

## CHAPTER 3

### **DIFFERENTIATION**

3.1 Differentiation of hyperbolic functions

3.2 Differentiation of inverse trigonometric functions

3.3 Differentiation of inverse hyperbolic functions

#### **\*Recall: Methods of differentiation**

- Chain rule
- Product differentiation
- Quotient differentiation
- Implicit differentiation

### 3.1 Differentiation of Hyperbolic Functions

**Recall: Definition:**

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

## Derivatives of hyperbolic functions

**Example 3.1:** Find the derivatives of

(a)  $\sinh x$

(b)  $\cosh x$

(c)  $\tanh x$

**Solution:**

$$(a) \frac{d}{dx} \sinh x = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right)$$

$$= \frac{1}{2} (e^x + e^{-x}) = \cosh x$$

$$(b) \frac{d}{dx} \cosh x = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right)$$

$$= \frac{1}{2} (e^x - e^{-x}) = \sinh x$$

$$(c) \frac{d}{dx} \tanh x = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

Using quotient diff:

$$\begin{aligned}
&= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2} \\
&= \frac{4}{(e^x + e^{-x})^2} = \left( \frac{2}{(e^x + e^{-x})} \right)^2 \\
&= \left( \frac{1}{\cosh x} \right)^2 = \operatorname{sech}^2 x
\end{aligned}$$

Using the same methods, we can obtain the derivatives of the other hyperbolic functions and these gives us the standard derivatives.

## Standard Derivatives (given in Test and Final)

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\coth x$	$-\operatorname{cosech}^2 x$

### **Example 3.2:**

1. Find the derivatives of the following functions:

a)  $y = \frac{x^3}{\sinh x}$

b)  $y = \tanh^3 2x$

c)  $y = e^{\cosh 4x^2}$

2. Find the derivatives of the following functions:

(a)  $y = \cosh(3x)$

(b)  $r = \sinh(2t^2 - 1)$

(c)  $g(x) = (x - 1)^3 \operatorname{sech}^2 x$

(d)  $y = \tanh(\ln x)$

### **3. (*Implicit differentiation*)**

Find  $\frac{dy}{dx}$  from the following expressions:

(a)  $x = y^2 \sinh 4x + \cosh y$

(b)  $y = \tanh(x + y)$

## 3.2 Differentiation Involving Inverse Trigonometric Functions

**Recall:** Definition of inverse trigonometric functions

Function	Domain	Range
$\sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\sec^{-1} x$	$ x  \geq 1$	$0 \leq y < \frac{\pi}{2} \cup \frac{\pi}{2} < y < \pi$
$\cot^{-1} x$	$-\infty \leq x \leq \infty$	$0 < y < \pi$
$\text{cosec}^{-1} x$	$ x  \geq 1$	$-\frac{\pi}{2} < y < 0 \cup 0 < y < \frac{\pi}{2}$

# Derivatives of Inverse Trigonometric Functions

*Standard Derivatives:*

$$1. \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$2. \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$3. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$4. \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$6. \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

### 3.2.1 Derivatives of $y = \sin^{-1} x$ . (*proof*)

Recall:  $y = \sin^{-1} x \Leftrightarrow x = \sin y$

for  $x \in [-1, 1]$  and  $y \in [-\pi/2, \pi/2]$ .

Because the sine function is differentiable on  $[-\pi/2, \pi/2]$ , the inverse function is also differentiable.

To find its derivative we proceed implicitly:

Given  $\sin y = x$ . Differentiating w.r.t.  $x$ :

$$\begin{aligned}\frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) \\ \cos y \frac{dy}{dx} &= 1 \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{\cos y}\end{aligned}$$

Since  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ ,  $\cos y \geq 0$ , hence

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

**Example 3.3:**

1. Differentiate each of the following functions.

(a)  $f(x) = \tan^{-1} \sqrt{x}$

(b)  $g(t) = \sin^{-1}(1 - t)$

(c)  $h(x) = \sec^{-1} e^{2x}$

2. Find the derivative of:

(a)  $y = (\tan^{-1} x^2)^4$

(b)  $f(x) = \ln(\sin^{-1} 4x)$

3. Find the derivative of  $y = \tan^{-1}(\tan(3t^2 - 1))$ .

4. Find the derivative  $\frac{dy}{dx}$  if

(a)  $x \tan^{-1} y = x^2 + y$

(b)  $\sin^{-1}(xy) + \frac{\pi}{2} = \cos^{-1} y$

## Summary

If  $u$  is a differentiable function of  $x$ , then

$$1. \frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$2. \frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$3. \frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$4. \frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \frac{du}{dx}$$

$$5. \frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$$

$$6. \frac{d}{dx}(\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$$

### 3. 3 Derivatives of Inverse hyperbolic Functions

*Recall: Inverse Hyperbolic Functions*

Function	Domain	Range
$y = \sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = \cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$y = \tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$y = \coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$y = \operatorname{cosech}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Function	Logarithmic form
$y = \sinh^{-1} x$	$\ln\left(x + \sqrt{x^2 + 1}\right)$
$y = \cosh^{-1} x$	$\ln\left(x + \sqrt{x^2 - 1}\right)$
$y = \tanh^{-1} x$	$\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right);  x  < 1$

**3.3.1 Proof:**  $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

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*Recall:*  $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$

To find its derivative we proceed implicitly:

➤ Given  $x = \sinh y$ . Differentiating w.r.t.  $x$ :

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$1 = \cosh y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cosh y}$$

➤ Since  $-\infty < y < \infty$ ,  $\cosh y \geq 0$ , so using the identity

$$\cosh^2 y - \sinh^2 y = 1:$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

➤ Other ways to obtain the derivatives are:

(a)  $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$  then

$$x = \frac{e^y - e^{-y}}{2}. \text{ Hence, find } \frac{dy}{dx}.$$

(b)  $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$

Hence, find  $\frac{dy}{dx}$ .

## Standard Derivatives (Given in Test and Final)

Function, $y$	Derivatives, $\frac{dy}{dx}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2 + 1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2 - 1}};  x  > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2};  x  < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2};  x  > 1$
$\operatorname{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}}; 0 < x < 1$
$\operatorname{cosech}^{-1} x$	$\frac{1}{ x \sqrt{1+x^2}}; x \neq 0$

## Generalised Form

$y = f(u);$ $u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\sinh^{-1} u$	$\frac{1}{\sqrt{u^2 + 1}} \frac{du}{dx}$
$\cosh^{-1} u$	$\frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx};  u  > 1$
$\tanh^{-1} u$	$\frac{1}{1-u^2} \frac{du}{dx};  u  < 1$
$\coth^{-1} u$	$\frac{1}{1-u^2} \frac{du}{dx};  u  > 1$
$\operatorname{sech}^{-1} u$	$-\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}; 0 < u < 1$
$\operatorname{cosech}^{-1} u$	$\frac{1}{ u \sqrt{1+u^2}} \frac{du}{dx}; u \neq 0$

**Example 3.4:** Find the derivatives of

$$(a) \ y = \sinh^{-1}(1 - 3x)$$

$$(b) \ y = \cosh^{-1}\left(\frac{1}{x}\right)$$

$$(c) \ y = e^x \operatorname{sech}^{-1} x$$

$$(d) \ y = \sinh^{-1}(\tan 3x)$$

$$(e) \ f(t) = \frac{\tanh^{-1} t^2}{1 - \sec t}$$

$$(f) \ y = \sqrt{\coth^{-1} u}$$

$$(g) \ y = \cos 4x \ \cosh^{-1} 4x$$

$$(h) \ y^3 - \sinh^{-1} xy = 0$$