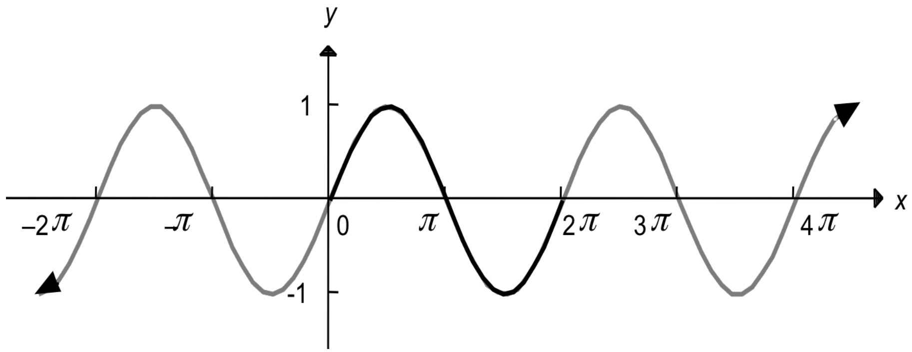
CHAPTER 2

FURTHER TRANSCENDENTAL FUNCTIONS

2.1 Review

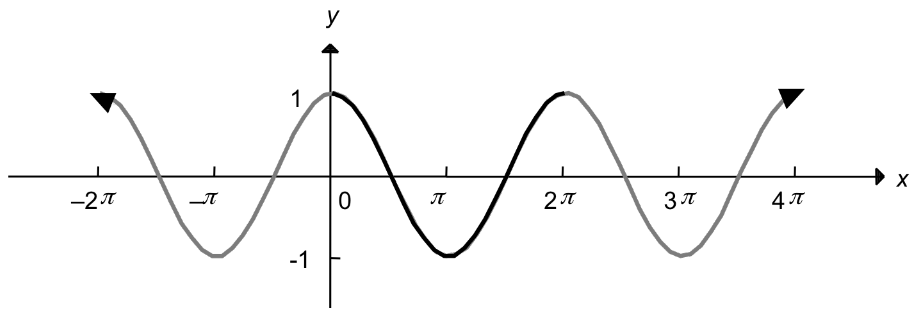
2.1.1 Graphs of Trigonometric Functions

Graph of $y = \sin x$



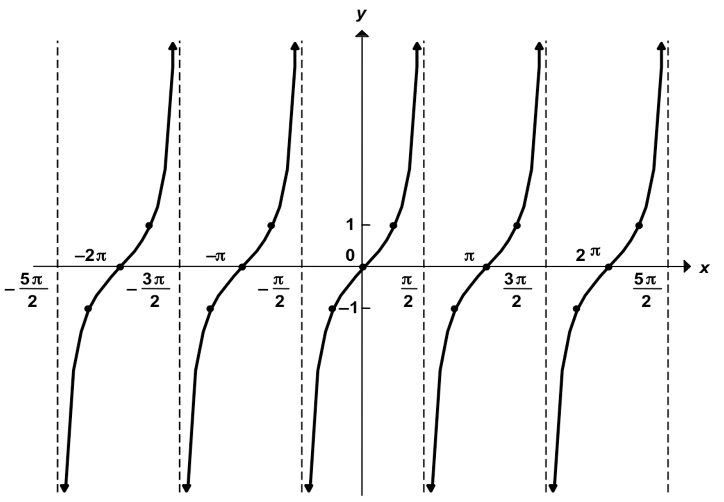
- Period: 2π
- Domain: All real numbers
- Range: [–1, 1]
- Symmetric with respect to the origin

Graph of $y = \cos x$



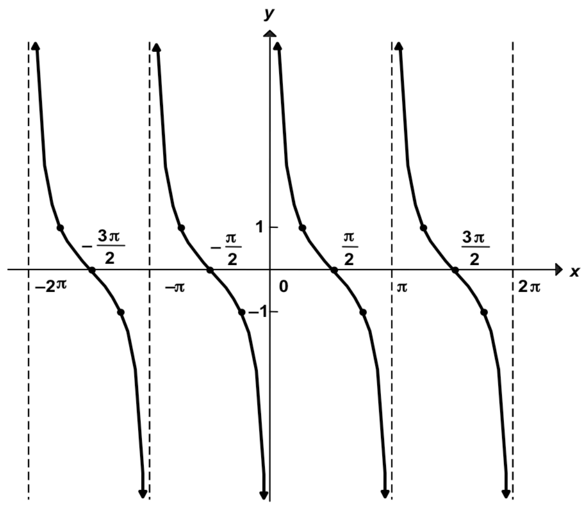
- Period: 2π
- Domain: All real numbers
- Range: [–1, 1]
- Symmetric with respect to the *y* axis

Graph of $y = \tan x$



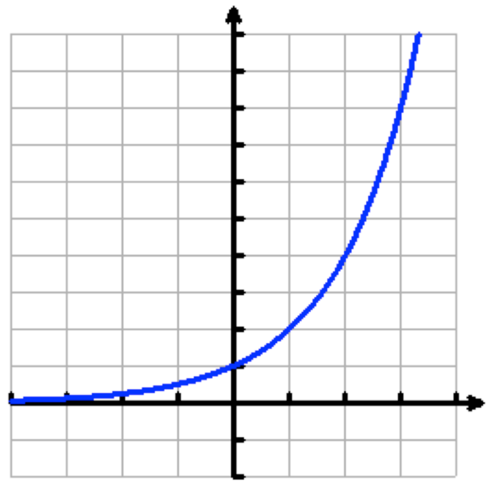
- Period: π
- Domain: All real numbers except $\pi/2 + k\pi, k$ is an integer
- Range: All real numbers
- Symmetric with respect to origin

Graph of $y = \cot x$

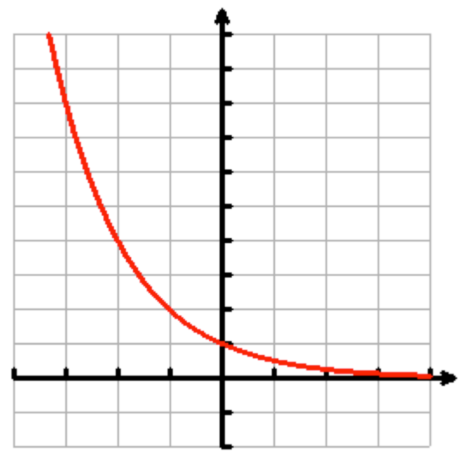


- Period: π
- Domain: All real numbers except $k\pi, k$ is an integer
- Range: All real numbers
- Symmetric with respect to origin

2.1.2 Graphs of Exponential Functions



Graph of $y = a^x, a > 1$



Graph of $y = a^x, 0 < a < 1$

- Domain: $(-\infty, \infty)$, Range: $(0, \infty)$
- Natural Exponential Function $f(x) = e^x$

2.1.3 Trigonometric Identities

TRIGONOMETRIC IDENTITIES

The six trigonometric functions:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y} = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{y} = \frac{1}{\tan\theta}$$

Sum or difference of two angles:

$$\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Double angle formulas:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Half angle formulas:

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Sum and product formulas:

$$\sin a\cos b = \tfrac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\cos a\sin b = \tfrac{1}{2}[\sin(a+b) - \sin(a-b)]$$

$$\cos a\cos b = \tfrac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$\sin a\sin b = \tfrac{1}{2}[\cos(a-b) - \cos(a+b)]$$

$$\sin a + \sin b = 2\sin\left(\tfrac{a+b}{2}\right)\cos\left(\tfrac{a-b}{2}\right)$$

$$\sin a - \sin b = 2\cos\left(\tfrac{a+b}{2}\right)\sin\left(\tfrac{a-b}{2}\right)$$

$$\cos a + \cos b = 2\cos\left(\tfrac{a+b}{2}\right)\cos\left(\tfrac{a-b}{2}\right)$$

$$\cos a - \cos b = -2\sin\left(\tfrac{a+b}{2}\right)\sin\left(\tfrac{a-b}{2}\right)$$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

where A is the angle of a scalene triangle opposite side a.

Radian measure: 8.1 p420

$$1° = \frac{\pi}{180}\text{ radians}$$

$$1\text{ radian} = \frac{180°}{\pi}$$

Reduction formulas:

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin(\theta) = -\sin(\theta-\pi) \qquad \cos(\theta) = -\cos(\theta-\pi)$$

$$\tan(-\theta) = -\tan\theta \qquad \tan(\theta) = \tan(\theta-\pi)$$

$$\mp\sin x = \cos(x \pm \tfrac{\pi}{2}) \qquad \pm\cos x = \sin(x \pm \tfrac{\pi}{2})$$

Complex Numbers:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \qquad \sin\theta = \tfrac{1}{j2}(e^{j\theta} - e^{-j\theta})$$

TRIGONOMETRIC VALUES FOR COMMON ANGLES

Degrees	Radians	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0°	0	0	1	0	Undefined	1	Undefined
30°	π/6	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
45°	π/4	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	π/3	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
90°	π/2	1	0	Undefined	0	Undefined	1
120°	2π/3	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
135°	3π/4	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	5π/6	1/2	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2
180°	π	0	-1	0	Undefined	-1	Undefined
210°	7π/6	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2
225°	5π/4	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	4π/3	$-\sqrt{3}/2$	-1/2	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$
270°	3π/2	-1	0	Undefined	0	Undefined	-1
300°	5π/3	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$	$-\sqrt{3}$	2	$-2\sqrt{3}/3$
315°	7π/4	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	11π/6	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2
360°	2π	0	1	0	Undefined	1	Undefined

2.1.4 Graphs of f and f^{-1}

Inverse Functions

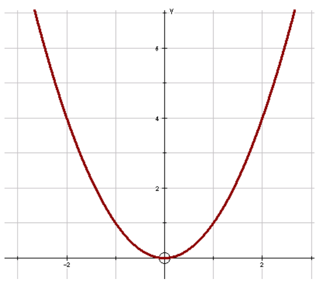
The inverse of a function f is denoted by f^{-1}. The inverse reverses the original function.
Hence, if $f(a) = b$ then $f^{-1}(b) = a$

Note: $f^{-1}(x)$ does **not** mean $1/f(x)$.

One to one Functions

If a function is to have an inverse which is also a function then it must be **one to one**.
This means that a horizontal line will never cut the graph more than once. i.e we cannot have $f(a) = f(b)$ if $a \neq b$,
Two different inputs (x values) are not allowed to give the same output (y value).

For instance $f(-2) = f(2) = 4$
$y = f(x) = x^2$ with domain $x \in \Re$ is not one to one.



Drawing the graph of the Inverse

The graph of $y = f^{-1}(x)$ is the reflection in the line $y = x$ of the graph of $y = f(x)$.

Example: Find the inverse of the function $y = f(x) = (x-2)^2 + 3$, $x \geq 2$
Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes showing the relationship between them.

Domain:
This is the function we considered earlier except that its domain has been restricted to $x \geq 2$ in order to make it one-to-one. We know that the Range of f is $y \geq 3$ and so the domain of f^{-1} will be $x \geq 3$.

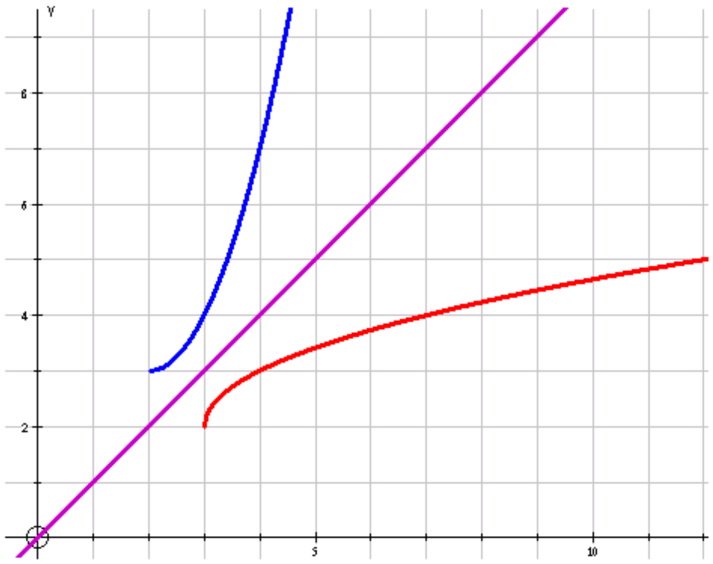
Rule:
Swap x and y to get $x = (y-2)^2 + 3$. Now make y the main subject:

$$x - 3 = (y-2)^2$$
$$\sqrt{(x-3)} = y-2$$
$$y = 2 + \sqrt{(x-3)}$$

Hence, the final answer is: $f^{-1}(x) = 2 + \sqrt{(x-3)}$, $x \geq 3$

Graphs

Reflect in $y = x$ to get the graph of the inverse function.



Note:
Remember with inverse functions everything swaps over.
Input and output (x and y) swap over
Domain and Range swap over
Reflecting in y = x swaps over the coordinates of a point so (a,b) on one graph becomes (b,a) on the other.

Note: we could also have
$-\sqrt{(x-3)} = y-2$
and $y = 2 - \sqrt{(x-3)}$
But this would not fit our function as y must be greater than 2 (see graph)

2.2.1 Definition of Hyperbolic Functions

- **Hyperbolic Sine,** pronounced **"shine".**

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

- **Hyperbolic Cosine,** pronounced **"cosh".**

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

- **Hyperbolic Tangent,** pronounced **"tanh".**

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$$

- **Hyperbolic Secant,** pronounced **"shek".**

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

- **Hyperbolic Cosecant,** pronounced **"coshek".**

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

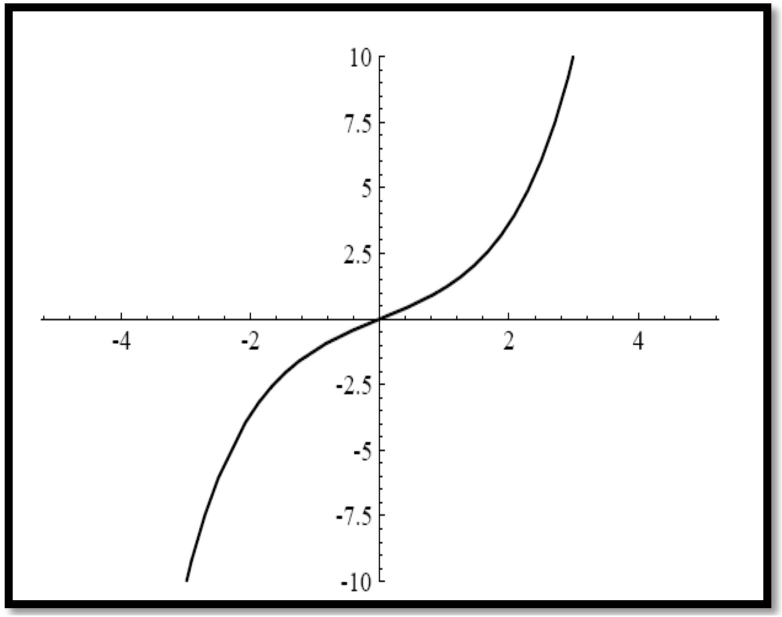
- **Hyperbolic Cotangent,** pronounced **"coth".**

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

2.2.2 Graphs of Hyperbolic Functions

Since the hyperbolic functions depend on the values of e^x and e^{-x}, its graphs is a combination of the exponential graphs.

(i) Graph of sinh *x*



From the graph, we see

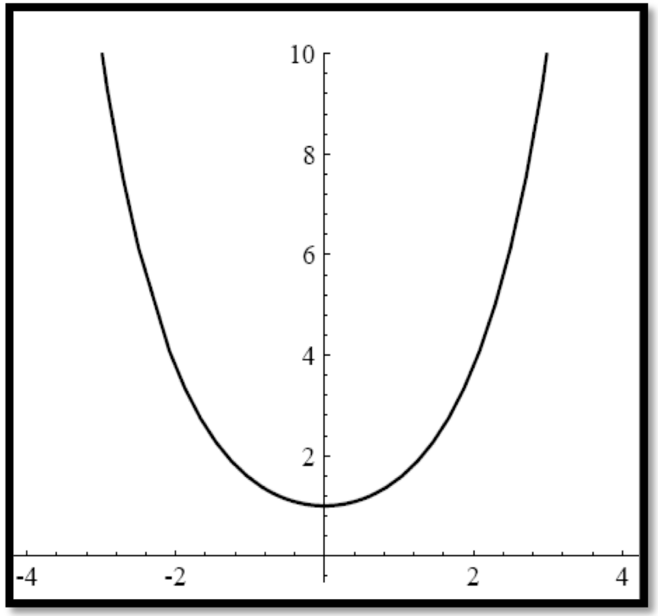
(i) sinh 0 = 0.

(ii) The domain is all real numbers

(iii) The curve is symmetrical about the origin, i.e.
sinh $(-x) = -\sinh x$

(iv) It is an increasing one-to-one function.

(ii) Graph of cosh *x*



(i) cosh 0 = 1

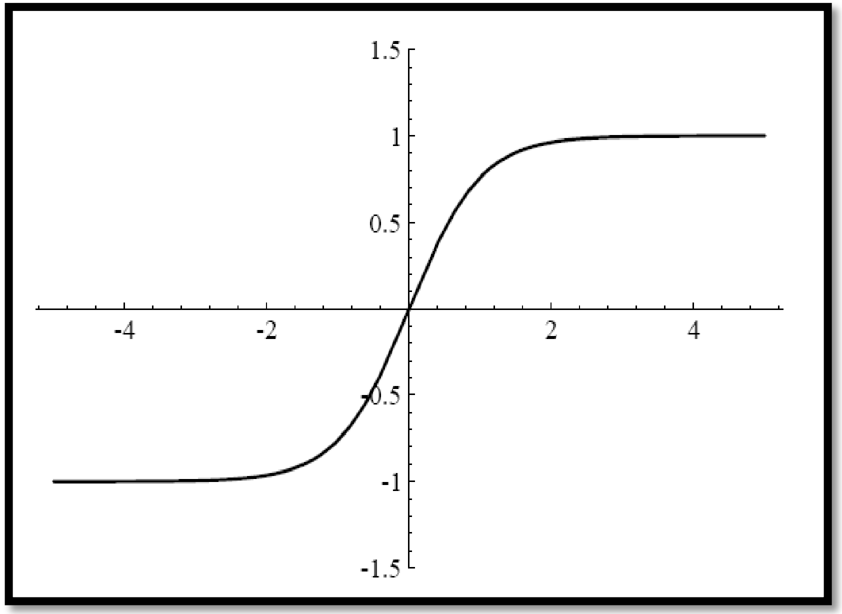
(ii) The domain is all real numbers.

(iii) The value of cosh x is never less than 1.

(iv) The curve is symmetrical about the y-axis, i.e.
cosh $(-x) = \cosh x$

(v) For any given value of cosh x, there are two values of x.

(iii) Graph of tanh x



We see

(i) $\tanh 0 = 0$

(ii) $\tanh x$ always lies between $y = -1$ and $y = 1$.

(iii) $\tanh(-x) = -\tanh x$

(iv) It has horizontal asymptotes $y = \pm 1$.

2.2.3 Hyperbolic Identities

For every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions.

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \operatorname{sec}h^2 x$
3. $\coth^2 x - 1 = \operatorname{cos}ech^2 x$
4. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
7. $\sinh 2x = 2 \sinh x \cosh x$
8. $\cosh 2x = \cosh^2 x + \sinh^2 x$

   $= 2\cosh^2 x - 1$

   $= 2\sinh^2 x + 1$
9. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Some of the hyperbolic identities follow exactly the trig. identities; others have a difference in sign.

Trig. Identities	**Hyperbolic Identities**
$\sec\theta \equiv \dfrac{1}{\cos\theta}$ $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$ $\cot\theta \equiv \dfrac{1}{\tan\theta}$	$\operatorname{sech}\theta = \dfrac{1}{\cosh\theta}$ $\operatorname{cosech}\theta = \dfrac{1}{\sinh\theta}$ $\coth\theta = \dfrac{1}{\tanh\theta}$
$\cos^2\theta + \sin^2\theta \equiv 1$ $1 + \tan^2\theta \equiv \sec^2\theta$ $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$	$\cosh^2\theta - \sinh^2\theta \equiv 1$ $1 - \tanh^2\theta \equiv \operatorname{sech}^2\theta$ $\coth^2\theta - 1 \equiv \operatorname{cosech}^2\theta$
$\sin 2A \equiv 2\sin A\cos A$ $\cos 2A \equiv \cos^2 A - \sin^2 A$ $\equiv 1 - 2\sin^2 A$ $\equiv 2\cos^2 A - 1$	$\sinh 2A \equiv 2\sinh A\cosh A$ $\cosh 2A \equiv \cosh^2 A + \sinh^2 A$ $\equiv 1 + 2\sinh^2 A$ $\equiv 2\cosh^2 A - 1$

Examples 2.1

1. Sketch the graph of the following functions. State the domain and range.
   a) $y = \sinh x + 2$
   b) $y = 2\tanh 3x$

2. By using definition of hyperbolic functions,

   a) Evaluate sinh(-4) and cosh(ln 2) to four decimal places.
   b) Show that $2\cosh^2 x - 1 = \cosh 2x$
   c) Show that $\cosh^2 x - \sinh^2 x = 1$

3. By using identities of hyperbolic functions, show that

$$\frac{1-\tanh^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$$

4. Solve the following for x, giving your answer in 4dcp.
   a) $2\cosh x - \sinh x = 2$
   b) $\cosh 2x - \sinh x = 1$