## CHAPTER 4

## INTEGRATION

4.1 Integration of hyperbolic functions
4.2 Integration of inverse trigonometric functions
4.3 Integration of inverse hyperbolic functions
4.4 Further Applications of Integrations

Recall: Methods involved:

- Substitution of $u$
- By parts
- Tabular method
- Partial fractions


### 4.1 Integrals of Hyperbolic Functions

## Integral Formulae

1. $\int \sinh x d x=\cosh x+C$
2. $\int \cosh x d x=\sinh x+C$
3. $\int \sec h^{2} x d x=\tanh x+C$
4. $\int \operatorname{cosech}^{2} x d x=-\operatorname{coth} x+C$
5. $\int \sec h x \tanh x d x=-\sec h x+C$
6. $\int \operatorname{cosech} x \operatorname{coth} x d x=-\operatorname{cosech} x+C$

## Example 1:

Integrate the following hyperbolic functions using appropriate technique (definition, identities, etc) and method (substitution, by parts, tabular, etc).
a) $\int \sinh 2 x \cosh 3 x d x$
b) $\int \frac{\cosh x}{2+3 \sinh x} d x$
c) $\int \sinh ^{3} x d x$
d) $\int x \cosh 2 x d x$
e) $\int \sinh \left(\frac{x}{2}\right) \cosh \left(\frac{x}{2}\right) d x$
f) $\int \sqrt{\tanh x} \sec h^{2} x d x$

### 4.2 Integration of Inverse Trigonometric Functions

Integration formulae of the Inverse Trigonometric Functions

| Differentiation | Integration |
| :--- | :--- |
| $\frac{d}{d x}\left(\sin ^{-1} x\right)=\frac{1}{\sqrt{1-x^{2}}}$ | $\int \frac{d x}{\sqrt{1-x^{2}}}=\sin ^{-1} x+C$ |
| $\frac{d}{d x}\left(\cos ^{-1} x\right)=\frac{-1}{\sqrt{1-x^{2}}}$ | $\int \frac{-d x}{\sqrt{1-x^{2}}}=\cos ^{-1} x+C$ |
| $\frac{d}{d x}\left(\tan ^{-1} x\right)=\frac{1}{1+x^{2}}$ | $\int \frac{d x}{1+x^{2}}=\tan ^{-1} x+C$ |
| $\frac{d}{d x}\left(\cot ^{-1} x\right)=\frac{-1}{1+x^{2}}$ | $\int \frac{-d x}{1+x^{2}}=\cot ^{-1} x+C$ |
| $\frac{d}{d x}\left(\sec ^{-1} x\right)=\frac{1}{\|x\| \sqrt{x^{2}-1}}$ | $\int \frac{d x}{\|x\| \sqrt{x^{2}-1}}=\sec ^{-1} x+C$ |
| $\frac{d}{d x}\left(\csc ^{-1} x\right)=\frac{-1}{\|x\| \sqrt{x^{2}-1}}$ | $\int \frac{-d x}{\|x\| \sqrt{x^{2}-1}}=\csc ^{-1} x+C$ |

## Example 2 :

1. Evaluate the following integrals
a) $\int_{0}^{1} \tan ^{-1} x d x$
b) $\int \frac{e^{\sin ^{-1} x}}{\sqrt{1-x^{2}}} d x$
c) $\int \frac{\sqrt{\tan ^{-1} x}}{1+x^{2}} d x$
2. Use partial fraction decomposition to solve

$$
\int_{0}^{1} \frac{x^{2}-2 x}{(2 x+1)\left(x^{2}+1\right)} d x .
$$

Example 3 : Evaluate the following integrals

1. a) $\int \frac{d x}{\sqrt{16-x^{2}}}$
b) $\int \frac{2 d x}{3+x^{2}}$
2. 

a) $\int \frac{d x}{\sqrt{1-4 x^{2}}}$
b) $\int \frac{d x}{4+3 x^{2}}$
3. Use completing the square technique to solve:
a) $\int \frac{d x}{\sqrt{-x^{2}+2 x+3}}$
b) $\int \frac{d x}{x^{2}-2 x+2}$
4. By using substitution $t=\tan \left(\frac{x}{2}\right)$, show that

$$
\int \frac{d x}{5+4 \cos x}=\frac{2}{3} \tan ^{-1}\left(\frac{1}{3} \tan \left(\frac{x}{2}\right)\right)+C
$$

### 4.3 Integration involving Inverse Hyperbolic Functions

Integration formulae of the Inverse Hyperbolic Functions:

| Differentiation | Integration |
| :--- | :--- |
| $\frac{d}{d x}\left(\sinh ^{-1} x\right)=\frac{1}{\sqrt{1+x^{2}}}$ | $\int \frac{d x}{\sqrt{1+x^{2}}}=\sinh ^{-1} x+C$ |
| $\frac{d}{d x}\left(\cosh ^{-1} x\right)=\frac{1}{\sqrt{x^{2}-1}}$ | $\int \frac{d x}{\sqrt{x^{2}-1}}=\cosh ^{-1} x+C$ |
| $\frac{d}{d x}\left(\tanh ^{-1} x\right)=\frac{1}{1-x^{2}}$ | $\int \frac{d x}{1-x^{2}}=\tanh ^{-1} x+C$ |

## Example 4:

1. Solve the following:
a) $\int \frac{d x}{\sqrt{3 x^{2}+2}}$
b) $\quad \int \frac{d x}{\sqrt{2(x-3)^{2}+1}}$
c)

2. Show that $\int \frac{x+1}{\sqrt{x^{2}+1}} d x=\sqrt{x^{2}+1}+\sinh ^{-1} x+C$.

### 4.4 Further Applications of Integrations

### 4.4.1 Arc Length in Cartesian Form

The length of the curve $(x(t), y(t))$ as $t$ varies from $t_{0}$ to $t_{1}$ is given by

$$
\mathcal{L}=\int_{t=t_{0}}^{t=t_{1}} \sqrt{\left(x^{\prime}(t)\right)^{2}+\left(y^{\prime}(t)\right)^{2}} d t
$$

## Example 5:

Consider the curve given by $x(t)=\cos t, y(t)=\sin t, 0 \leq t \leq \pi$. Find the length of the curve.

Its length is:

If we wish to find the length of a curve which is the graph of a function

$$
y=f(x), a \leq x \leq b,
$$

we let

$$
x(t)=t, \quad y(t)=f(x(t))=f(x)
$$

and we get

$$
x^{\prime}(t)=1 \text { and } y^{\prime}(t)=f^{\prime}(x(t)) x^{\prime}(t)=f^{\prime}(x),
$$

so we have a simple formula for the length:

$$
\mathcal{L}=\int_{x=a}^{x=b} \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x=\int_{a}^{b} \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x=\int_{a}^{b} \sqrt{1+\left(y^{\prime}\right)^{2}} d x
$$

Similarly, if we have a curve $x=g(\mathrm{y}), c \leq y \leq d$, we get

$$
\mathcal{L}=\int_{y=c}^{y=d} \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y=\int_{c}^{d} \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y=\int_{c}^{d} \sqrt{1+\left(x^{\prime}\right)^{2}} d y
$$

## Example 6:

Find the length of the curve
a) $y=\frac{1}{3}\left(x^{2}+2\right)^{\frac{3}{2}}, 0 \leq x \leq 3$.
b) $x=\frac{2}{3}(y-1)^{\frac{3}{2}}, 1 \leq y \leq 4$.

## Example 7:

Find the length of the arc of the parabola $y^{2}=x$ from $(0,0)$ to $(1,1)$.

$$
\text { Ans: } L=\frac{\sqrt{5}}{2}+\frac{\ln (\sqrt{5}+2)}{4}
$$

## Arc Length with Polar Coordinates

We now need to move into the Calculus II applications of integrals and how we do them in terms of polar coordinatec In this section we'll look at the arc length of the curve given by,

$$
\alpha \leq \theta \leq \beta
$$

where we also assume that the curve is traced out exactly once. Just as we did with the tangent lines in polareeerdinates we'll first write the curve in terms of a set of parametric equations,
$x=r \cos \theta$
$=f(\theta) \cos \theta$
$y=r \sin \theta$

$$
=f(\theta) \sin \theta
$$

and we can now use the parametric formula for finding the arc length.

$$
\begin{aligned}
\frac{d x}{d \theta} & =f^{\prime}(\theta) \cos \theta-f(\theta) \sin \theta & \frac{d y}{d \theta} & =f^{\prime}(\theta) \sin \theta+f(\theta) \cos \theta \\
& =\frac{d r}{d \theta} \cos \theta-r \sin \theta & & =\frac{d r}{d \theta} \sin \theta+r \cos \theta
\end{aligned}
$$

We'll need the following for our $d s$.

$$
\begin{aligned}
\left(\frac{d x}{d \theta}\right)^{2}+\left(\frac{d y}{d \theta}\right)^{2}= & \left(\frac{d r}{d \theta} \cos \theta-r \sin \theta\right)^{2}+\left(\frac{d r}{d \theta} \sin \theta+r \cos \theta\right)^{2} \\
= & \left(\frac{d r}{d \theta}\right)^{2} \cos ^{2} \theta-2 r \frac{d r}{d \theta} \cos \theta \sin \theta+r^{2} \sin ^{2} \theta \\
& +\left(\frac{d r}{d \theta}\right)^{2} \sin ^{2} \theta+2 r \frac{d r}{d \theta} \cos \theta \sin \theta+r^{2} \cos ^{2} \theta \\
= & \left(\frac{d r}{d \theta}\right)^{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right)+r^{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right) \\
= & r^{2}+\left(\frac{d r}{d \theta}\right)^{2}
\end{aligned}
$$

The arc length formula for polar coordinates is then,

$$
L=\int d s
$$

where,

$$
d s=\sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta
$$

### 4.4.2 Arc Length in Polar Coordinates

The length of a curve with polar equation $r=f(\theta), a \leq \theta \leq b$, is

$$
L=\int_{a}^{b} \sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta
$$

## Example 8:

a) Find the length of the curve $r=\theta, 0 \leq \theta \leq 1$.

b) Find the length of the cardioid $r=1-\cos \theta, 0 \leq \theta \leq 2 \pi$.

## Surface of Revolution

A surface of revolution is a surface generated by rotating a twodimensional curve about an axis. The resulting surface therefore always has azimuthal symmetry. Examples of surfaces of revolution include the apple, cone, frustum, cylinder,lemon, sphere, etc.
cone

conical frustum

cylinder

oblate spheroid

prolate spheroid

zone


### 4.4.3 Area of Surface of Revolution in Cartesian Form

Let's start with some simple surfaces. The lateral surface area of a circular cylinder with radius $r$ and height $h$ is taken to be $A=2 \pi r h$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2 \pi r$ and $h$.


FIGURE 1


Likewise, we can take a circular cone with base radius $r$ and slant height $l$, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius $l$ and central angle $\theta=\frac{2 \pi r}{l}$. We know that, in general, the area of a sector of a circle with radius
$l$ and angle $\theta$ is $\frac{1}{2} l^{2} \theta$. So in this case it is

$$
A=\frac{1}{2} l^{2} \theta=\frac{1}{2} l^{2}\left(\frac{2 \pi r}{l}\right)=\pi r l
$$

Therefore, we define the lateral surface area of a cone to be $A=\pi r l$.


FIGURE 3
What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original
curve by a polygon. When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area. By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of bands, each formed by rotating a line segment about an axis. To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3. The area of the band (or frustum of a cone) with slant height $l$ and upper and lower radii $r_{1}$ and $r_{2}$ is found by subtracting the areas of two cones:

$$
A=\pi r_{2}\left(l_{1}+l\right)-\pi r_{1} l_{1}=\pi\left[\left(r_{2}-r_{1}\right) l_{1}+r_{2} l\right] \text { [1] }
$$

From similar triangles we have

$$
\frac{l_{1}}{r_{1}}=\frac{l_{1}+l}{r_{2}}
$$

which gives

$$
r_{2} l_{1}=r_{1} l_{1}+r_{1} l \quad \text { or } \quad\left(r_{2}-r_{1}\right) l_{1}=r_{1} l
$$

Putting this in Equation 1, we get

$$
A=\pi\left(r_{1} l+r_{2} l\right)
$$

or

$$
\begin{equation*}
A=2 \pi r l \tag{2}
\end{equation*}
$$

where $r=\frac{1}{2}\left(r_{1}+r_{2}\right)$ is the average radius of the band.

So, for the purposes of the derivation of the formula, let's look at rotating the continuous function $y=f(x)$ in the interval $[a, b]$ about the $x$-axis. We'll also need to assume that the derivative is continuous on $[a, b]$. Below is a sketch of a
function and the solid of revolution we get by rotating the function about the $x$ axis.



We can derive a formula for the surface area much as we derived the formula for arc length. We'll start by dividing the interval into $n$ equal subintervals of width $\Delta x$. On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of the each interval. Here is a sketch of that for our representative function using $n=4$.


Now, rotate the approximations about the $x$-axis and we get the following solid.


The approximation on each interval gives a distinct portion of the solid and to make this clear each portion is colored differently. Each of these portions are called frustums. To find the surface area of frustums is already discussed above.

(a) Surface of revolution

(b) Approximating band

FIGURE 4

Now we apply this formula to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve $y=f(x), a \leqslant x \leqslant b$, about the $x$-axis, where $f$ is positive and has a continuous derivative. In order to define its surface area, we divide the interval $[a, b]$ into $n$ subintervals with endpoints $x_{0}, x_{1}, \ldots, x_{n}$ and equal width $\Delta x$, as we did in determining arc length. If $y_{i}=f\left(x_{i}\right)$, then the point $P_{i}\left(x_{i}, y_{i}\right)$ lies on the curve. The part of the surface between $x_{i-1}$ and $x_{i}$ is approximated by taking the line segment $P_{i-1} P_{i}$ and rotating it about the $x$-axis. The result is a band with slant height $l=\left|P_{i-1} P_{i}\right|$ and average radius $r=\frac{1}{2}\left(y_{i-1}+y_{i}\right)$ so, by Formula 2, its surface area is

$$
2 \pi \frac{y_{i-1}+y_{i}}{2}\left|P_{i-1} P_{i}\right|
$$

As in the proof of Theorem 7.4.2, we have

$$
\left|P_{i-1} P_{i}\right|=\sqrt{1+\left[f^{\prime}\left(x_{i}^{*}\right)\right]^{2}} \Delta x
$$

where $x_{i}^{*}$ is some number in $\left[x_{i-1}, x_{i}\right]$. When $\Delta x$ is small, we have $y_{i}=f\left(x_{i}\right) \approx f\left(x_{i}^{*}\right)$ and also $y_{i-1}=f\left(x_{i-1}\right) \approx f\left(x_{i}^{*}\right)$, since $f$ is continuous. Therefore

$$
2 \pi \frac{y_{i-1}+y_{i}}{2}\left|P_{i-1} P_{i}\right| \approx 2 \pi f\left(x_{i}^{*}\right) \sqrt{1+\left[f^{\prime}\left(x_{i}^{*}\right)\right]^{2}} \Delta x
$$

and so an approximation to what we think of as the area of the complete surface of revolution is

$$
\begin{equation*}
\sum_{i=1}^{n} 2 \pi f\left(x_{i}^{*}\right) \sqrt{1+\left[f^{\prime}\left(x_{i}^{*}\right)\right]^{2}} \Delta x \tag{3}
\end{equation*}
$$

This approximation appears to become better as $n \rightarrow \infty$ and, recognizing (3) as a Riemann sum for the function $g(x)=2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}}$, we have

$$
\lim _{n \rightarrow x} \sum_{i=1}^{n} 2 \pi f\left(x_{i}^{*}\right) \sqrt{1+\left[f^{\prime}\left(x_{i}^{*}\right)\right]^{2}} \Delta x=\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x
$$

Therefore, in the case where $f$ is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating the curve $(x(t), y(t)), t_{1} \leq t \leq t_{2}$ about the $x$ - and $y$-axes respectively:

$$
\begin{aligned}
& S_{x}=\int_{t_{1}}^{t_{2}} 2 \pi y(t) \sqrt{\left(x^{\prime}(t)\right)^{2}+\left(y^{\prime}(t)\right)^{2}} d t \\
& S_{y}=\int_{t_{1}}^{t_{2}} 2 \pi x(t) \sqrt{\left(x^{\prime}(t)\right)^{2}+\left(y^{\prime}(t)\right)^{2}} d t
\end{aligned}
$$

If the curve is the graph of a function $y=f(x), a \leq x \leq b$, then the area of the surface obtained by revolving the curve about the $\boldsymbol{x}$-axis is

$$
S_{x}=\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x
$$

and the area of the surface obtained by revolving the curve about the $\boldsymbol{y}$-axis is

$$
S_{y}=\int_{a}^{b} 2 \pi x \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x
$$

If the curve is the graph of a function $x=g(y), c \leq x \leq d$, then the area of the surface obtained by revolving the curve about the $x$-axis is

$$
S_{x}=\int_{c}^{d} 2 \pi y \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y
$$

and the area of the surface obtained by revolving the curve about the $\boldsymbol{y}$-axis is

$$
S_{y}=\int_{c}^{d} 2 \pi g(y) \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y
$$

These formulas can be remembered by thinking of $2 \pi y$ or $2 \pi x$ as the circumference of a circle traced out by the point $(x, y)$ on the curve as it is rotated about the $x$-axis or $y$-axis, respectively (see Figure 5).

(a) Rotation about $x$-axis: $S=\int 2 \pi y d s$

(b) Rotation about $y$-axis: $S=\int 2 \pi x d s$

## Example 9:

a) Find the area of the surface obtained by rotating the curve $y^{2}=4 x+4,0 \leq x \leq 8$, about the $x$-axis.

b) Find the area of the surface obtained by rotating the curve $x=1+2 y^{2}, 1 \leq y \leq 2$, about the $x$-axis.

### 4.4.4 Area of a Surface of Revolution in Polar Form

The areas of the surfaces generated by revolving the curve $r=f(\theta), a \leq \theta \leq b$ about the $x$ - and $y$-axis are given by the following formulas:

- Revolution about $x$-axis, $(y \geq 0)$ :

$$
S_{x}=\int_{a}^{b} 2 \pi r \sin \theta \sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta
$$

- Revolution about $y$-axis, $x \geq 0$ :

$$
S_{y}=\int_{a}^{b} 2 \pi r \cos \theta \sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta
$$

## Example 10:

Find the area of the surface generated by revolving

$$
r=\sqrt{\cos 2 \theta}, 0 \leq \theta \leq \frac{\pi}{4}
$$

about the $x$-axis.


Ans: $2 \pi-\frac{2 \pi}{\sqrt{2}}$

Summary Formula for Area of Revolution:

| Type of <br> Equation | Revolve about x-axis | Revolve about y-axis |
| :---: | :---: | :---: |
| Parametric <br> $x=f(t)$, <br> $y=g(t)$ | $S_{x}=\int_{t_{1}}^{t_{2}} 2 \pi y(t) \sqrt{\left(x^{\prime}(t)\right)^{2}+\left(y^{\prime}(t)\right)^{2}} d t$ | $S_{y}=\int_{t_{1}}^{t_{2}} 2 \pi x(t) \sqrt{\left(x^{\prime}(t)\right)^{2}+\left(y^{\prime}(t)\right)^{2}} d t$ |
| $y=f(x)$ | $S_{x}=\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x$ | $S_{y}=\int_{a}^{b} 2 \pi x \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x$ |
| $x=g(y)$ | $S_{x}=\int_{c}^{d} 2 \pi y \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y$ | $S_{y}=\int_{c}^{d} 2 \pi g(y) \sqrt{1+\left(g^{\prime}(y)\right)^{2}} d y$ |

$$
\begin{array}{|l|l|l|}
\hline \text { Polar form } \\
r=f(\theta) & S_{x}=\int_{a}^{b} 2 \pi r \sin \theta \sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta \quad S_{y}=\int_{a}^{b} 2 \pi r \cos \theta \sqrt{r^{2}+\left(\frac{d r}{d \theta}\right)^{2}} d \theta \\
\hline
\end{array}
$$

## Appendix:

1. Partial fraction decomposition.

| S.No. | Form of the rational function | Form of the partial fraction |
| :---: | :--- | :--- |
| 1. | $\frac{p x+q}{(x-a)(x-b)}, a \neq b$ | $\frac{\mathrm{~A}}{x-a}+\frac{\mathrm{B}}{x-b}$ |
| 2. | $\frac{p x+q}{(x-a)^{2}}$ | $\frac{\mathrm{~A}}{x-a}+\frac{\mathrm{B}}{(x-a)^{2}}$ |
| 3. | $\frac{\mathrm{A}}{(x-a)(x-b)(x-c)}+\frac{\mathrm{B}}{x-b}+\frac{\mathrm{C}}{x-c}$ |  |
| 4. | $\frac{\mathrm{A} x^{2}+q x+r}{(x-a)^{2}(x-b)}$ |  |
| 5. | $\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B}}{(x-a)^{2}}+\frac{\mathrm{C}}{x-b}$ |  |
| $(x-a)\left(x^{2}+b x+c\right)$ | $\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B} x+\mathrm{C}}{x^{2}+b x+c}$, |  |
| where $x^{2}+b x+c$ cannot be factorised further |  |  |

2. Integrations involving $\sqrt{A x^{2}+B x+C}$

| Expression | Substitution |
| :---: | :---: |
| $\sqrt{x^{2}+k^{2}}$ | $x=k \tan \theta$ or $x=k \sinh \theta$ |
| $\sqrt{x^{2}-k^{2}}$ | $x=k \sec \theta$ or $x=k \cosh \theta$ |
| $\sqrt{k^{2}-x^{2}}$ | $x=k \sin \theta$ or $x=k \tanh \theta$ |

