

2.3 Inverse Functions

Definition 2.3 (Inverse Functions)

If $f : X \rightarrow Y$ is a one-to-one function with the domain X and the range Y , then there exists an inverse function,

$$f^{-1} : Y \rightarrow X$$

where the domain is Y and the range is X such that

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

Thus, $f^{-1}(f(x)) = x$ for all values of x in the domain f .

Note:

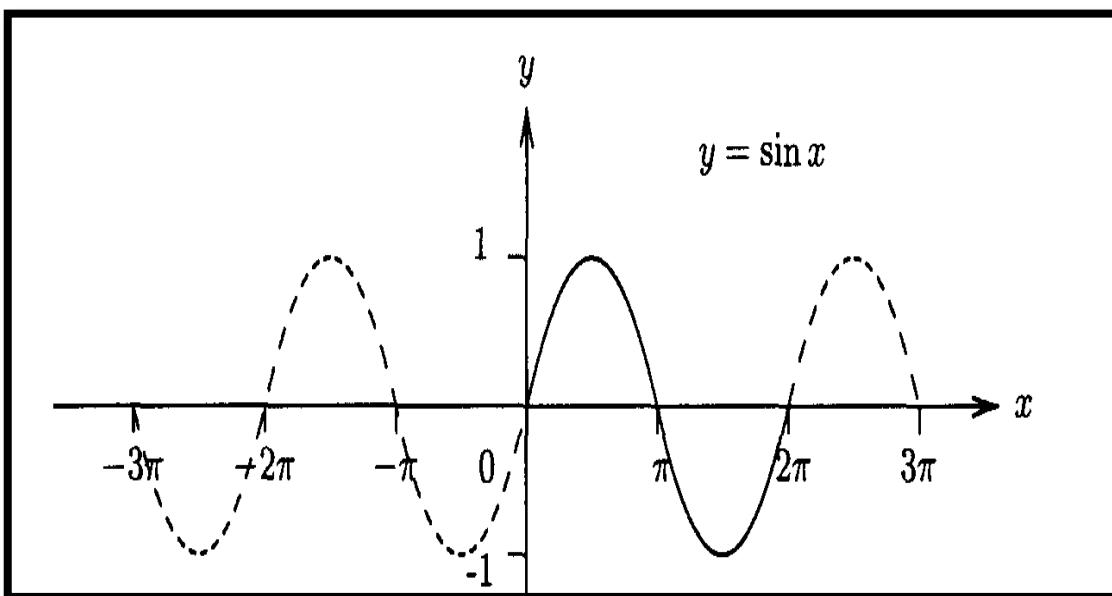
The graph of inverse function is reflections about the line $y = x$ of the corresponding non-inverse function.

2.3.1 Inverse Trigonometric Functions

Trigonometric functions are **periodic** hence they are **not one-to one**. However, if we **restrict the domain** to a chosen interval, then the restricted function is one-to-one and invertible.

(i) Inverse Sine Function

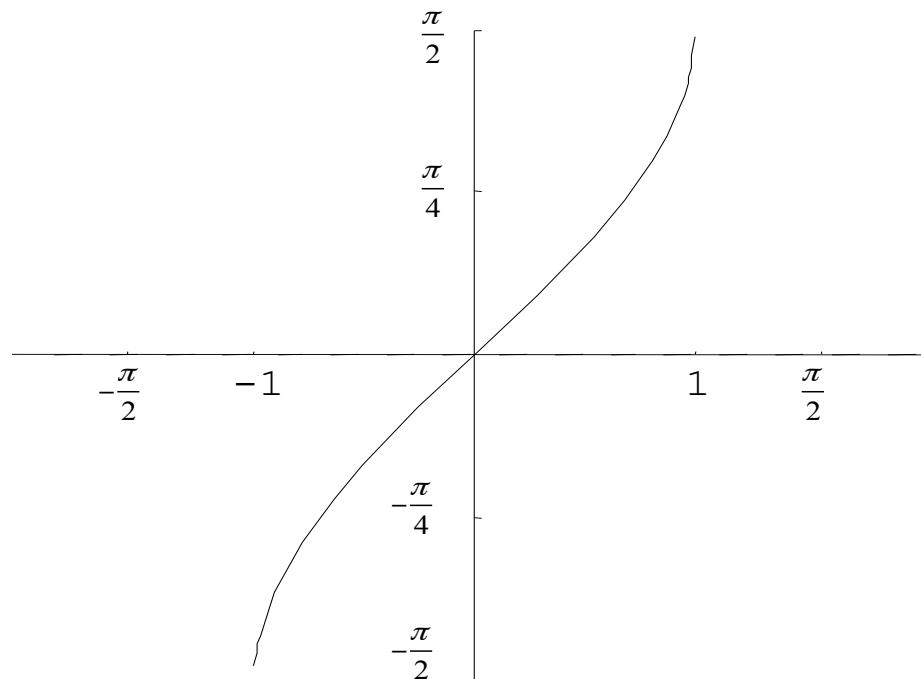
Look at the graph of $y = \sin x$ shown below



The function $f(x) = \sin x$ is not one to one. But if the domain is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $f(x)$ is one to one.

The inverse sine function is defined as $y = \sin^{-1} x \Leftrightarrow x = \sin y$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $-1 \leq x \leq 1$.

The graph of $y = \sin^{-1} x$ is shown below

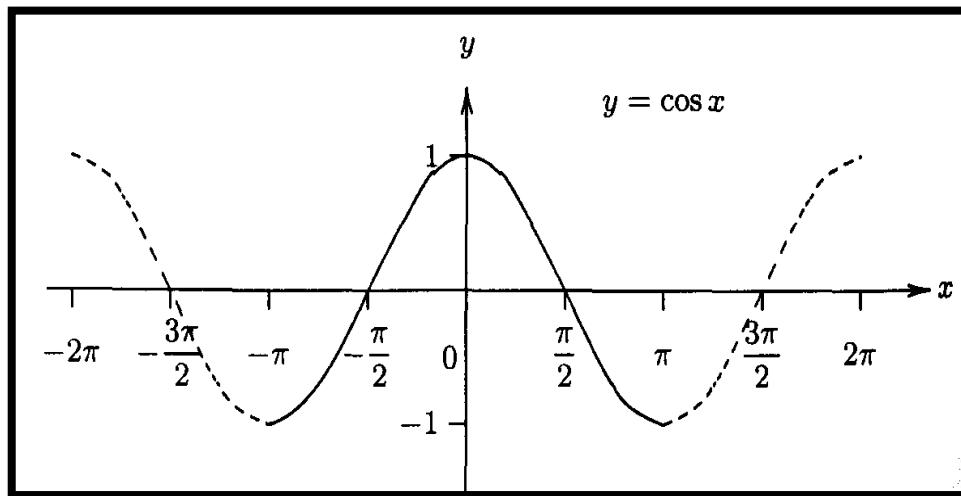


$$f(x) = \sin^{-1} x$$

$$f(x) = \arcsin x$$

ii) Inverse Cosine Function

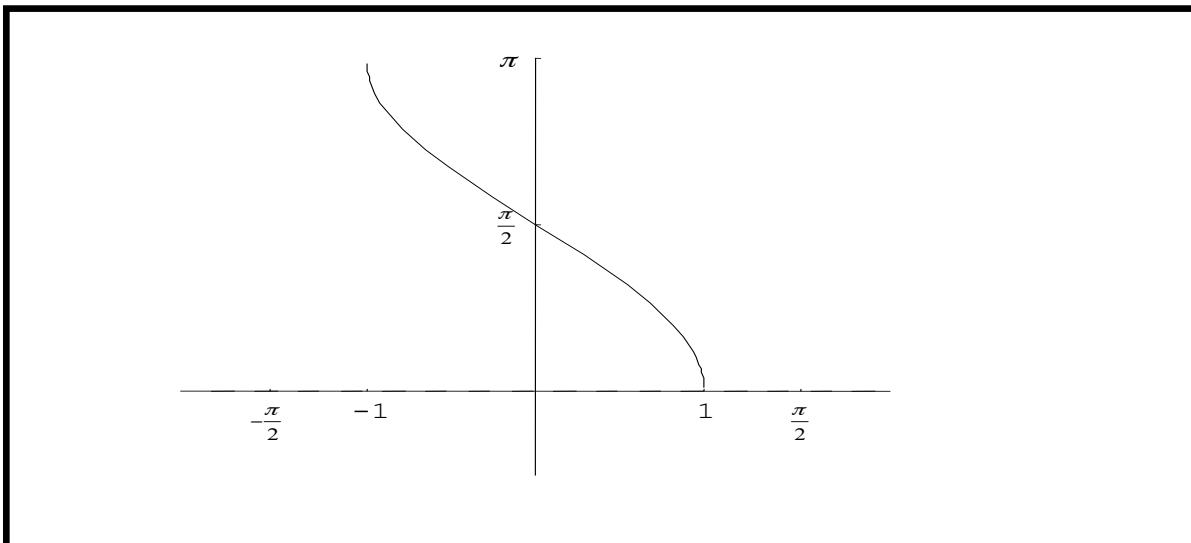
Look at the graph of $y = \cos x$ shown below



The function $f(x) = \cos x$ is not one to one. But if the domain is restricted to $[0, \pi]$, then $f(x)$ is one to one.

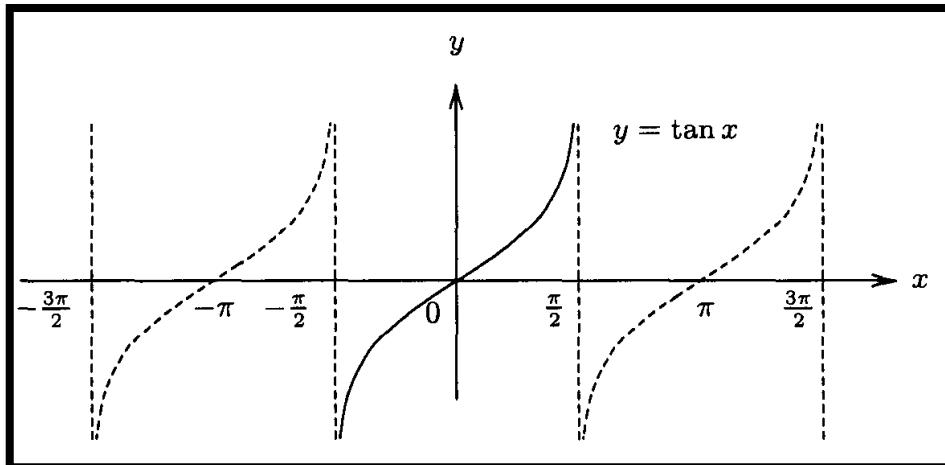
The inverse cosine function is defined as
$$y = \cos^{-1} x \quad \text{for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$
$$x = \cos y$$
where $0 \leq y \leq \pi$ and $-1 \leq x \leq 1$.

The graph of $y = \cos^{-1} x$ is shown below



(iii) Inverse Tangent Function

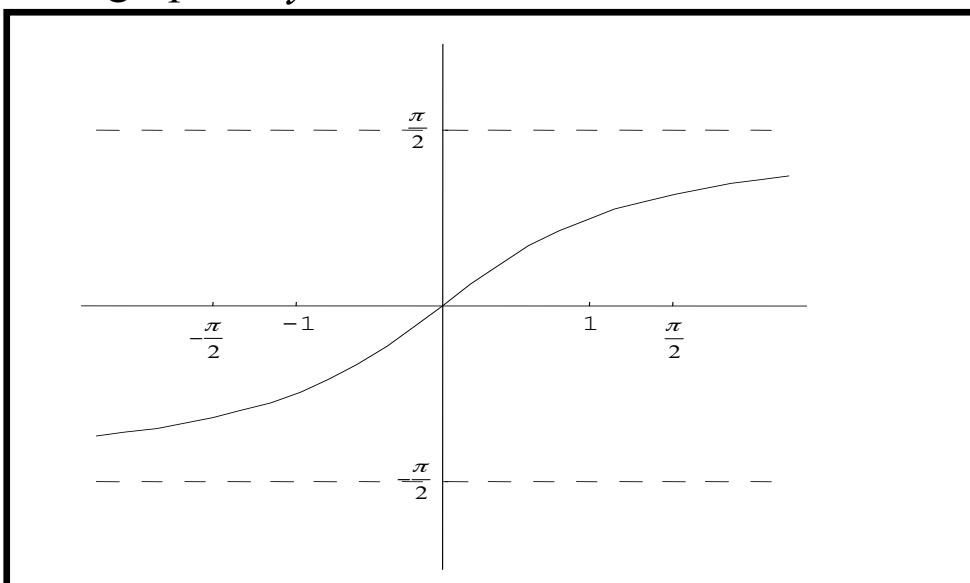
Look at the graph of $y = \tan x$ shown below



The function $f(x) = \tan x$ is not one to one. But if the domain is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $f(x)$ is one to one.

The inverse tangent function is defined as
 $y = \tan^{-1} x$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $-\infty \leq x \leq \infty$.

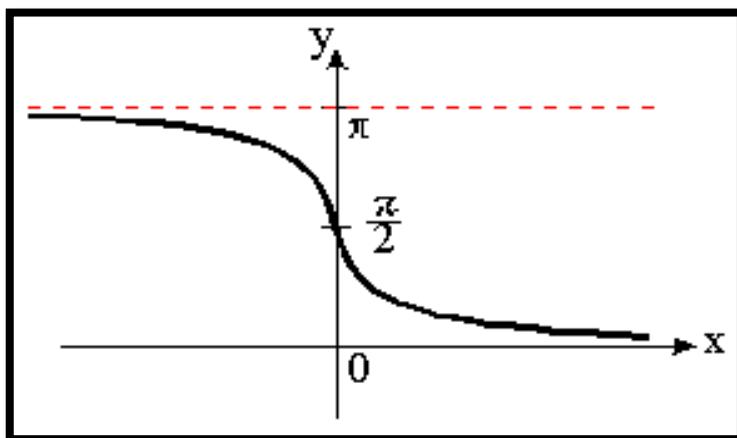
The graph of $y = \tan^{-1} x$ is shown below



(iv) Inverse Cotangent Function

Domain:

Range:



(vi) Inverse Cosecant Function

Domain:

Range:

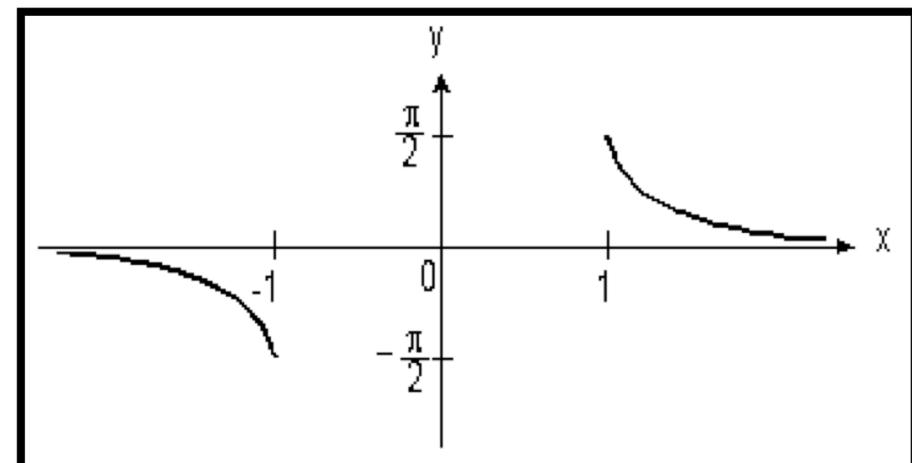


Table of Inverse Trigonometric Functions

Functions	Domain	Range
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \csc^{-1} x$	$ x \geq 1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \sec^{-1} x$	$ x \geq 1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

➤ $\sin^{-1} x \neq \frac{1}{\sin x}$ whereas $(\sin x)^{-1} = \frac{1}{\sin x}$.

2.3.2 Inverse Trigonometric Identities

The definition of the inverse functions yields several formulas.

Inversion formulas

$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

$$\sin^{-1}(\sin y) = y \quad \text{for } -90^\circ \leq y \leq 90^\circ$$

$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

$$\cos^{-1}(\cos y) = y \quad \text{for } 0^\circ \leq y \leq 180^\circ$$

$$\tan(\tan^{-1} x) = x \quad \text{for all } x$$

$$\tan^{-1}(\tan y) = y \quad \text{for } -90^\circ \leq y \leq 90^\circ$$

- These formulas are valid only on the specified domain

Basic Relation

Reciprocal Identities

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$	for $0 \leq x \leq 1$	$\csc^{-1} x = \sin^{-1} \left(\frac{1}{x} \right)$	for $ x \geq 1$
$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$	for $0 \leq x \leq 1$	$\sec^{-1} x = \cos^{-1} \left(\frac{1}{x} \right)$	for $ x \geq 1$
$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$	for $0 \leq x \leq 1$	$\cot^{-1} x = \tan^{-1} \left(\frac{1}{x} \right)$	for all x

Negative Argument Formulas

$\sin^{-1}(-x) = -\sin^{-1} x$	$\sec^{-1}(-x) = \pi - \sec^{-1} x$	$\cos^{-1}(-x) = \pi - \cos^{-1} x$
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Examples 2.2:

1. Evaluate the given functions.

(i) $\sin(\sin^{-1} 0.5)$ (ii) $\sin(\sin^{-1} 3)$

(iii) $\sin^{-1}(\sin 45^\circ)$ (iv) $\sin^{-1}(\sin 135^\circ)$

2. Evaluate the given functions.

(i) $\text{arcsec}(-2)$ (ii) $\csc^{-1}(\sqrt{2})$ (iii) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

3. Show that

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\cos(\sin^{-1}x) = \sqrt{1-x^2}$ (iii) $\sin^{-1}(-x) = -\sin^{-1}x$

4. Given that $2\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{2}$, find the value of x .

2.3.3 Inverse Hyperbolic Functions

The three basic inverse hyperbolic functions are $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$.

Definition (*Inverse Hyperbolic Function*)

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y \quad \text{for all } x \text{ and } y \in \mathfrak{R}$$

$$y = \cosh^{-1} x \Leftrightarrow x = \cosh y \quad \text{for } x \geq 1 \text{ and } y \geq 0$$

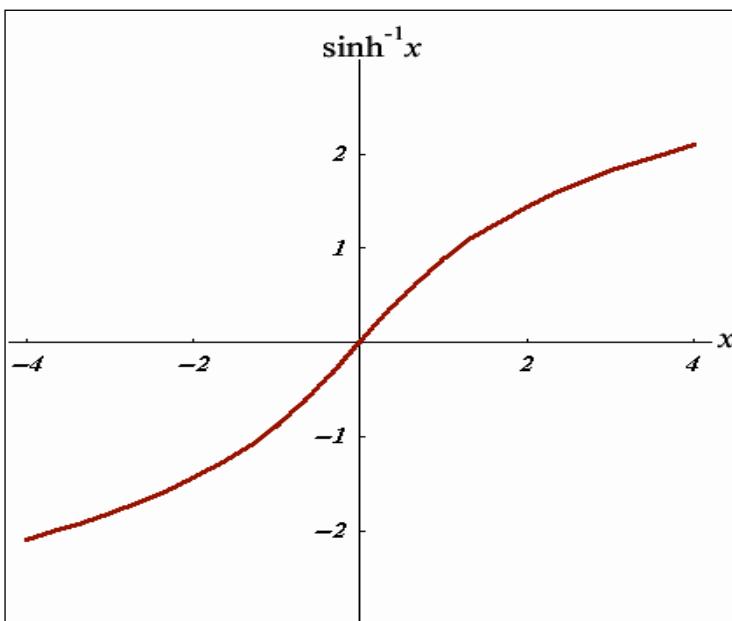
$$y = \tanh^{-1} x \Leftrightarrow x = \tanh y \quad \text{for } -1 \leq x \leq 1, y \in \mathfrak{R}$$

Graphs of Inverse Hyperbolic Functions

(i) $y = \sinh^{-1} x$

Domain:

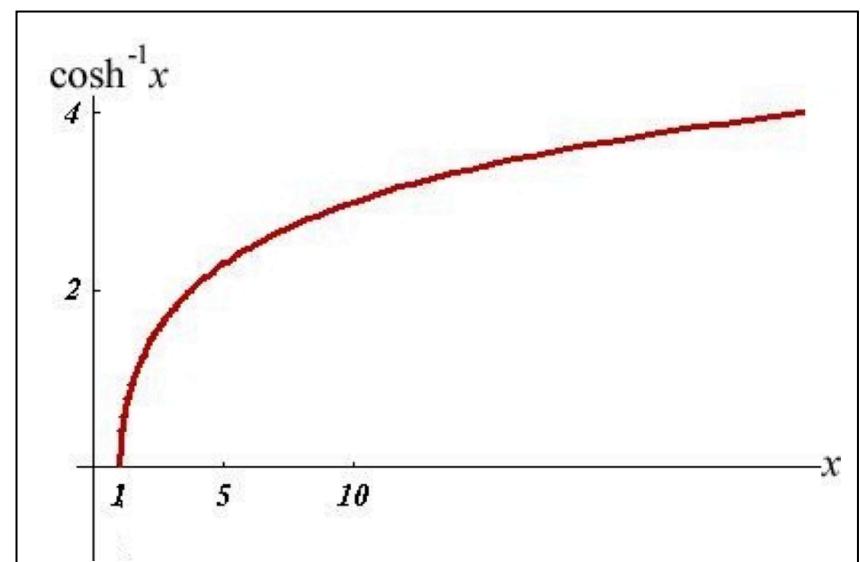
Range:



(ii) $y = \cosh^{-1} x$

Domain:

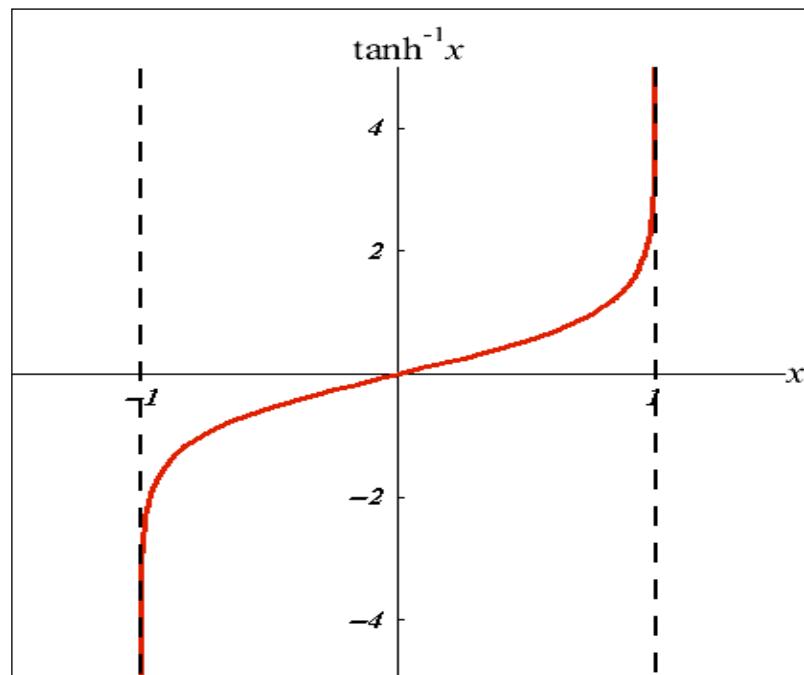
Range:



(iii) $y = \tanh^{-1} x$

Domain:

Range:



2.3.4 Log Form of the Inverse Hyperbolic Functions

It may be shown that

$$(a) \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$(b) \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$(c) \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$(d) \coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$$

$$(e) \operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$(f) \operatorname{cosech}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Inverse Hyperbolic Cosine (Proof)

If we let $y = \cosh^{-1} x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

Hence,

$$2x = e^y + e^{-y}$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0$$

Hence, (using formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$)

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

Since $e^y > 0$,

$$\therefore e^y = x + \sqrt{x^2 - 1}$$

Taking natural logarithms,

$$y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

Proof for $\sinh^{-1} x$

$$y = \sinh^{-1} x$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\therefore 2x = e^y - e^{-y} \text{ (multiply with } e^y)$$

$$2xe^y = e^{2y} - 1$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

Since $e^y > 0$,

$$\therefore e^y = x + \sqrt{x^2 + 1}$$

Taking natural logarithms,

$$y = \sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right)$$

In the same way, we can find the expression for $\tanh^{-1} x$ in logarithmic form.

Examples 2.3:

1. Prove that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

2. Evaluate

a) $\sinh^{-1}(0.5)$

b) $\cosh^{-1}(0.5)$

c) $\tanh^{-1}(-0.6)$

d)

3. Solve the following equations:

a) $\sinh^{-1} x = \ln 2$

b) $\cosh^{-1} 5x = \sinh^{-1} 4x$