CHAPTER 5:

IMPROPER INTEGRALS

5.1 L'Hopital Rule

5.1.1 L'Hopital Rule for 0/0

5.1.2 L'Hopital Rule for ∞/∞

5.2 Improper Integrals

5.2.1 Improper Integral Type 1

5.2.2 Improper Integral Type 2

5.1 L'Hopital Rule

If you are doing any limit and you get something in the form of 0/0 or ∞/∞, then you should probably try to use L'Hopital rule. The basic idea of L'Hospital rule is simple. Consider the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If both the numerator and the denominator are finite at a and $g(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

Example 1:

$$\lim_{x \to 3} \frac{x^2 + 1}{x + 2} = \frac{10}{5} = 2.$$

But what happen if both the numerator and the denominator tend to zero???

It is not clear what the limit is. In fact, depending on what functions $f(x)$ and $g(x)$ are, the limit can be anything at all!

5.1.1 L'Hopital Rule for 0/0

Suppose $\lim f(x) = \lim g(x) = 0$. Then

1. If $$\lim \frac{f'(x)}{g'(x)} = L,$$

   then $$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

2. If $\lim \dfrac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, then

   so does $\lim \dfrac{f(x)}{g(x)}$.

Example 2:

Find $\lim_{x \to 0} \frac{\sin x}{x}$ by L'Hopital rule.

Example 3:

Find $\lim_{x \to 1} \frac{2 \ln x}{x - 1}$.

Example 4:

Find $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Example 5:

Find $\lim_{x \to 0} \frac{1 - \cos x}{x}$.

Note: If the numerator and the denominator both tend to $+\infty$ or $-\infty$, L'Hopital rule still applies.

5.1.2 L'Hopital Rule for ∞/∞

Suppose $\lim f(x)$ and $\lim g(x)$ are both infinite. Then

1. If $\lim \frac{f'(x)}{g'(x)} = L$,

   then $$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

2. If $\lim \frac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, then so does $\lim \frac{f(x)}{g(x)}$.

Example 6:

Find $\lim_{x\to\infty} \frac{x^2}{e^x}$.

Example 7:

Find $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$.

5.2 Improper Integrals

The definite integral

$$\int_a^b f(x)\,dx$$

is known as *improper integral* if either

1) one or both limits are infinite, or

2) $f\left(x\right)$ is undefined at certain points on/in the interval.

Note: We called case: 1) as Type I
2) as Type II

5.2.1 Improper Integral Type 1

1) If $f(x)$ is continuous in the interval $[a,\infty)$,

then $$\int_a^\infty f(x)\, dx = \lim_{T\to\infty} \int_a^T f(x)\, dx.$$

2) If $f(x)$ is continuous in the interval $(-\infty, b]$,

then $$\int_{-\infty}^b f(x)\, dx = \lim_{T\to-\infty} \int_T^b f(x)\, dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

3) If $f(x)$ is continuous in the interval $(-\infty,\infty)$,

then $$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$$

with any real number c.

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

Example 8:

Determine whether the following integrals are convergent or divergent:

1) $\int_0^{\infty} e^{-2x}\, dx$

2) $\int_0^{\infty} xe^{-x}\, dx$

3) $\int_{-\infty}^{2} \frac{dx}{5-2x}$

4) $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$

5.2.2 Improper Integral Type 2

1) If $f(x)$ is continuous on $[a,b)$, and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{T\to b^-} \int_a^T f(x)\,dx.$$

2) If $f(x)$ is continuous on $(a,b]$, and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{T\to a^+} \int_T^b f(x)\,dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

3) If $f(x)$ has discontinuity at c, where $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

Example 10:

Determine whether $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ converge or diverge.

Example 11:

Determine whether $\int_1^2 \frac{dx}{1-x^2}$ converge or diverge.

Example 12:

Find $\int_{-1}^{1} \ln x\,dx$ if possible.