

### Example 4E

Given the one parameter family of maps  $f_\mu : \mathbb{R} \rightarrow \mathbb{R}$  where

$$F(\mu, x) = f_\mu(x) = \frac{9}{5}x - \frac{4}{5}x^2 - \mu.$$

Find the fixed point and points of prime period 2. Plot the bifurcation diagram representing the dynamics behaviour of the maps.

*Solution:*

Finding fixed points:

$$\begin{aligned} f_\mu(x) &= \frac{9}{5}x - \frac{4}{5}x^2 - \mu = x \\ x - \frac{9}{5}x + \frac{4}{5}x^2 + \mu &= 0 \\ \frac{4}{5}x^2 - \frac{4}{5}x + \mu &= 0 \\ \frac{4}{5}\left(x^2 - x + \frac{5}{4}\mu\right) &= 0 \\ x &= \frac{1 \pm \sqrt{1 - 5\mu}}{2} \end{aligned}$$

The fixed points are:

$$\text{If } \mu = \frac{1}{5}, x = \frac{1}{2}.$$

$$\text{If } \mu < \frac{1}{5}, \text{ there are two fixed points, } P_1 = \frac{1 + \sqrt{1 - 5\mu}}{2}, P_2 = \frac{1 - \sqrt{1 - 5\mu}}{2}.$$

$$\text{If } \mu > \frac{1}{5}, \text{ there is no fixed point.}$$

Check the stability of the fixed points:

$$f'_\mu(x) = \frac{9}{5} - \frac{8}{5}x \quad \text{and} \quad f''_\mu(x) = -\frac{8}{5}$$

$$\text{At } x = \frac{1}{2},$$

$$f'_\mu\left(\frac{1}{2}\right) = \frac{9}{5} - \frac{8}{5}\left(\frac{1}{2}\right) = 1 \Rightarrow x = \frac{1}{2} \text{ is non-hyperbolic.}$$

$$f''_\mu\left(\frac{1}{2}\right) = -\frac{8}{5} < 0, \text{ hence } x = \frac{1}{2} \text{ is semi-stable from above.}$$

Stability for  $x = P_1$ ,

$$\begin{aligned} f'_\mu \left( \frac{1 + \sqrt{1 - 5\mu}}{2} \right) &= \frac{9}{5} - \frac{8}{5} \left( \frac{1 + \sqrt{1 - 5\mu}}{2} \right) \\ &= \frac{9}{5} - \frac{4}{5} - \frac{4\sqrt{1 - 5\mu}}{5} \\ &= 1 - \frac{4\sqrt{1 - 5\mu}}{5} \end{aligned}$$

$$\begin{aligned} \left| f'_\mu \left( \frac{1 + \sqrt{1 - 5\mu}}{2} \right) \right| &= \left| 1 - \frac{4\sqrt{1 - 5\mu}}{5} \right| < 1 \\ -1 &< 1 - \frac{4\sqrt{1 - 5\mu}}{5} < 1 \\ -\frac{21}{20} &< \mu < \frac{1}{5} \\ -1.05 &< \mu < 0.2 \end{aligned}$$

$x = P_1 = \frac{1 + \sqrt{1 - 5\mu}}{2}$  is an attractor if  $-1.05 < \mu < 0.2$ .

At  $\mu = -\frac{21}{20}$ ,  $P_1 = \frac{7}{4}$ , and  $f'_\mu(P_1) = -1 \Rightarrow x = P_1 = \frac{7}{4}$  is non-hyperbolic.

$$-2f''''_\mu \left( \frac{7}{4} \right) - 3 \left( f''_\mu \left( \frac{7}{4} \right) \right)^2 < 0 \Rightarrow P_1 = \frac{7}{4} \text{ is a weak attractor.}$$

Stability for  $x = P_2$ ,

$$\begin{aligned} f'_\mu \left( \frac{1 - \sqrt{1 - 5\mu}}{2} \right) &= \frac{9}{5} - \frac{8}{5} \left( \frac{1 - \sqrt{1 - 5\mu}}{2} \right) \\ &= \frac{9}{5} - \frac{4}{5} + \frac{4\sqrt{1 - 5\mu}}{5} \\ &= 1 + \frac{4\sqrt{1 - 5\mu}}{5} > 1 \end{aligned}$$

Hence,  $x = P_2$  is a repellor.

Determine the prime period 2 of the system. Need to solve

$$f_\mu^2(x) = x$$

$$\begin{aligned} \frac{9}{5} \left( \frac{9}{5}x - \frac{4}{5}x^2 - \mu \right) - \frac{4}{5} \left( \frac{9}{5}x - \frac{4}{5}x^2 - \mu \right)^2 - \mu &= x \\ f_\mu^2(x) - x &= 0 \\ \frac{56}{25}x - \frac{36}{25}x^2 - \frac{14}{5}\mu - \frac{4}{5} \left( \frac{9}{5}x - \frac{4}{5}x^2 - \mu \right)^2 &= 0 \\ -\frac{64}{125}x^4 + \frac{288}{125}x^3 + \left( -\frac{504}{125} - \frac{32}{25}\mu \right)x^2 + \left( \frac{56}{25} + \frac{72}{25}\mu \right)x - \frac{14}{5}\mu - \frac{4}{5}\mu^2 &= 0 \\ -\frac{8}{125} \left( x^2 - x + \frac{5}{4}\mu \right) (8x^2 - 28x + 35 + 10\mu) &= 0 \end{aligned}$$

$\left( x^2 - x + \frac{5}{4}\mu \right)$  must be a factor since two solutions of  $f_\mu^2(x) = x$  are  $\text{Per}_1(f_\mu)$ .

$$x = \{P_1, P_2\} = \frac{1 \pm \sqrt{1 - 5\mu}}{2} \quad (8x^2 - 28x + 35 + 10\mu) = 0.$$

The second equation gives

$$x = \frac{28 \pm \sqrt{(28)^2 - 4(8)(35 + 10\mu)}}{16} = \frac{28 \pm \sqrt{-336 - 320\mu}}{16} = \frac{7 \pm \sqrt{-21 - 20\mu}}{4}.$$

Thus  $f_\mu$  has 2-cycle  $\{q_u^+, q_u^-\}$  if  $-21 - 20\mu > 0 \Rightarrow \mu < -\frac{21}{20}$ .

For  $\mu < -\frac{21}{20}$ : 2-cycle with  $q_\mu^\pm = \frac{7 \pm \sqrt{-21 - 20\mu}}{4}$ .

We have to check

$$\begin{aligned} |(f_\mu^2)'(q_\mu^-)| &= |(f_\mu^2)'(q_\mu^+)| \\ &= |f'_\mu(q_\mu^-)f'_\mu(q_\mu^+)| \\ f'_\mu(q_\mu^-) &= \frac{9}{5} - \frac{8}{5} \left( \frac{7 - \sqrt{-21 - 20\mu}}{4} \right) \\ &= -1 + \frac{2}{5}\sqrt{-21 - 20\mu} \\ f'_\mu(q_\mu^+) &= -1 - \frac{2}{5}\sqrt{-21 - 20\mu} \end{aligned}$$

$$\begin{aligned}
\text{so } |f'_\mu(q_\mu^-)f'_\mu(q_\mu^+)| &= \left| 1 - \frac{4}{25}(-21 - 20\mu) \right| \\
&= \left| \frac{109}{25} + \frac{16}{5}\mu \right| \\
|f'_\mu(q_\mu^-)f'_\mu(q_\mu^+)| < 1 &\iff -1 < \frac{109}{25} + \frac{16}{5}\mu < 1 \\
&\iff -\frac{67}{40} < \mu < -\frac{21}{20}
\end{aligned}$$

Hence  $\{q_\mu^+, q_\mu^-\}$  is an attracting 2-cycle for  $-1.67 < \mu < -1.05$ .