2}^{16} x^2\,dz\,dy\,dx.$

(b) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{(x^2+y^2)^2}^{9} y^2\,dz\,dx\,dy.$

(c) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{(x^2+y^2)^2}^{9} y^2\,dz\,dx\,dy.$

(d) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx.$

12. Find the volume of the following solid using spherical coordinates.

(a) The sphere $x^2 + y^2 + z^2 = 9$.

(b) The solid bounded above by the sphere $\rho = 4$, and below by the cone $\phi = \pi/3$.

(c) The solid that lies above the xy-plane, outside the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 9$.

13. Evaluate the following integrals using spherical coordinates.

(a) $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx.$

(b) $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx.$

(c) $\displaystyle\iiint_G \frac{1}{\sqrt{x^2 + y^2 + z^2}}\, dV,$ where G is a sphere $x^2 + y^2 + z^2 \leq 9$.

14. Find the centroid of the solid that lies below the sphere $\rho = 4$ and above the cone $\phi = \pi/3$.

15. Find the centroid of the solid bounded by the surface $z = r$ and the plane $z = 9$.

16. Find the moment of inertia of the solid created by intersection of the sphere $\rho \leq 1$ with the cone $\phi = \pi/3$ and z-axis if the density of the solid is $\delta(x, y, z) = 2$.

17. Find the coordinate, $\bar{z}$ of the centroid of a solid bounded by the planes $x = 0, z = 0, y = x, y = 1,$ and $z = 2 - x$.

18. Show that the volume of a common solid of the intersection between a sphere $x^2+y^2+z^2 = 2$ and a parboloid $z = x^2 + y^2$ is

$$\frac{\pi}{6}\left(8\sqrt{2} - 7\right).$$

Obtain the centroid of the solid.

19. By using the cylinder coordinates, find the volume of the solid bounded on top by the sphere $x^2 + y^2 + z^2 = 16$ and it's sides by a cylinder $x^2 + y^2 = 4y$ and $z = 0$.

20. By using the spherical coordinates, evaluate

$$\iiint_G z\, dV,$$

where G is the solid bounded below by a cone $z = \sqrt{x^2 + y^2}$ and above by a sphere $x^2 + y^2 + z^2 = 2z$.

21. By using the cylindrical coordinates, evaluate

$$\iiint_G \sqrt{x^2 + y^2}\, dV,$$

where G is the solid bounded below by the plane $z = 0$ and above by the plane $z = 3$ and it's sides by the surface $x^2 + (y - 1)^2 = 1$.

22. By using the spherical coordinates, evaluate

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_{\sqrt{x^2+y^2}}^5 dz\, dy\, dx.$$

23. Transform the integral

$$\int_{-\sqrt{12}}^{\sqrt{12}} \int_{-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \int_2^{\sqrt{16-x^2-y^2}} dz\, dy\, dx$$

into each of the following coordinates systems:

(a) cylindrical, (b) spherical.

Hence, evaluate the integral using only one of the coordinates system.

24. Evaluate the integral $\iiint\limits_G z\, dV$,

where G is the tetrahedron in the first octant bounded by the plane $x + 2y + 3z = 6$.

25. Find the volume of the solid bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the $xy-$plane and laterally by the cylinder $x^2 + y^2 = 2y$.

26. Use cylindrical coordinates to find the volume of the solid below the surface $x^2+y^2+z^2 = 4$ and above the plane $z = 1$.

27. Use spherical coordinates to evaluate

$$\int_0^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{\frac{3}{2}}\, dz\, dx\, dy.$$

28. Evaluate the following triple integrals in an appropriate coordinate system

$$\iiint\limits_G (x^2 + y^2)^{\frac{3}{2}}\, dV,$$

where G is the solid in the first octant bounded by paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

29. Evaluate the following iterated integrals by changing the coordinate system.

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^4 \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx.$$

30. The mass of a solid G with density function $\sigma(x, y, z)$ is given by

$$m = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x \sigma(x, y, z)\, dz\, dx\, dy.$$

(a) Sketch the solid G.

(b) Find the centroid of the solid G, if the density is constant.

31. Find the volume of the solid cut from sphere $\rho = a$ by the cone $\phi = \alpha$, where $0 < \alpha < \pi/2$. Hence, deduce the volume of the solid for the cases when $\alpha = 0,\ \pi/2,\ \pi$.

32. Find the smaller volume cut from the sphere $\rho = 2$ by the plane $z = \sqrt{2}$.

33. (a) Use triple integrals to show that the volume of the solid above by a sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z = \sqrt{2}$ is $\frac{2}{3}(8 - 5\sqrt{2})\pi$.

(b) By using the spherical coordinates, evaluate

$$\iiint_G z\,dV,$$

where G is the solid bounded above by a sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z = \sqrt{2}$.

Hence, by using the result in part (a), find the center of gravity of the solid assuming constant density.

34. Find the mass of the solid G bounded by $x^2 + y^2 + z^2 \leq 4$, $x \geq 0$, $y \geq 0$, $z \geq 0$, if the density is equal to xyz.

35. Show that the volume of the solid G bounded by $z = 4 - x^2 - y^2$ and the xy–plane is 8π. Hence find the centroid of G assuming constant density σ. ANS $(0,\ 0,\ \frac{4}{3})$.

36. Show that the volume of the solid G in first octant bounded by $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, where a, b, c are constants is $\frac{1}{6}abc$. Hence find the centroid of G assuming constant density σ. ANS $(\frac{1}{4}a,\ \frac{1}{4}b,\ \frac{1}{4}c)$.

37. Evaluate

$$\iiint_G \sqrt{x^2 + y^2 + z^2}\,dx\,dy\,dz,$$

where G is the solid bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2 + y^2}$. Give a physical interpretation of the integral.

38. Evaluate

$$\iiint_G \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}},$$

where G is the solid bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ where $a > b > 0$. Give a physical interpretation of the integral.

39. Find the

(a) volume, (b) centroid,

of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the plane $z = b$ where $a > b > 0$, assuming constant density σ.