Chapter 6: Multivariable Functions

6.0 Review: Graph of Functions

6.1 Functions of Two Variables

- Domain and Range

6.2 3D Cartesian Coordinate System

6.3 Graphs in 3D System

- Some Common Surfaces
- Sketching Common Surfaces
- Level Curves/Contour lines

6.4 Functions of Three Variables

- Domain and Range
- Level Surfaces

6.0 Review: Graph of Functions

You may know these functions and how does it looks like?

$y = \sin x$ $\quad$ $y = \ln x$ $\quad$ $y = x^2$

$y = x^3$ $\quad$ $y = \sqrt{x}$ $\quad$ $y = -\sqrt{x}$

$y = \dfrac{1}{x}$ $\quad$ $y = e^x$ $\quad$ $y = 3$

$y = x$ $\quad$ $y = 2x + 1$

$y^2 + x^2 = a^2$; a is constant

$y = \sqrt{a^2 - x^2}$ $\quad$ $y = -\sqrt{a^2 - x^2}$

$x = \sqrt{a^2 - y^2}$ $\quad$ $x = -\sqrt{a^2 - y^2}$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $a < b$? $a > b$?

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $\quad$ $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

You may know these.....

GRAPH OF FUNCTIONS

$y = \sin x$

$y = \ln x$

quadratic

$y = x^2$

cubic

$y = x^3$

reciprocal:

$y = \frac{1}{x}$

$y = e^x$

$y = x$

$y = 2x + 1$

$y = 3$

Circle:

$y^2 + x^2 = a^2$

$y = \sqrt{a^2 - x^2}$

$y = -\sqrt{a^2 - x^2}$

$x = \sqrt{a^2 - y^2}$

$x = -\sqrt{a^2 - y^2}$

PARABOLA:

$x = y^2$

$y = \sqrt{x}$

$y = -\sqrt{x}$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

6.1 Functions of Two Variables

$$z = f(x, y)$$

The function f is called a **real-valued function of two variables** or simply function of two variables.

We refer to x and y as the **independent variables** and z as the **dependent variable**.

- **Domain and Range**

$$(x, y) \in \textbf{Domain}$$

$$z \in \textbf{Range}$$

Domain (D) : the set of all possible inputs (x, y) of the functions $f(x, y)$.

Range (R) : the set of outputs z that result when (x, y) varies over the domain D.

1.1.5 Domain and Range of $z = f(x, y)$

$$\text{Domain} : \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}, \underline{???}\}$$

$\uparrow$

any constraint

??? $f(x, y)$ **may consist:**

***Sometimes we need to sketch the domain of the function given.**

Range – z-values that results when (x,y) varies over the domain

(i) z positive ?

(ii) z negative ?

(iii) z zero ?

(iv) z has maximum value ?

(v) z has minimum value ?

$$\text{Range}: \{z \mid z \in \mathbb{R},\ \underline{???}\}$$

put the limitation of z here!!

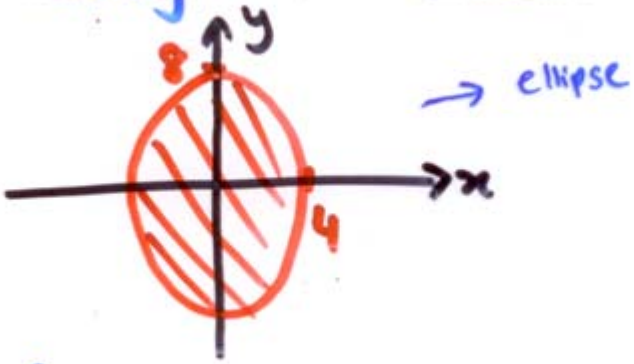
Example

Describe the domain and the range of

$$z = \sqrt{64 - 4x^2 - y^2}\ .$$

Solution

The sketching of Domain



Domain : $\{(x,y) / x \in \mathbb{R}, y \in \mathbb{R}, \ 64 - 4x^2 - y^2 \geq 0\}$

or

Domain : $\{(x,y) / x \in \mathbb{R}, y \in \mathbb{R}, \ \frac{x^2}{16} + \frac{y^2}{64} \leq 1\}$

Range : $\{ z / z \in \mathbb{R}, 0 \leq z \leq \sqrt{64} \}$

or

Range : $\{ z | z \in \mathbb{R}, 0 \leq z \leq 8 \}$

Example

Find the domain and range of $z = x^2\sqrt{y} - 1$.

Solution

Constraint : $y \geq 0$

Domain : $\{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R},\ y \geq 0\}$

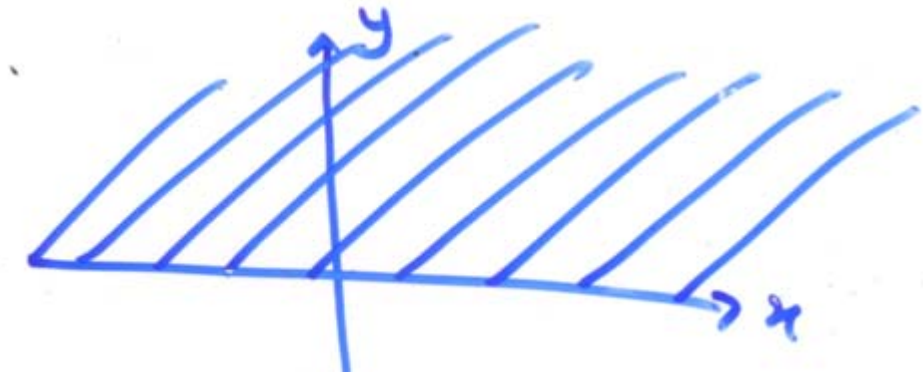
Range : ?

$x^2\sqrt{y} \geq 0$ (always +ve)

$\therefore \quad z \geq -1$

$\Rightarrow$ Range : $\{z \mid z \in \mathbb{R},\ z \geq -1\}$

Sketching of domain



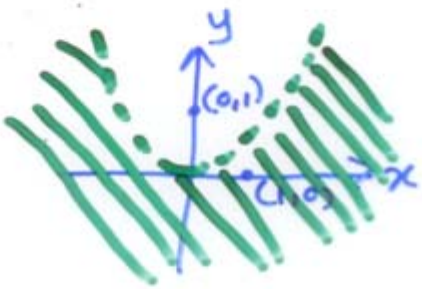
Example

Find the domain and the range of $z = \ln(x^2 - y)$.

Solution

constraint : $x^2 - y > 0 \Rightarrow y < x^2$

$\therefore$ Domain : $\{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}, y < x^2\}$



Sketching of the Domain

For Range z can be zero
can be +ve
can be -ve

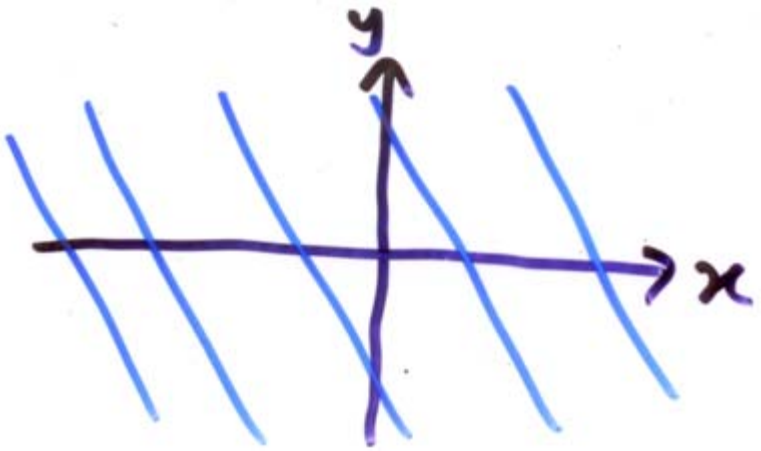
$\therefore$ Range : $\{z \mid z \in \mathbb{R}, -\infty < z < +\infty\}$

Example

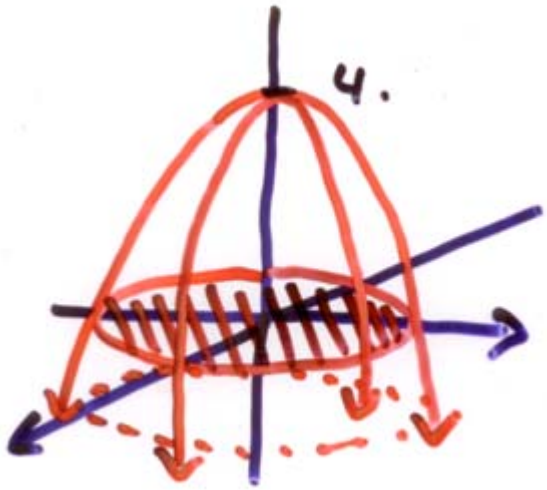
Find the domain and the range of $z = 4 - x^2 - y^2$.

Solution

Domain $:\{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$



Range $:\{z \mid z \in \mathbb{R},\ z \leq 4\}$



Example 1

Let $f(x,y) = 1 - x + y$.

(a) Evaluate $f(-1,2)$, $f(0,-3)$ and $f(r,s)$.

(b) Describe the domain and the range of f.

Solution

(a) By substitution,

$$f(-1,2) = 1 - (-1) + 2 = 4$$

$$f(0,-3) = 1 - 0 + (-3) = -2$$

$$f(r,s) = 1 - r + s$$

(b) The ***domain*** is a set of all ordered pairs (x,y) for which $1 - x + y$ is defined. There is no restriction on the independent variables. This means we can have all real values of x and y as inputs. Thus the domain of f consists of all points in the entire xy-plane. This normally can be written as

$$D_f = \{(x,y) \mid x, y \in \mathbb{R}\}$$

The ***range*** which is the set of outputs, is all single real numbers, that is

$$R_f : \{z \mid z \in \mathbb{R}\} \quad \text{or} \quad R_f : (-\infty, \infty)$$

Example 2

Let. $f(x,y) = 3x^2\sqrt{y} - 1$.

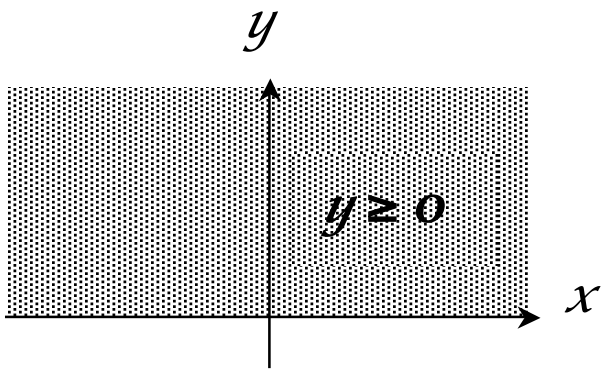
i. Describe and sketch the domain.
ii. Determine the range.

Solution

(i) The domain is a set of all ordered pairs (x,y) for which $3x^2\sqrt{y} - 1$ is defined. We must have the restriction $y \geq 0$ in order for the square root to be defined. Thus the domain of *f* consists of all points in the *xy*-plane that are on or above the *x*-axis. This can be written as

$$D_f = \{(x,y) | y \geq 0\}$$

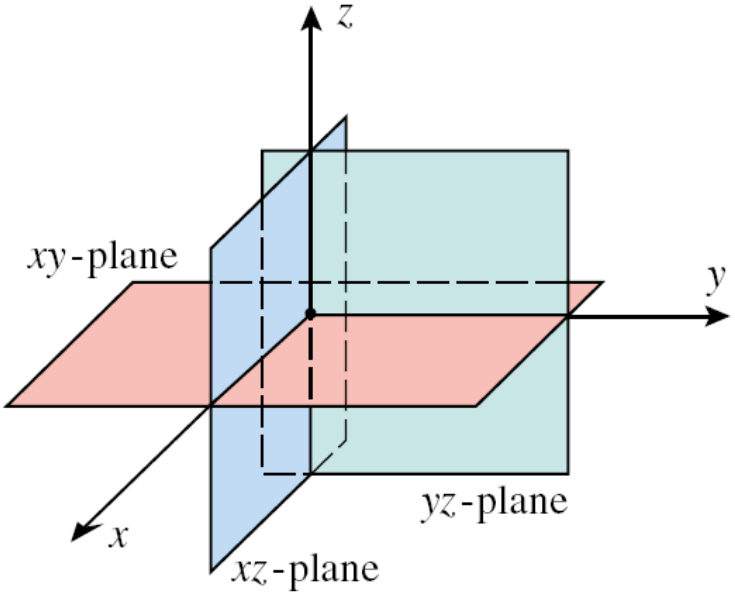
The graph of the domain:



(ii) The range is all real numbers greater than or equal to –1, that is

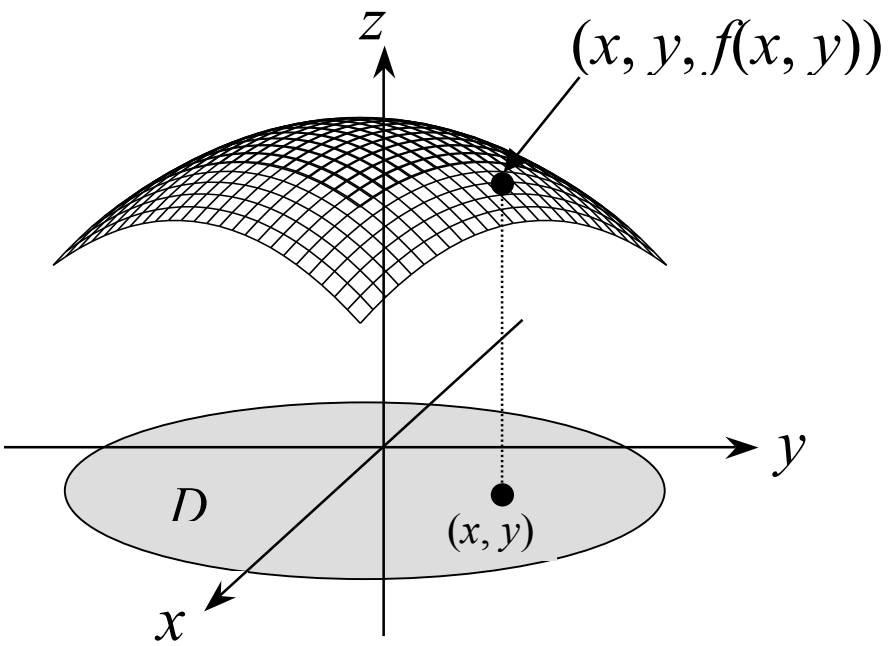
$$R_f : \{z | z \geq -1\} \text{ or } R_f : [-1, \infty)$$

6.2 3D Cartesian Coordinate System



6.3 Graphs in 3D System

The **graph of the function *f* of two variables** is the set of all points (x, y, z) in three-dimensional space, where the values of (x, y) lie in the domain of f and $z = f(x, y)$.



Note:

- $y = f(x)$, the graph is a curve in the xy-plane $\mathbb{R}^2$ consisting of all ordered pairs (x,y)
- $z = f(x,y)$, the graph is a surface in $\mathbb{R}^3$ consisting of (x,y,z) for (x, y) in the domain of f
- $w = f(x,y,z)$, the graph is 3-dimensional inside $\mathbb{R}^4$

• Some Common Surfaces

The graph of an equation in $\mathbb{R}^3$ is called a **surface**. Four types of surface in space:

♦ Planes

Std equation: $ax + by + cz = d$

♦ Spheres

Std equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

♦ Cylinders

parabolic, circular, hyperbolic and elliptic

♦ Quadric Surfaces

3-D analogs of conic sections

Std equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J$$

- **6 types of quadric surfaces:**

For simplicity, we only gave the equation for the quadric surfaces that are centred on the origin.

- Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- hyperboloid of two sheets:

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- elliptic cone:

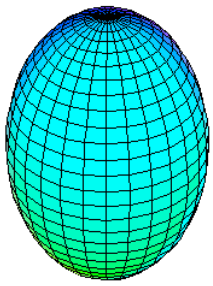
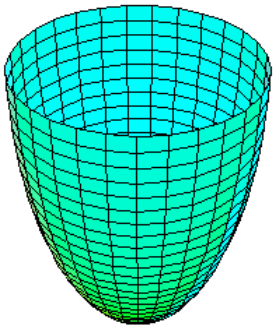
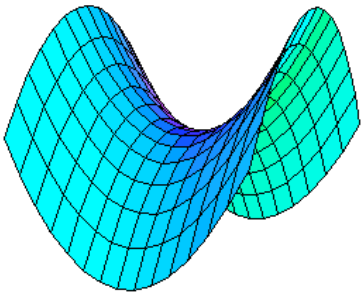
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

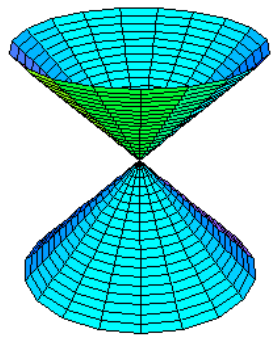
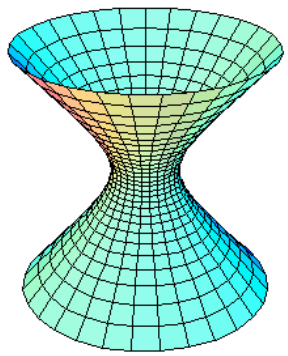
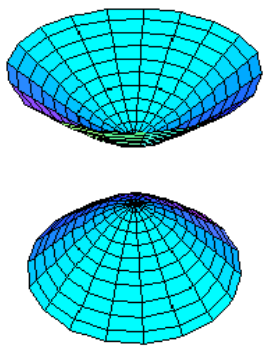
- elliptic paraboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

- hyperbolic paraboloid:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces in the coordinate planes and planes parallel to these are ellipses. If $a = b = c$, the ellipsoid is a sphere.	
Elliptic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ The trace in the *xy*-plane is a point (the origin), and the traces in planes parallel to and above the *xy*-plane are ellipses. The traces in the *yz*- and *xz*-planes are parabolas.	
Hyperbolic Paraboloid (saddle surface): $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ The trace in the *xy*-plane is a pair of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas. The traces in the *yz*- and *xz*-planes are parabolas, as are the traces in planes parallel to these.	

Elliptic cone: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ The trace in the *xy*-plane is a point (the origin), and the traces in planes parallel to these are ellipses. The traces in the *yz*- and *xz*-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.	
Hyperboloid of One Sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The trace in the *xy*-plane and planes parallel to these are ellipses. The traces in the *yz*-plane and *xz*-plane are hyperbolas. The axis symmetry corresponds to the variable whose coefficient is negative.	
Hyperboloid of Two Sheets: $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ There is no trace in the *xy*-plane. In planes parallel to the *xy*-plane the traces are ellipses. In the *yz*- and *xz*-planes, the traces are hyperbolas.	

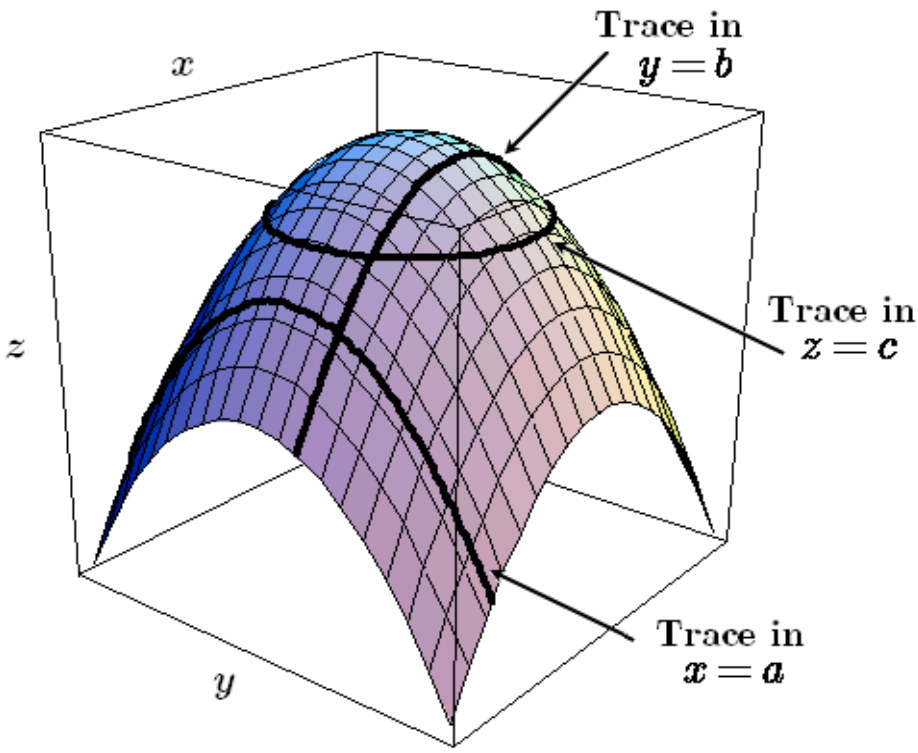
- **Sketching Common Surfaces**

The following ways can be use to sketch the surface $z = f(x,y)$ in 3-space:

Step 1: Determine the variables (domain and range – defined the surface).

Step 2: Sketch the traces - curves of intersection in coordinate planes and sometimes in parallel planes (based on the variables exist) or using algebraic manipulation to identify the standard equation of the function.

Step 3: Make the projection onto the trace-plane which is parallel to the variables that not exist.



Example 3 (Planes-one variable)

Sketch the following equation:

(a) $x = 0, x = 3.$

(b) $y = 0, y = -1$

(c) $z = 0, z = 5.$

Example 4 (Planes-two variables)

Sketch the following equation:

(a) $y = -x + 6.$

(b) $2y = 4z + 5.$

(c) $z + x = 4.$

Example 5 (Tetrahedron-3 variables)

Sketch the following equation:

(a) $x + y + z = 1.$

(b) $z = 6 - 3y + 2x.$

Example 6 (Curved Surface)

(a) $y = x^2$.

(b) $x^2 + y^2 = 25$.

(c) $y^2 - z^2 = 9$.

Example 7 (Curved Surface)

(a) $z = x^2 + y^2$.

(b) $z = \sqrt{x^2 + y^2}$.

(c) $\dfrac{x^2}{9} + \dfrac{y^2}{4} - z^2 = 1$.

- **Level Curves / Contour Lines**

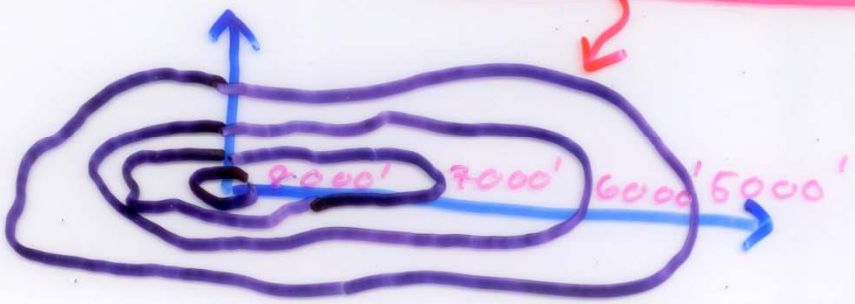
When the plane $z = c$ intersects the surface $z = f(x, y)$, the result is the space curve with the equation $f(x, y) = c$.

- The intersection curve is called the **trace** of the graph *f* in the plane $z = c$.
- The projection of this curve on the *xy*-plane is called a **level curve**.
- A collection of such curves is a **contour map/plot**.

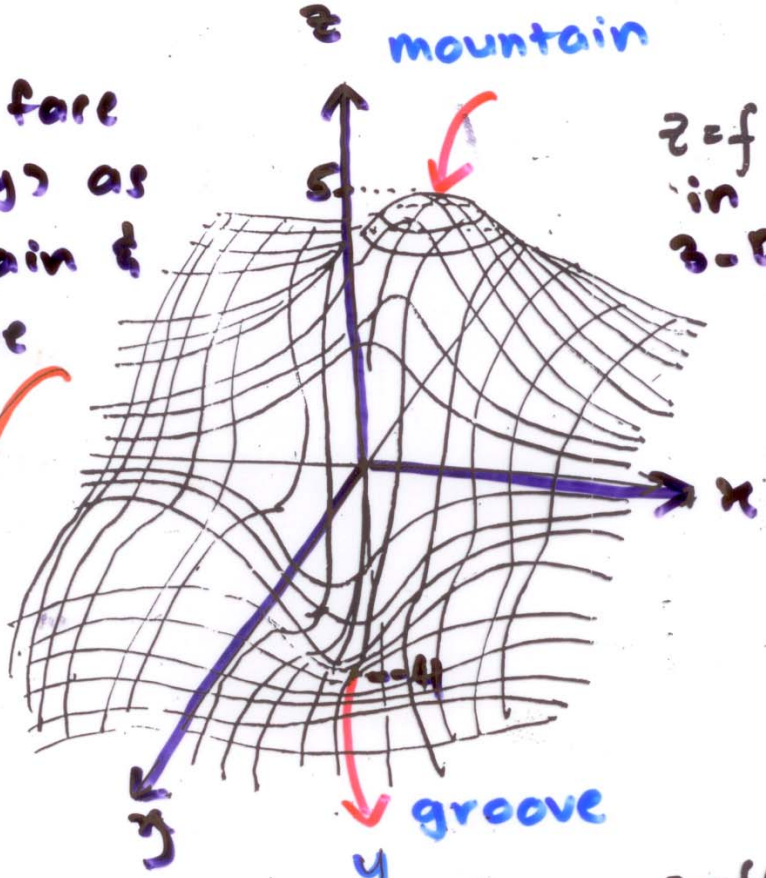
Eg: CONTOUR MAP

Level curves yield a contour map of $z = f(x,y)$

bird's eye view

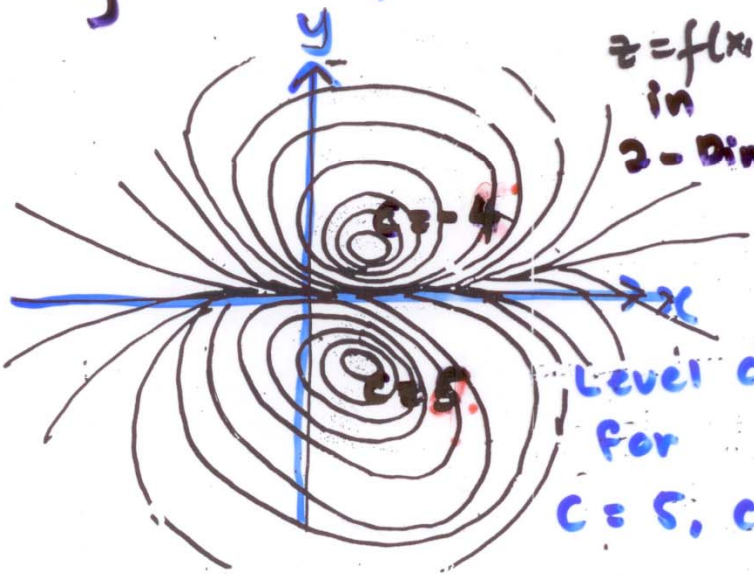


the surface $z = f(x,y)$ as a mountain & a groove



$z = f(x,y)$ in 3-Dimension

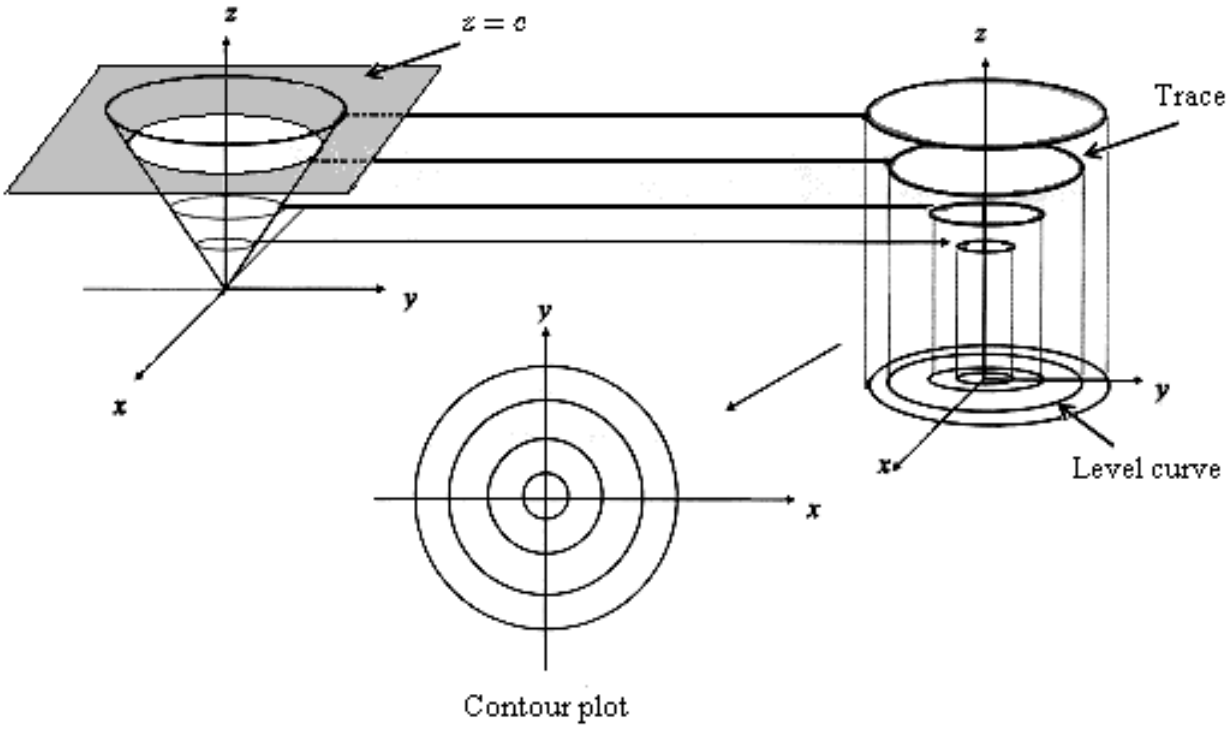
contour map



$z = f(x,y)$ in 2-Dimension

Level curves for $c = 5$, $c = -4$ etc...

Relationship Between Graphs of Surfaces and Level Curves



Definition

The level curves of a function f of two variables are the curves with equations $f(x,y) = c$, where c is a constant (in the range of f). A set of level curves for $z = f(x,y)$ is called a contour plot of f.

Example

Sketch the contour lines/level curves and the graphs

(i) $z = x^2 + y^2$, $c = 0, 1, 2, 3, 4, 9$

(ii) $z = \sqrt{x^2 + y^2}$, $c = 0, 1, 4, 9$

(iii) $z = 6 - x^2 - y$, $c = 0, 2, 4, 6$

Solution

(i) $z = x^2 + y^2$, $c = 0, 1, 2, 3, 4, 9$

Sketching the level curves

- first, replace z with the value of c

- second, plot the graph on the xy-plane

$c = 0$: $x^2 + y^2 = 0$

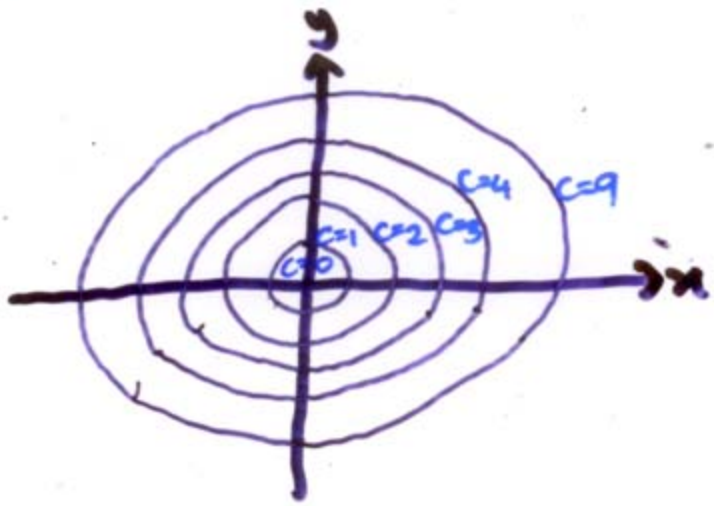
$c = 1$: $x^2 + y^2 = 1$

$c = 2$: $x^2 + y^2 = 2$

$c = 3$: $x^2 + y^2 = 3$

$c = 4$: $x^2 + y^2 = 4$

$c = 9$: $x^2 + y^2 = 9$

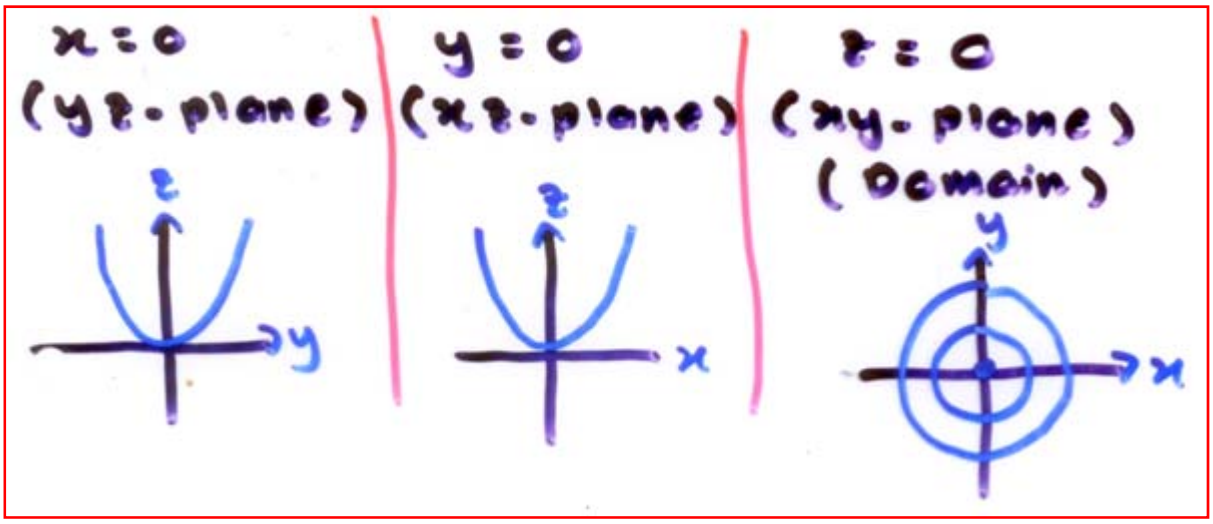


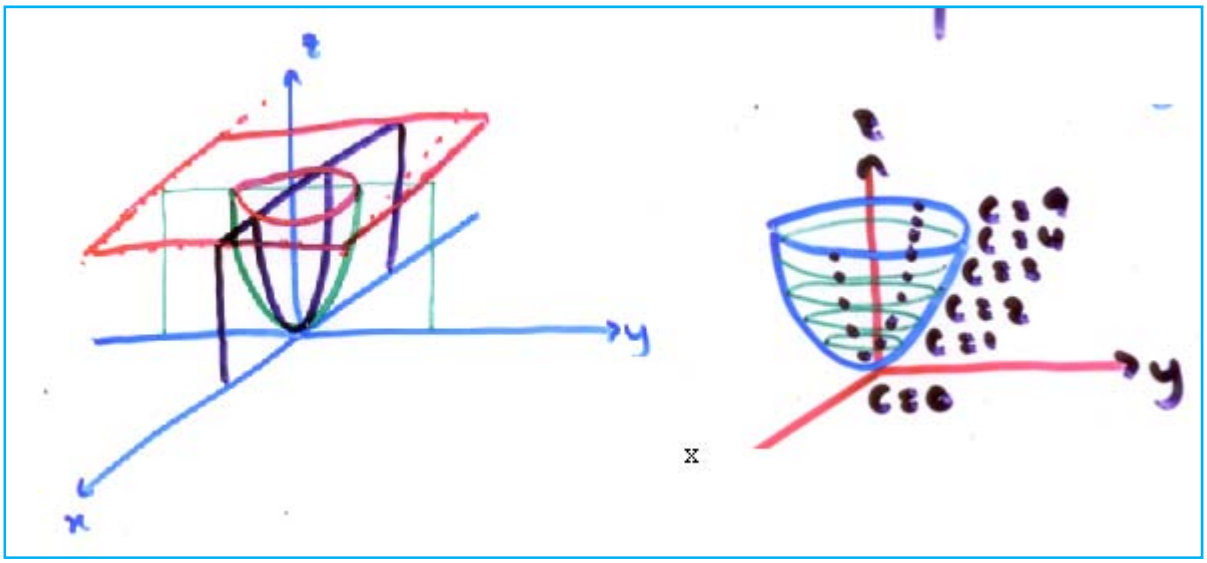
The traces in the coordinate planes:

- yz-plane, $x = 0$: the quadratic curve, $z = y^2$

- xz-plane, $y = 0$: the quadratic curve, $z = x^2$

- xy-plane, $z = 0$: a point (the origin)





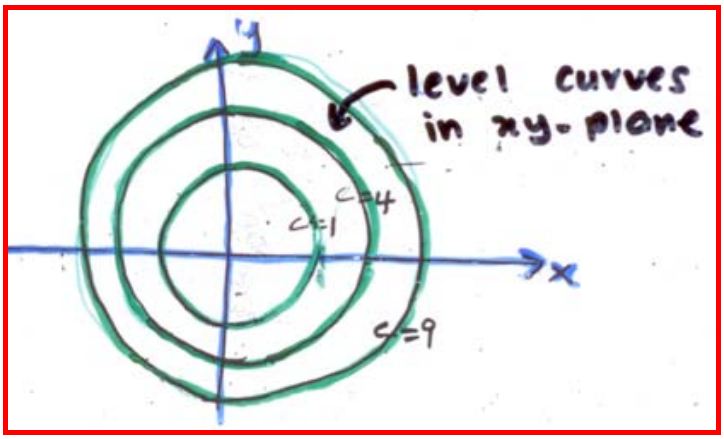
(ii) $z = \sqrt{x^2 + y^2}$, $c = 0, 1, 4, 9$.

$c = 0 \qquad : \sqrt{x^2 + y^2} = 0$

$c = 1 \qquad : \sqrt{x^2 + y^2} = 1$

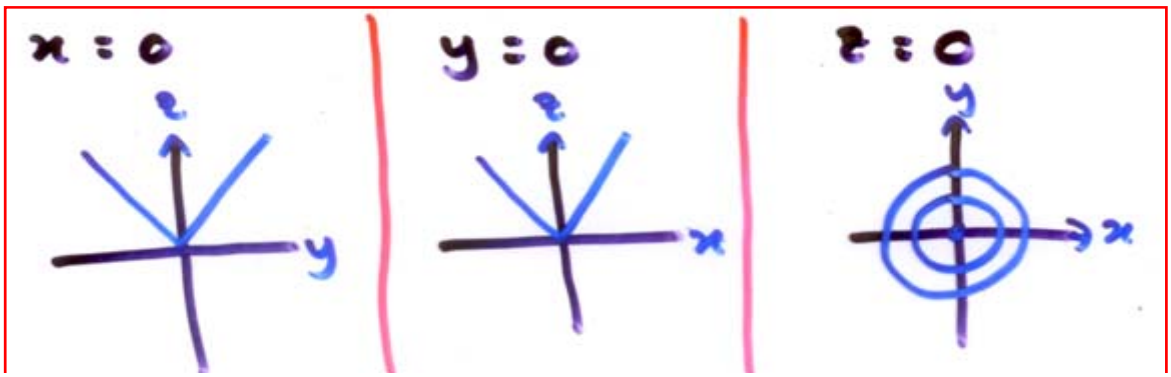
$c = 4 \qquad : \sqrt{x^2 + y^2} = 4$

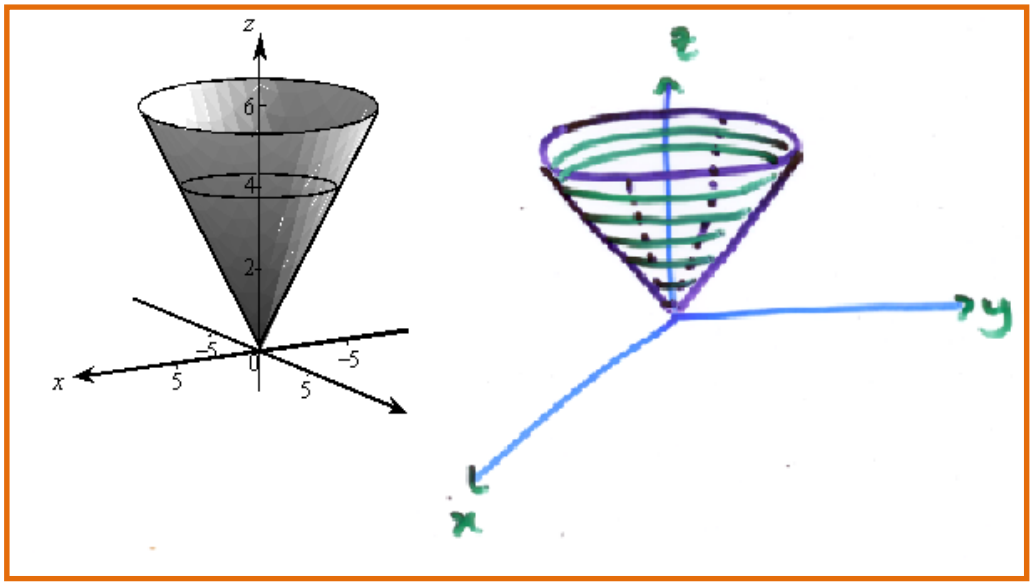
$c = 9 \qquad : \sqrt{x^2 + y^2} = 9$



The traces in the coordinate planes:

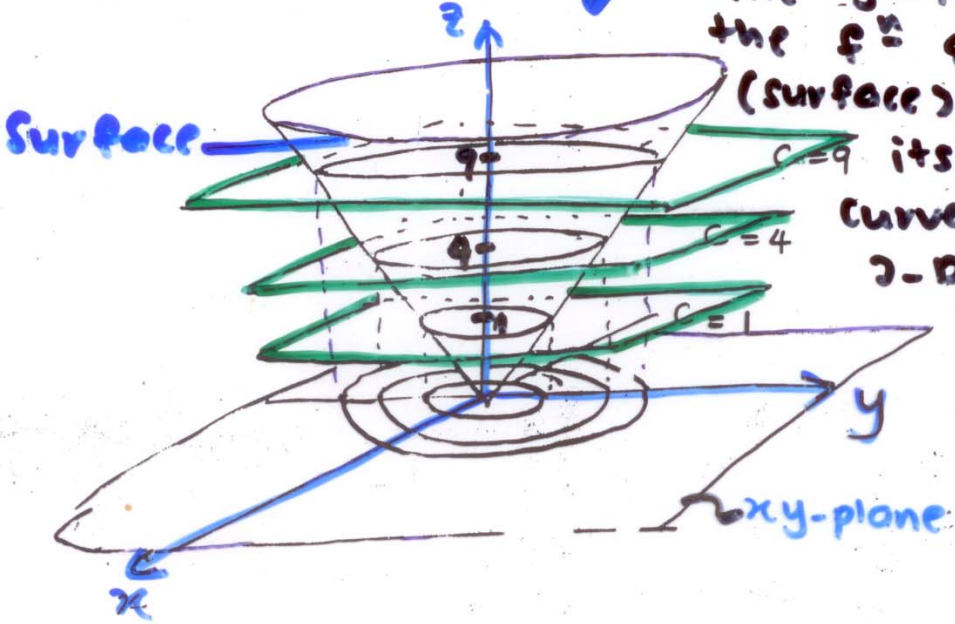
- yz-plane, $x = 0$: the straight line, $z = y$
- xz-plane, $y = 0$: the straight line, $z = x$
- xy-plane, $z = 0$: a point (the origin)
- parallel to xy-plane, $z = 4$: the circle $x^2 + y^2 = 4^2$


x = 0
z
y
y = 0
z
x
z = 0
y
x


z
6
4
2
x
-5
5
0
-5
5
z
y
x

eg. $z = \sqrt{x^2+y^2}$

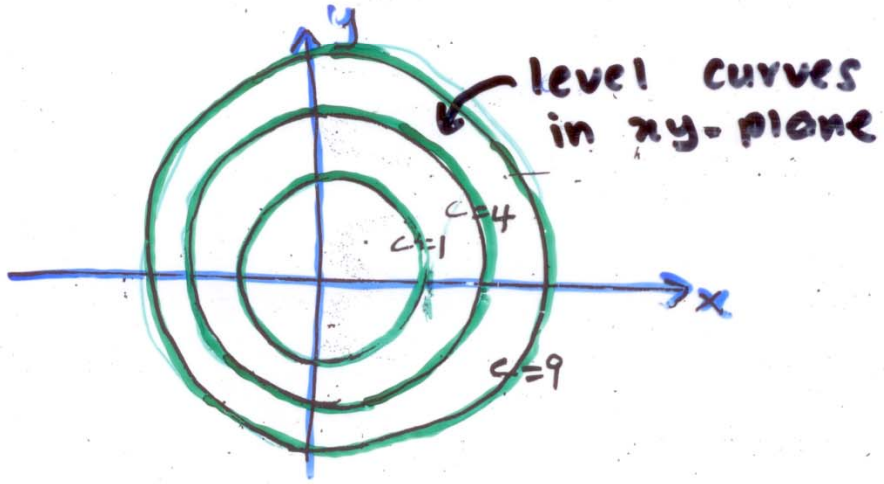
Relation between the graph of the f^n given (surface) and its level curves in 2-D



$c=1 \;:\; 1 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 1$

$c=4 \;:\; 4 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 16$

$c=9 \;:\; 9 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 81$



(ii) $z = 6 - x^2 - y \;\; , c = 0, 2, 4, 6$.

Sketching the level curves

- first, replace z with the value of c

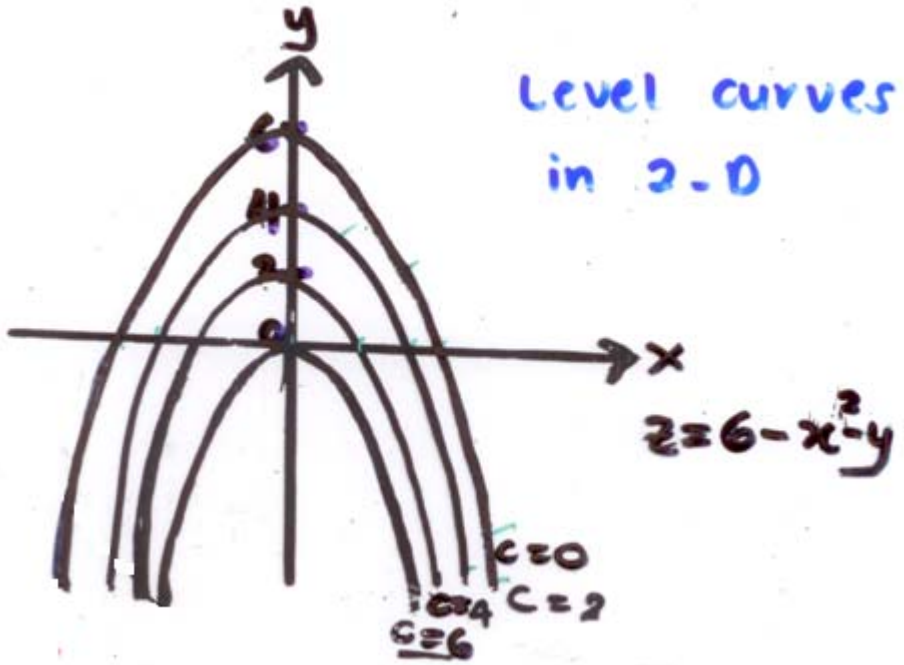
- second, plot the graph on the xy-plane

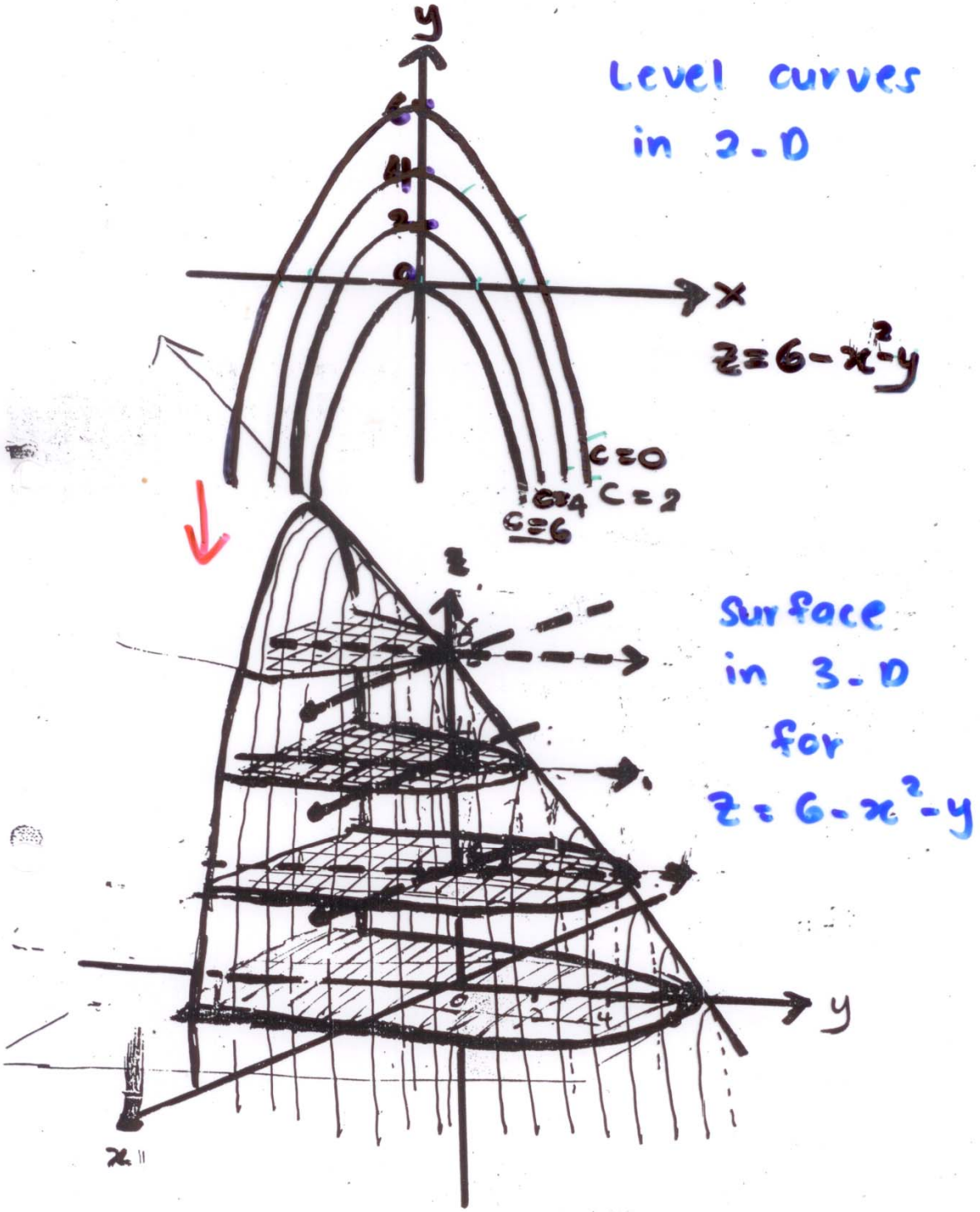
$c = 0 \qquad : 6 - x^2 - y = 0 \Rightarrow y = -x^2 + 6$

$c = 2 \qquad : 6 - x^2 - y = 2 \Rightarrow y = -x^2 + 4$

$c = 4 \qquad : 6 - x^2 - y = 4 \Rightarrow y = -x^2 + 2$

$c = 6 \qquad : 6 - x^2 - y = 6 \Rightarrow y = -x^2$




Level curves
in 2-D
y
x
6
4
2
0
$z = 6 - x^2 - y$
c=0
c=2
c=4
c=6
z
Surface
in 3-D
for
$z = 6 - x^2 - y$
y
x

Example 8

Find the domain and range of the following functions; also sketch the level curves for some values of $z = c, xz$-trace and yz-trace; and hence sketch the surface $z = f(x, y)$.

(a) $f(x,y) = 9 - x^2 - y^2$; $c = 0, 3, 6, 8$.

(b) $z = x^2 + y^2 - 4x + 6y + 13$; $c = 1, 4, 9$.

6.4 Functions of 3 Variables

Basic ideas of functions of two variables can be extended to the study of functions of three variables.

(a) The graph of a one variable function is 2-space, a two variables function is in 3-space thus we expect a three variable function will be in 4-space.

(b) It is difficult to visualise graphs in 4-space. However, we can draw the level surfaces of the graph to ascertain the properties and behaviour of three variables functions.

- **Domain and Range**

A **function *f* of three variables** is a rule that assigns to each ordered triple (x, y, z) in some domain D in space a unique real number $w = f(x, y, z)$. The range consists of the output values for w.

Example 9

Let $f(x,y,z) = x^2 + y^2 + z^2$.

i. Evaluate $f(1, 2, 1)$, $f(1, 0, 1)$, $f(-3, 2, 1)$.

ii. Determine the domain and range.

Solution

i. By substitution,

$$f(1, 2, 1) = 1^2 + 2^2 + 1^2 = 6$$
$$f(1, 0, 1) = 1^2 + 0^2 + 1^2 = 2$$
$$f(-3, 2, 1) = (-3)^2 + 2^2 + 1^2 = 14$$

ii. The domain is a set of all ordered triplets (x, y, z) for which $x^2 + y^2 + z^2$ is defined.

$x^2 + y^2 + z^2 \geq 0$ for all points in space. Thus the domain is the entire 3-space.

Domain : $\{(x, y, z) \mid x, y, z \in \mathbb{R})$

Range: $0 \leq f(x, y, z) < \infty$

- **Level Surfaces**

The graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ lying in four-dimensional space.

(a) Graphs cannot be sketch effectively in three-dimensional frame of reference.

(b) Can obtain insight of how function behaves by looking at its three-dimensional level surfaces.

Level surfaces are the three dimensional analog of level curves. If $f(x, y, z)$ is a function of three variables and c is a constant then $f(x, y, z) = c$ is a surface in 3-space. It is called a contour surface or a level surface.

Example 10

Describe the level surfaces of the following function:

(a) $f(x, y, z) = x^2 + y^2 + z^2 ; c = 0, c < 0, c > 0.$

(b) $f(x, y, z) = z^2 - x^2 - y^2 ; c = 0, c < 0, c > 0.$

(c) $f(x, y, z) = x^2 + y^2 ; c = 4, 9.$

(d) $f(x, y, z) = 4x^2 + y^2 + 4z^2 ; c = 1, 4.$