2.5 Comparison Tests

In this section, we discuss methods to determine the convergence of a series by comparing the series with the series that we know of its convergence. The series that we use for comparison are usually the geometric series or the p-series.

The proof of comparison theorems depends on the following theorem, which is a restatement of Theorem 1.9 in term of series.

Theorem 2.8 *Assume $a_n \geq 0$ for all $n = 1, 2, 3, \ldots$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of its partial sum is bounded.*

Note that Theorem 2.8 is on the series of psitive terms. Since $a_n \geq 0$ for all $n = 1, 2, 3, \ldots$, then the sequence of its partial sum is automatically increasing.

Based on the above theorem we obtain the following theorem.

Theorem 2.9 *(Direct Comparison Test)*

Assume $a_n \geq 0$ and $b_n \geq 0$ for all $n = 1, 2, 3, \ldots$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all $n = 1, 2, 3, \ldots$, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all $n = 1, 2, 3, \ldots$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

(i) Let S_n be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$ and T_n be the sequence of partial sum of $\sum_{n=1}^{\infty} b_n$, that is

$$S_n = a_1 + a_2 + \ldots + a_n \quad \text{and} \quad T_n = b_1 + b_2 + \ldots + b_n.$$

Since $\sum_{n=1}^{\infty} b_n$ converges, let its sum is T. If $a_n \leq b_n$ for every $n = 1, 2, 3, \ldots$, then

$$S_n \leq T_n < T.$$

By Theorem 2.8, $\sum_{n=1}^{\infty} a_n$ converges.

(ii) From part (i), we have for $n = 1, 2, 3, \ldots$

$$S_n \geq T_n,$$

since $a_n \geq b_n$. Since $\sum_{n=1}^{\infty} b_n$ diverges, T_n tends to infinity as n tends infinity, which implies S_n tends to infinity as n tends infinity. Hence $\sum_{n=1}^{\infty} a_n$ diverges. ♣

Example 2.9 Determine whether each series converges or diverges.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{1}{(2+5^n)}, \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{1}{n\cos\left(\frac{1}{n}\right)}.$$

Solution

(i) Observe that $\dfrac{1}{2+5^n} < \dfrac{1}{5^n} = \left(\dfrac{1}{5}\right)^n$ for $n = 1, 2, 3, \ldots$.

Also we have that $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a geometric series that converges, since $r = \frac{1}{5} < 1$. Hence by direct comparison test, the series $\sum_{n=1}^{\infty} 1/(2+5^n)$ converges.

(ii) Observe that $\dfrac{1}{n} < \dfrac{1}{n\cos\left(\frac{1}{n}\right)}$ for $n = 1, 2, 3, \ldots$.

Also we have that $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series that diverges. Hence by direct comparison test the series $\sum_{n=1}^{\infty} 1/[n\cos\left(\frac{1}{n}\right)]$ diverges. ♣

If the limit of the ratio between the terms of the series that we want to determine its convergency and the corresponding terms of series that we want to compare can be easily found, then the following theorem is useful.

Theorem 2.10 *(Limit Comparison Test)*

Assume $a_n > 0$ and $b_n > 0$ fo all $n = 1, 2, 3, \ldots$.

(i) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = k$, $0 < k < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges.

(ii) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(iii) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

(i) Since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = k$, then there exist a positive integer N such that

$$\left|\frac{a_n}{b_n} - k\right| < \frac{k}{2}, \quad \text{that is} \quad \frac{k}{2} < \frac{a_n}{b_n} < \frac{3k}{2}$$

for all $n \geq N$. Hence $kb_n < 2a_n$ and $2a_n < 3kb_n$ for all $n \geq N$. By using the direct comparison test we obtain the statement of the theorem.

The proof of part (ii) and (iii) are left as exercises for the readers. ♣

Example 2.10 Determine whether each series converges or diverges.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{7n^3 - n^2 + 1}{n^5 + n^3 + 2}, \qquad \text{(ii)} \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right).$$

Solution

(i) With $a_n = \dfrac{7n^3 - n^2 + 1}{n^5 + n^3 + 2}$, we want to find b_n.

Since we want to find limit as n tends to infinity, we investigate the approximate value of a_n when n is large. When n is large, the value of a_n will be influenced mainly by the highest power n in the numerator and the denominator. Hence we have

$$a_n = \frac{7n^3 - n^2 + 1}{n^5 + n^3 + 2} \approx \frac{7n^3}{n^5} = \frac{7}{n^2}.$$

By taking $b_n = 7/n^2$, and after using L'Hôpital's rule we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{7n^5 - n^4 + n^2}{7(n^5 + n^3 + 2)} = 1 > 0.$$

But the series

$$\sum_{n=1}^{\infty} b_n = 7 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges since it is p-series with $p = 2$. By using part (i) of the limit comparison theorem, we conclude that the series

$$\sum_{n=1}^{\infty} \frac{7n^3 - n^2 + 1}{n^5 + n^3 + 2}$$

converges.

(ii) Since $a_n = \sin(1/n)$, we take $b_n = 1/n$. Then we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0.$$

But the series $\sum_{n=1}^{\infty} 1/n$ is the $p-$series that diverges. By using part (i) of the limit comparison theorem, we conclude that the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges. ♣

Example 2.11 Determine whether each series converges or diverges.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{\ln n}{4\sqrt{n^3}}, \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+100}.$$

Solution

(i) Whe n is large, we have

$$a_n = \frac{\ln n}{4\sqrt{n^3}} = \frac{\ln(n^{1/4})}{n^{3/2}} \approx \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}.$$

Hence we take $b_n = 1/n^{5/4}$ to obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[\frac{(\ln n)}{4\sqrt{n^3}} \div \frac{1}{n^{5/4}}\right] = \lim_{n\to\infty} \frac{\ln n}{4n^{1/4}}.$$

By using L'Hôpital's rule, we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{4n^{1/4}} = \lim_{n\to\infty} \frac{1/n}{n^{-3/4}} = \lim_{n\to\infty} \frac{1}{n^{1/4}} = 0.$$

But the series $\sum_{n=1}^{\infty} 1/n^{5/4}$ is the p-series that converges. By using part (ii) of the limit comparison test, we conclude that the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{4\sqrt{n^3}}$$

converges.

(ii) When n is large we have

$$a_n = \frac{1+n\ln n}{n^2+100} \approx \frac{n\ln n}{n^2} = \frac{\ln n}{n} > \frac{1}{n}.$$

Therefore we take $b_n = 1/n$ to obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[\frac{(1+n\ln n)}{(n^2+100)} \div \frac{1}{n}\right] = \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+100}.$$

By using L'Hôpital's rule, we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+100} = \infty.$$

But the series $\sum_{n=1}^{\infty} 1/n$ is the p-series that diverges. By using part (iii) of the limit comparison test, we conclude that the series

$$\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+100}$$

diverges. ♣