CHAPTER 2: Partial Derivatives

2.1 Definition of a Partial Derivative

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Chapter 2: Partial Derivatives

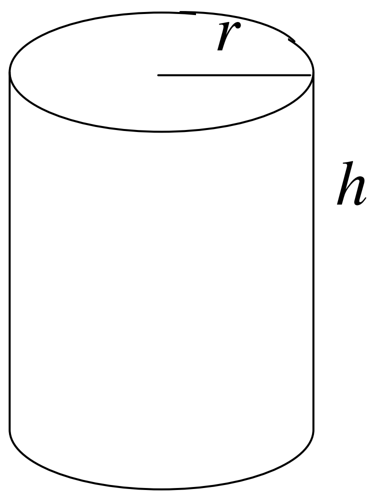
2.1 Definition of a Partial Derivative

- The process of differentiating a function of several variables with respect to one of its variables while keeping the other variables fixed is called **partial differentiation**.

- The resulting derivative is a **partial derivative** of the function.

See illustration

As an illustration, consider the surface area of a right-circular cylinder with radius *r* and height *h*:



We know that the surface area is given by $S = 2\pi r^2 + 2\pi rh$. This is a function of two variables *r* and *h*.

Suppose *r* is held fixed while *h* is allowed to vary. Then,

$$\left[\frac{dS}{dh}\right]_{r \text{ const.}} = 2\pi r$$

This is the **"partial derivative of *S* with respect to *h*"**. It describes the rate with which a cylinder's surface changes if its height is increased and its radius is kept constant.

Likewise, suppose *h* is held fixed while *r* is allowed to vary. Then,

$$\left[\frac{dS}{dr}\right]_{h \text{ const.}} = 4\pi r + 2\pi h$$

This is the **"partial derivative of *S* with respect to *r*"**. It represents the rate with which the surface area changes if its radius is increased and its height is kept constant.

In standard notation, these expressions are indicated by

$$S_h = 2\pi r,\ S_r = 4\pi r + 2\pi h$$

Thus in general, the partial derivative of $z = f(x,y)$ with respect to *x*, is the rate at which *z* changes in response to changes in *x*, holding *y* constant. Similarly, we can view the partial derivative of *z* with respect to *y* in the same way.

Note

Just as the ordinary derivative has different interpretations in different contexts, so does a partial derivative. We can interpret derivative as a rate of change and the slope of a tangent line.

Recall: Derivative of a single variable f is defined formally as,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The definition of the partial derivatives with respect to x and y are defined similarly.

Definition 2.1

If $z = f(x, y)$, then the (first) partial derivatives of f with respect to x and y are the functions f_x and f_y respectively defined by

$$f_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

2.1.1 Notation

For $z = f(x, y)$, the partial derivatives f_x and f_y are also denoted by the symbols:

$$\frac{\partial f}{\partial x}, \frac{\partial z}{\partial x}, \frac{\partial}{\partial x} f(x, y), \; f_x(x, y) \text{ or } z_x$$

$$\frac{\partial f}{\partial y}, \frac{\partial z}{\partial y}, \frac{\partial}{\partial y} f(x, y), \; f_y(x, y) \text{ or } z_y$$

The values of the partial derivatives at the point (a, b) are denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) \text{ and } \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Note

- The stylized "d" symbol in the notation is called **roundback d**, **curly d** or **del d**.
- It is not the usual derivative d (dee) or δ (delta d).

Illustration

- Finding and evaluating partial derivative of a function of two variables
- Finding partial derivative of a function of three variables
- Finding partial derivative of an implicitly defined function

Example 2.7

If
$f(x,y) = x^3y + x^2y^2 + 4x$,
find

i. $\dfrac{\partial f}{\partial x}$ ii. $\dfrac{\partial f}{\partial y}$ iii. $f_y(1,-2)$

Prompts/Questions

- What do the notations stand for?
  - Which variable is changing?
  - Which variable is held constant?
- Which variables give the value of a derivative?

Solution

(a) For f_x, hold y constant and find the derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\ x^3 y + x^2 y^2 \ = 3x^2 y + 2xy^2 \ + 4$$

(b) For f_y, hold x constant and find the derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\ x^3 y + x^2 y^2 \ = x^3 \ + 2x^2 y$$

(c) $f_y(1,-2) = (1)^3 + 2(1)^2(-2) = -3$

For a function $f(x, y, z)$ of three variables, there are three partial derivatives:

$$f_x, \quad f_y \quad \text{and} \quad f_z$$

The partial derivative f_x is calculated by holding y and z constant. Likewise, for f_y and f_z.

Example 2.2

Let $f(x, y, z) = x^2 + 2xy^2 + yz^3$, find:

(a) f_x (b) f_y (c) f_z

Solution

(a) $f_x(x, y, z) = 2x + 2y^2$

(b) $f_y(x, y, z) = 4xy + z^3$

(c) $f_z(x, y, z) = 3yz^2$

The rules for differentiating functions of a single variable holds in calculating partial derivatives.

Example 2.3

Find $\dfrac{\partial f}{\partial y}$ if $f(x, y) = \ln(x + y)$.

Solution

We treat x as a constant and f as a composite function:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\ln(x + y)] = \frac{1}{x + y}\frac{\partial}{\partial y}(x + y)$$

$$= \frac{1}{x + y}(0 + 1)$$

$$= \frac{1}{x + y}$$

Example 2.3a

Determine the partial derivatives of the following functions with respect to each of the independent variables:

(a) $z = (x^2 + 3y)^5$

(b) $w = ze^{3x-7y}$

Example 2.3b

Determine the partial derivatives of the following functions with respect to each of the independent variables:

a) $z = x\sin(2x^2 + 5y)$

b) $f(x, y) = \dfrac{2y}{y + \cos x}$

Example 2.4

If $z = f(x^2 + y^2)$, show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$$

Example 2.5

Find $\dfrac{\partial z}{\partial x}$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of two independent variables x and y.

Solution

We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y)$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0, \qquad y \text{ constant}$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\therefore \qquad \frac{\partial z}{\partial x} = \frac{z}{yz - 1}$$

Example 2.5a

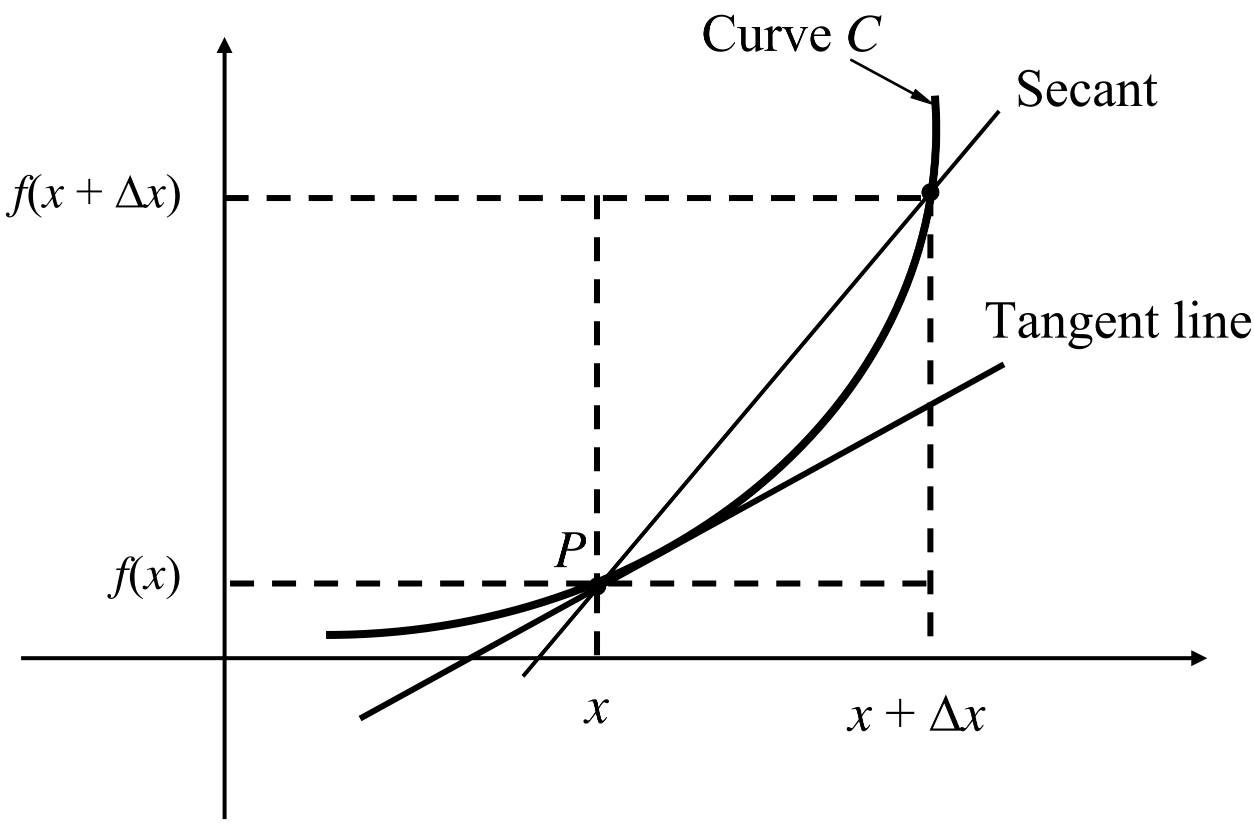
If

$$\cos(x+2z)+3y^2+2xyz=0$$

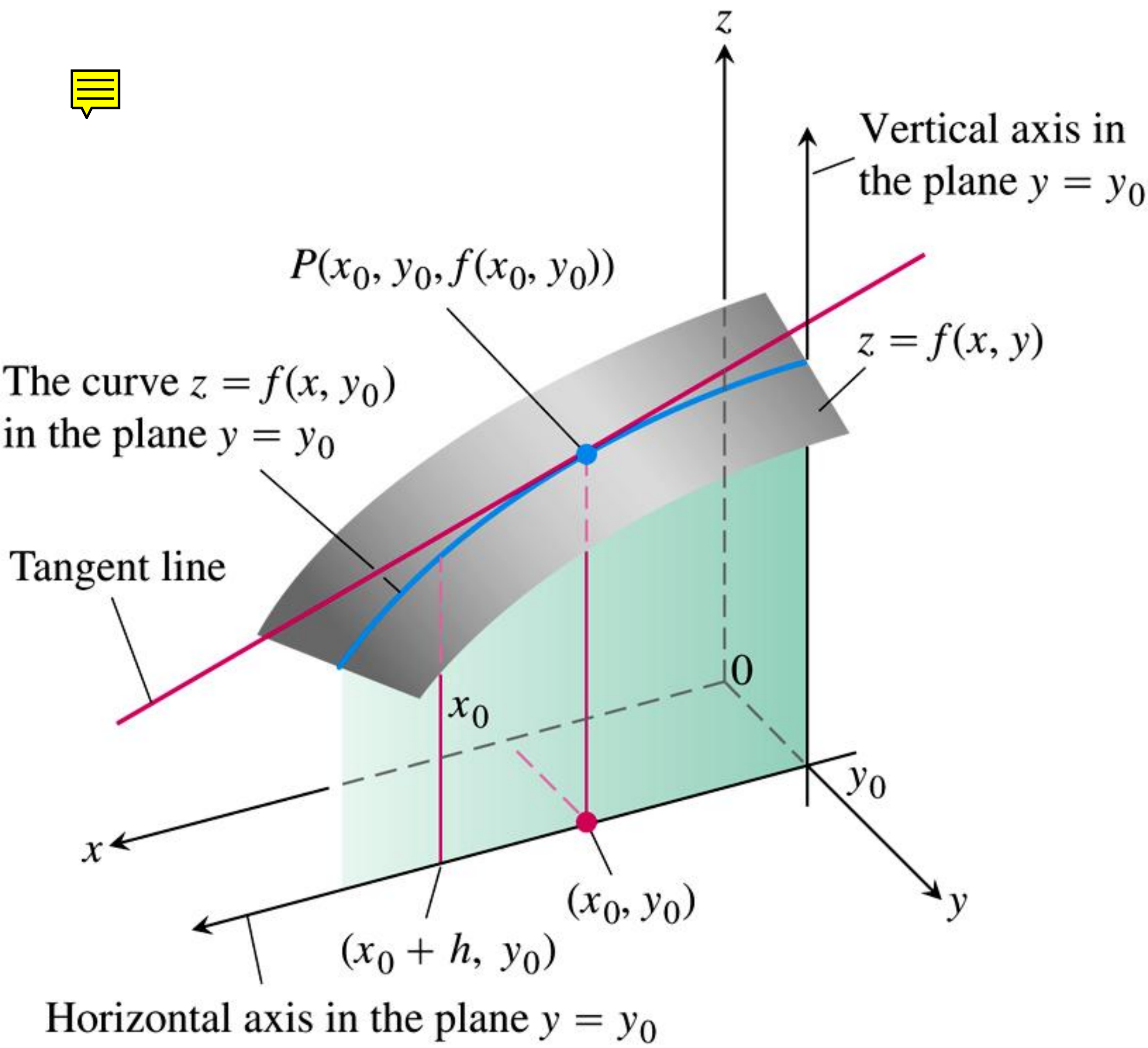
defines z as a function of two independent variables x and y. Determine expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of x, y and z.

2.1.2 Partial Derivative as a Slope

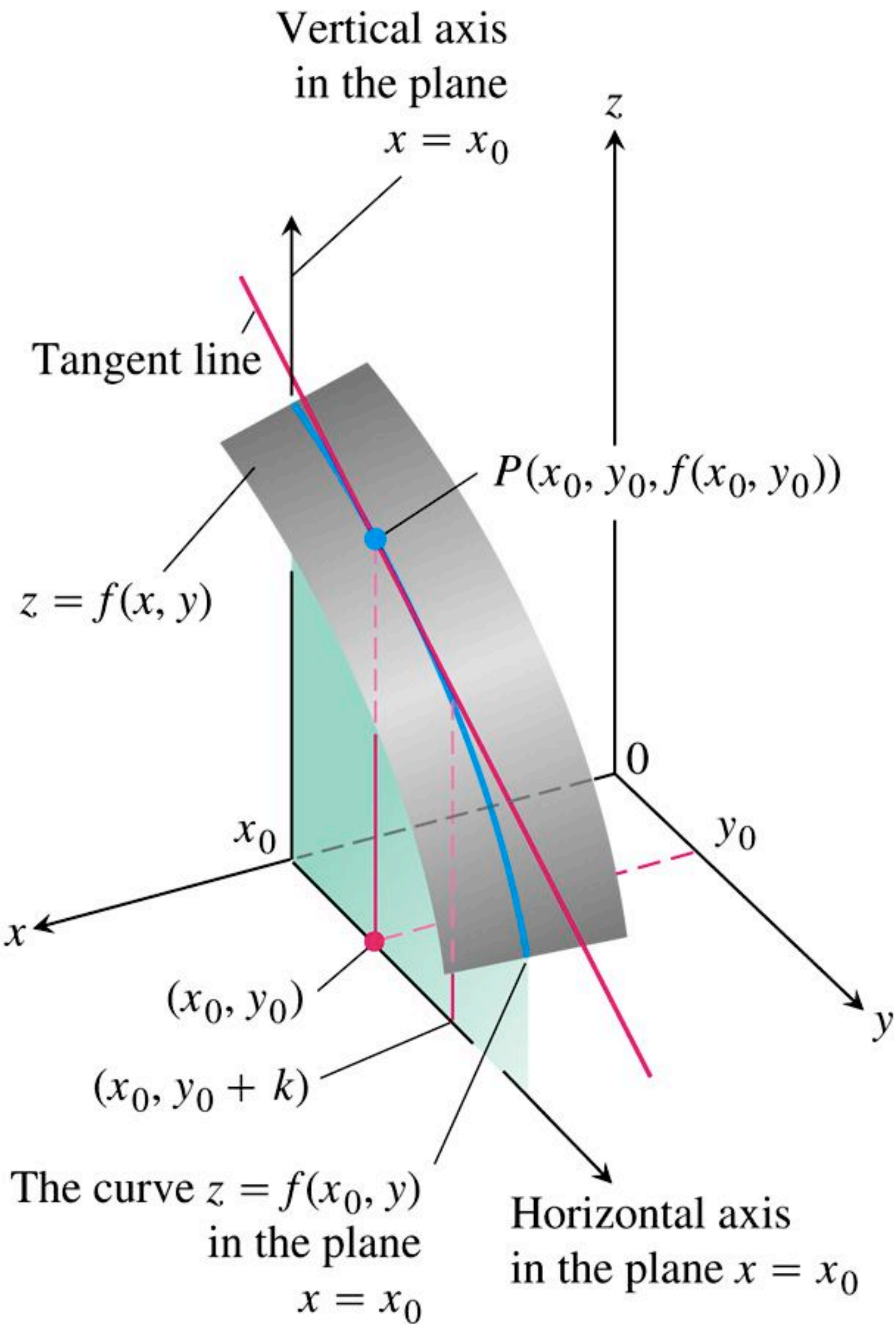
To understand the concept let's take a look at the one-variable case:



At P, the tangent line to the curve C has slope $f'(x)$.



The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$.



The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$.

Example 2.6

Find the slope of the line that is parallel to the xz-plane and tangent to the surface $z = x\sqrt{x + y}$ at the point $P(1, 3, 2)$.

Solution

Given $f(x, y) = x\sqrt{x + y}$

WANT: $f_x(1, 3)$

$$f_x(x, y) = (x + y)^{1/2} + x\left(\frac{1}{2}\right)(x + y)^{-1/2}(1 + 0)$$

$$= \sqrt{x + y} + \frac{x}{2\sqrt{x + y}}$$

Thus the required slope,

$$f_x(1, 3) = \sqrt{1 + 3} + \frac{1}{2\sqrt{1 + 3}} = \frac{9}{4}$$

2.1.3 Partial Derivative as a Rate of Change

The derivative of a function of one variable can be interpreted as a rate of change. Likewise, we can obtain the analogous interpretation for partial derivative.

- A **partial derivative** is the rate of change of a multi-variable function when we allow only one of the variables to change.

- Specifically, the partial derivative $\frac{\partial f}{\partial x}$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.

Example 2.7

The volume of a gas is related to its temperature T and its pressure P by the gas law $PV = 10T$, where V is measured in cubic inches, P in pounds per square inch, and T in degrees Celsius. If T is kept constant at 200, what is the rate of change of pressure with respect to volume at $V = 50$?

Solution

WANT: $\left.\dfrac{\partial P}{\partial V}\right|_{T=200,V=50}$

Given $PV = 10T$.

$$\frac{\partial P}{\partial V} = \frac{-10T}{V^2}$$

$$\therefore \quad \left.\frac{\partial P}{\partial V}\right|_{T=200,V=50} = \frac{(-10)(200)}{(50)^2} = -\frac{4}{5}$$

2.1.4 Higher Order Partial Derivatives

The partial derivative of a function is a function, so it is possible to take the partial derivative of a partial derivative.

If z is a function of two independent variables, x and y, the possible partial derivatives of the second order are:

- **second partial derivative** – taking two consecutive partial derivatives with respect to the same variable

- **mixed partial derivative** - taking partial derivatives with respect to one variable, and then take another partial derivative with respect to a different variable

Standard Notations

Given $z = f(x, y)$

Second partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

Mixed partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

Remark

- The mixed partial derivaties can give the same result whenever $f,\ f_x,\ f_y,\ f_{xy}$ and f_{yx} are all continuous.
- Partial derivaties of the third and higher orders are defined analogously, and the notation for them is similar.

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right] = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right]\right) = f_{yyxx}$$

The order of differentiation is immaterial as long as the derivatives through the order in question are continuous.

Example 2.8

Let $z = 7x^3 - 5x^2y + 6y^3$. Find the indicated partial derivatives.

i. $\dfrac{\partial^2 z}{\partial x \partial y}$ ii. $\dfrac{\partial^2 z}{\partial y \partial x}$

iii. $\dfrac{\partial^2 z}{\partial x^2}$ iv. $f_{xy}(2,1)$

Prompts/Questions

- What do the notations represent?
- What is the order of differentiation?
  - With respect to which variable do you differentiate first?

Solution

Keeping *y* fixed and differentiating w.r.t. *x*, we obtain $\dfrac{\partial z}{\partial x} = 21x^2 - 10xy$.

Keeping *x* fixed and differentiating w.r.t. *y*, we obtain $\dfrac{\partial z}{\partial y} = -5x^2 + 18y^2$.

(i) $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-5x^2 + 18y^2) = -10x$$

(ii) $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(21x^2 - 10xy) = -10x$$

(iii) $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial z}\right) = \frac{\partial}{\partial x}(21x^2 - 10xy) = 42x - 10y$$

$$\text{(iv) } f_{xy}(2,1) = \left.\frac{\partial^2 z}{\partial y \partial x}\right|_{(2,1)} = -10(2) = -20$$

Example 2.9

Determine all first and second order partial derivatives of the following functions:

i. $z = y\sin x + x\cos y$

ii. $z = e^{xy}(2x - y)$

iii. $f(x,y) = x\cos y + ye^x$

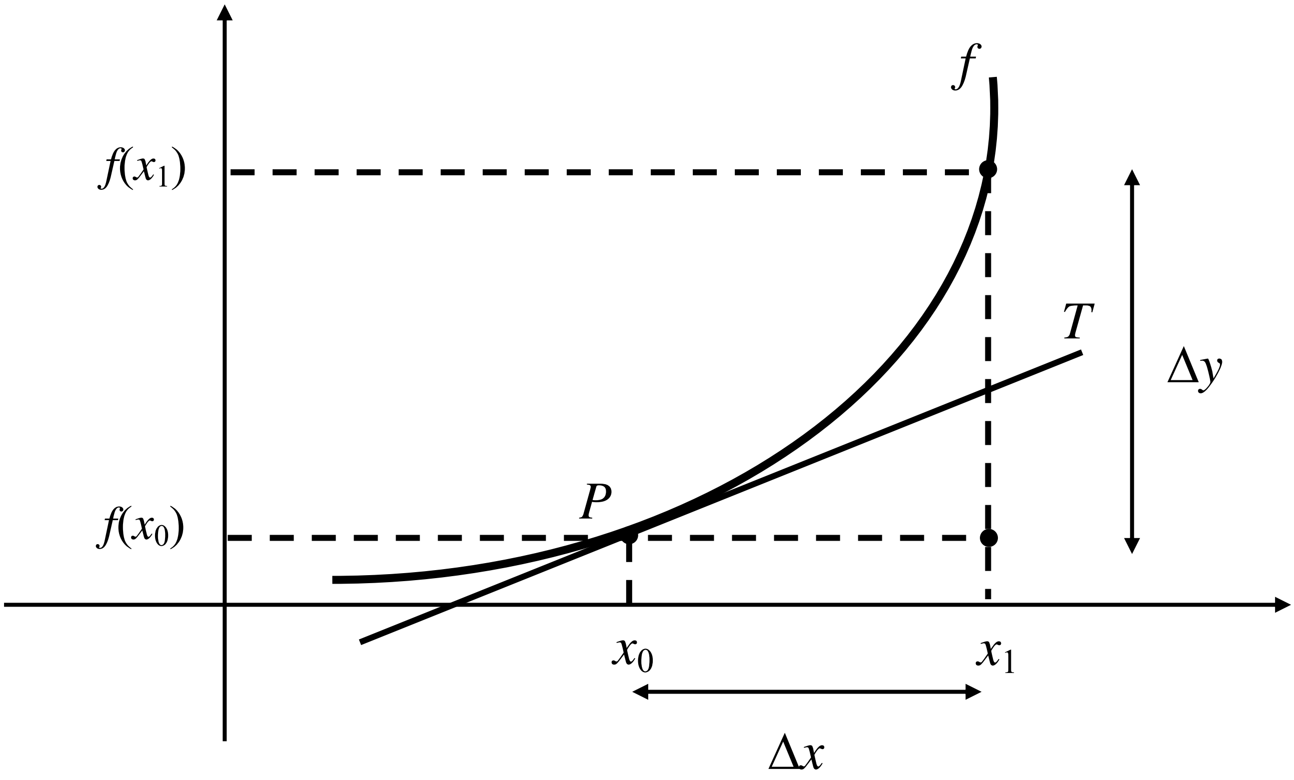
Prompts/Questions

- What are the first partial derivatives of f?
  - Which derivative rules or techniques do you need?
- How many second-order derivatives are there?

2.2 Increments and Differential

2.2.1 Functions of One Variable – A Recap

Tangent Line approximation



If f is differentiable at $x = x_0$, the tangent line at $P(x_0, f(x_0))$ has slope $m = f'(x_0)$ and equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

If x_1 is near x_0, then $f(x_1)$ must be close to the point on the tangent line, that is

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0)$$

This expression is called the **linear approximation formula**.

Incremental Approximation

We use the notation Δx for the difference $x_1 - x_0$ and the corresponding notation Δy for $f(x_1) - f(x_0)$. Then the linear approximation formula can be written as

$$f(x_1) - f(x_0) \approx f'(x_0)\Delta x$$

or equivalently

$$\Delta y \approx f'(x_0)\Delta x$$

Definition 2.2

If f is differentiable and the increment Δx is sufficiently small, then the increment Δy, in y, due to an increment of Δx, in x is given by

$$\Delta y \approx \frac{dy}{dx} \Delta x$$

or

$$\Delta f \approx f'(x) \Delta x$$

Note

This version of approximation is sometimes called the **incremental approximation formula** and is used to study propagation of error.

The Differential

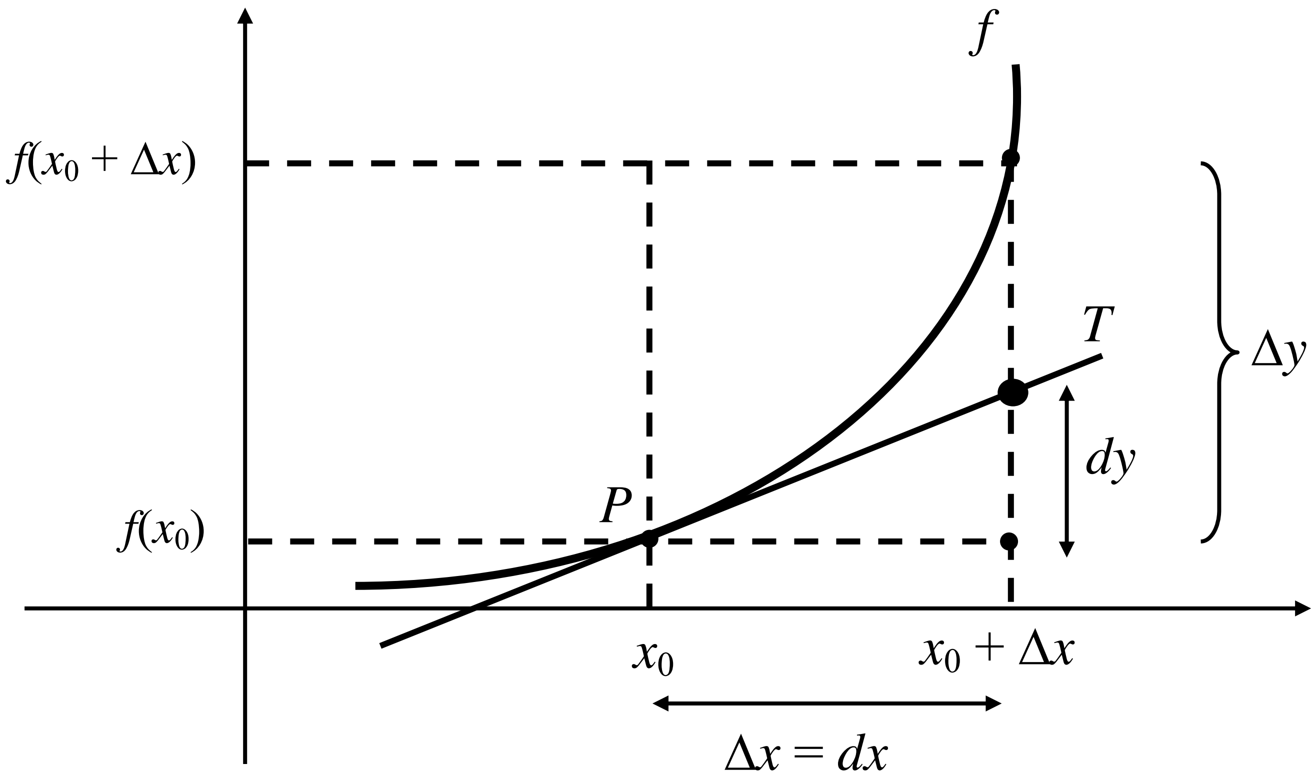
dx is called the **differential of *x*** and we define dx to be Δx, an arbitrary increment of x. Then, if f is differentiable at x, we define the corresponding **differential of *y***, dy as

$$dy = \frac{dy}{dx} dx$$

or equivalently $df = f'(x)\, dx$

Thus, we can estimate the change Δf, in f by the value of the differential df provided dx is the change in x.

$$\Delta f \approx df$$



- $\Delta x = dx$
- Δy is the rise of f (the change in y) that occurs relative to $\Delta x = dx$
- dy is the rise of tangent line relative to $\Delta x = dx$

The true change: $\Delta f = f(x_0 + \Delta x) - f(x_0)$

The differential estimate: $df = f'(x)dx$

2.2.2 Functions of Two Variables

Let $z = f(x, y)$, where x and y are independent variables.

If x is subject to a small increment (or a small error) of Δx, while y remains constant, then the corresponding increment of Δz in z will be

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x$$

Similarly, if y is subject to a small increment of Δy, while x remains constant, then the corresponding increment of Δz in z will be

$$\Delta z \approx \frac{\partial z}{\partial y}\Delta y$$

It can be shown that, for increments (or errors) in both x and y,

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

The formula for a function of two variables may be extended to functions of a greater number of independent variables.

For example, if $w = f(x, y, z)$ of three variables, then

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z$$

Definition 2.3

Let $z = f(x, y)$ where f is a differentiable function and let dx and dy be independent variables. The differential of the dependent variable, dz is called the **total differential of z** is defined as

$$dz = df(x, y) = f_x(x, y)dx + f_y(x, y)dy$$

Thus, $\Delta z \approx dz$ provided dx is the change in x and dy is the change in y.

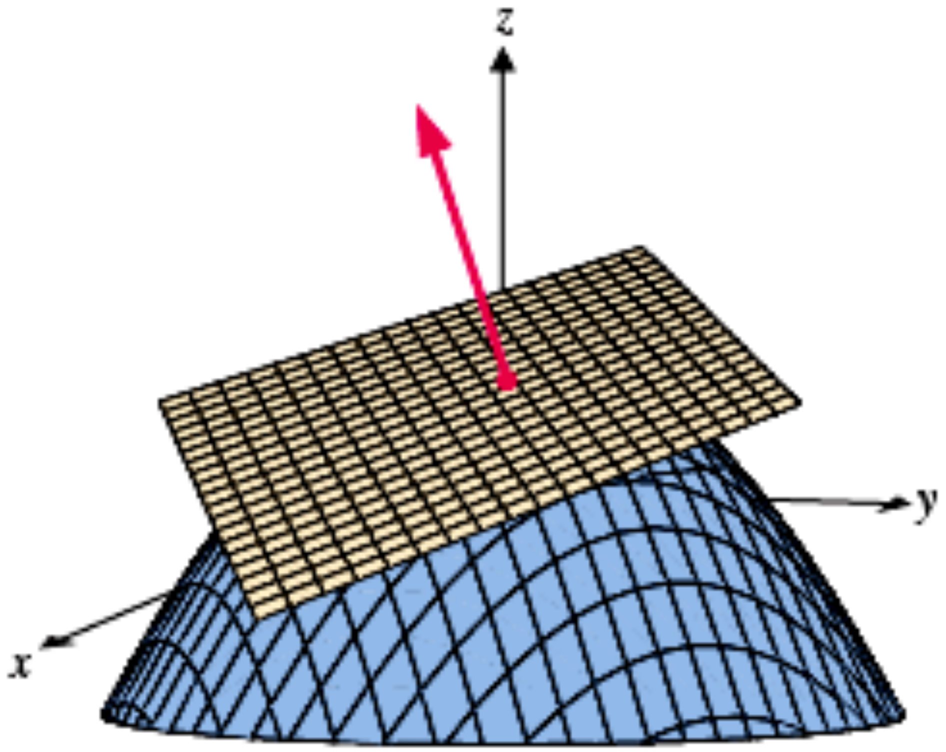
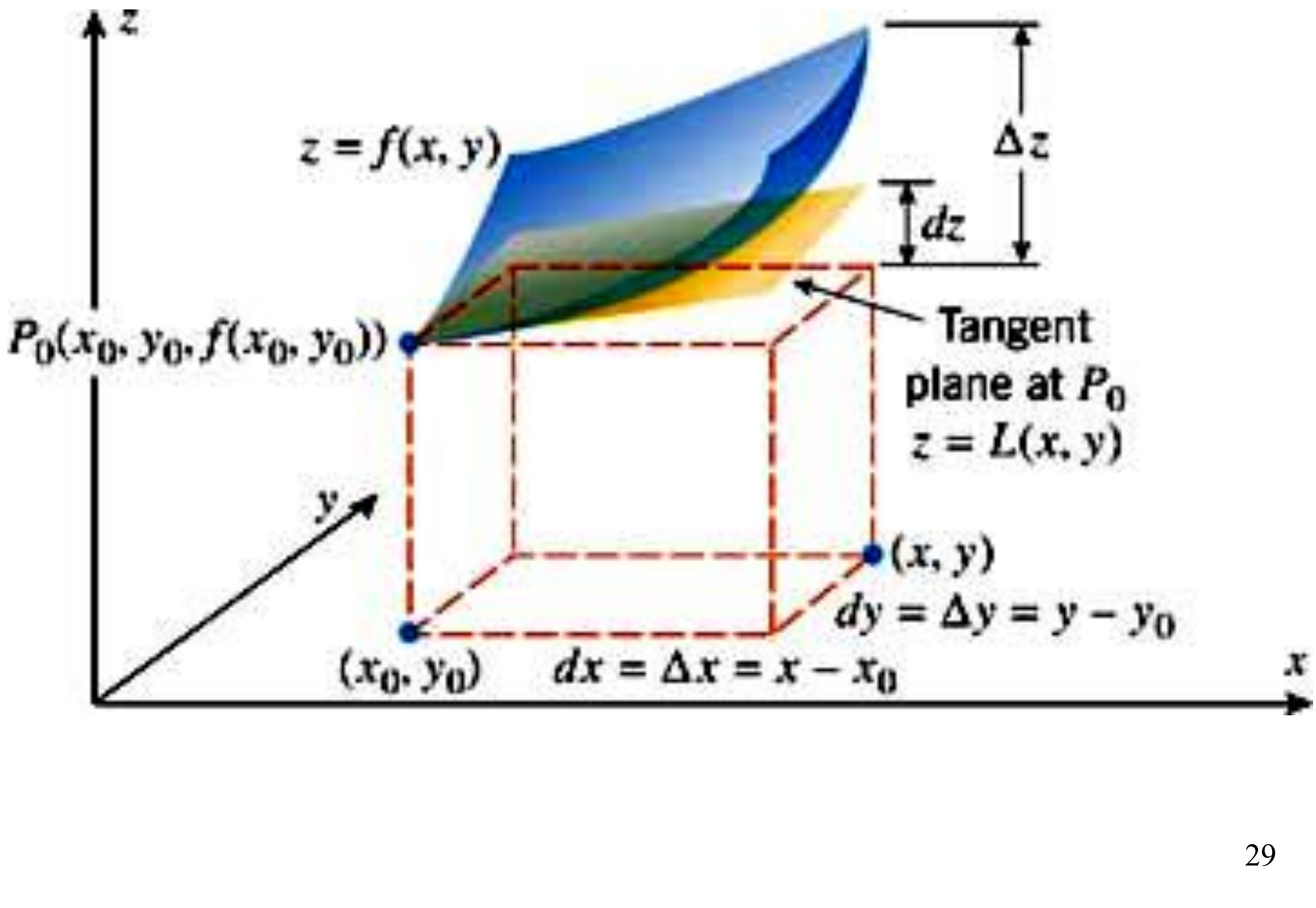


FIGURE 12.26c
Tangent plane and normal vector



Example 2.9

Let $f(x, y) = 2x^3 + xy - y^3$. Compute Δz and dz as (x, y) changes from (2, 1) to (2.03, 0.98).

Solution

$$\begin{aligned}\Delta z &= f(2.03, 0.98) - f(2, 1)\\ &= 2(2.03)^3 + (2.03)(0.98) - (0.98)^3\\ &\qquad - [2(2)^2 + 2(1) - 1^3]\\ &= 0.779062\end{aligned}$$

$$\begin{aligned}dz &= f_x(x, y)dx + f_y(x, y)dy\\ &= (6x^2 + y)\Delta x + (x - 3y^2)\Delta y\end{aligned}$$

At (2, 1) with $\Delta x = 0.03$ and $\Delta y = -0.02$,

$$dz = (25)(0.03) + (-1)(-0.02) = 0.77$$

Example 2.10

A cylindrical tank is 4 ft high and has a diameter of 2 ft. The walls of the tank are 0.2 in. thick. Approximate the volume of the interior of the tank assuming that the tank has a top and a bottom that are both also 0.2 in. thick.

Solution

WANT: interior volume of tank, V

KNOW: radius, $r = 12$ in., height, $h = 48$ in.

$$\Delta V \approx dV = V_r dr + V_h dh,$$

$$dr = -0.2 = dh$$

Volume of tank, $V = \pi r^2 h$

$$\Rightarrow V_r = 2\pi r h \text{ and } V_h = \pi r^2$$

$$\Delta V \approx V_r dr + V_h dh = (2\pi r h)dr + (\pi r^2)dh$$

Since $r = 12$ in., $h = 48$ in., and $dr = -0.2 = dh$ we have,

$$\begin{aligned} \Delta V &\approx 2\pi(12)(48)(-0.2) + \pi(12)^2(-0.2) \\ &\approx -814.3 \text{ in}^3 \end{aligned}$$

Thus the interior volume of the tank is

$$V \approx \pi(12)^2(48) - 814.3 \approx 20{,}900.4 \text{ in}^3$$

Example 2.11

Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in. but that the radius and height are off by the amounts $dr = 0.03$ and $dh = -0.1$. Estimate the resulting absolute, relative and percentage changes in the volume of the can.

Solution

WANT: Absolute change, $\Delta V \approx dV$

Relative change, $\dfrac{\Delta V}{V} \approx \dfrac{dV}{V}$

Percentage change, $\dfrac{dV}{V} \times 100$

Absolute change,

$$dV = V_r dr + V_h dh = 2\pi r h dr + \pi r^2 dh$$

$$= 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) = 0.2\pi$$

Relative change,

$$\frac{dV}{V} = \frac{0.2\pi}{\pi r^2 h} = \frac{0.2\pi}{\pi(1)^2(5)} = 0.04$$

Percentage change,

$$\frac{dV}{V} \times 100 = 0.04 \times 100 = 4\%$$

Example 2.12

1. The dimensions of a rectangular block of wood were found to be 100 mm, 120 mm and 200 mm, with a possible error of 5 mm in each measurement. Find approximately the greatest error in the surface area of the block and the percentage error in the area caused by the errors in the individual measurements.

2. The pressure P of a confined gas of volume V and temperature T is given by the formula $P = k\left(\frac{T}{V}\right)$ where k is a constant. Find approximately, the maximum percentage error in P introduced by an error of $\pm$ 0.4% in measuring the temperature and an error of $\pm$ 0.9% in measuring the volume.

Example 2.13

The radius and height of a right circular cone are measured with errors of at most 3% and 2% respectively. Use differentials to estimate the maximum percentage error in computing the volume.

2.3 Chain Rule

2.3.1 Partial Derivatives of Composite Functions

Recall: The chain rule for composite functions of one variable

If y is a differentiable function of x and x is a differentiable function of a parameter t, then the **chain rule** states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

- The corresponding rule for two variables is essentially the same except that it involves both variables.

Note

The rule is used to calculate the rate of increase (positive or negative) of composite functions with respect to t.

Assume that $z = f(x, y)$ is a function of x and y and suppose that x and y are in turn functions of a single variable t,

$$x = x(t), \quad y = y(t)$$

Then $z = f(x(t), y(t))$ is a composition function of a parameter t.

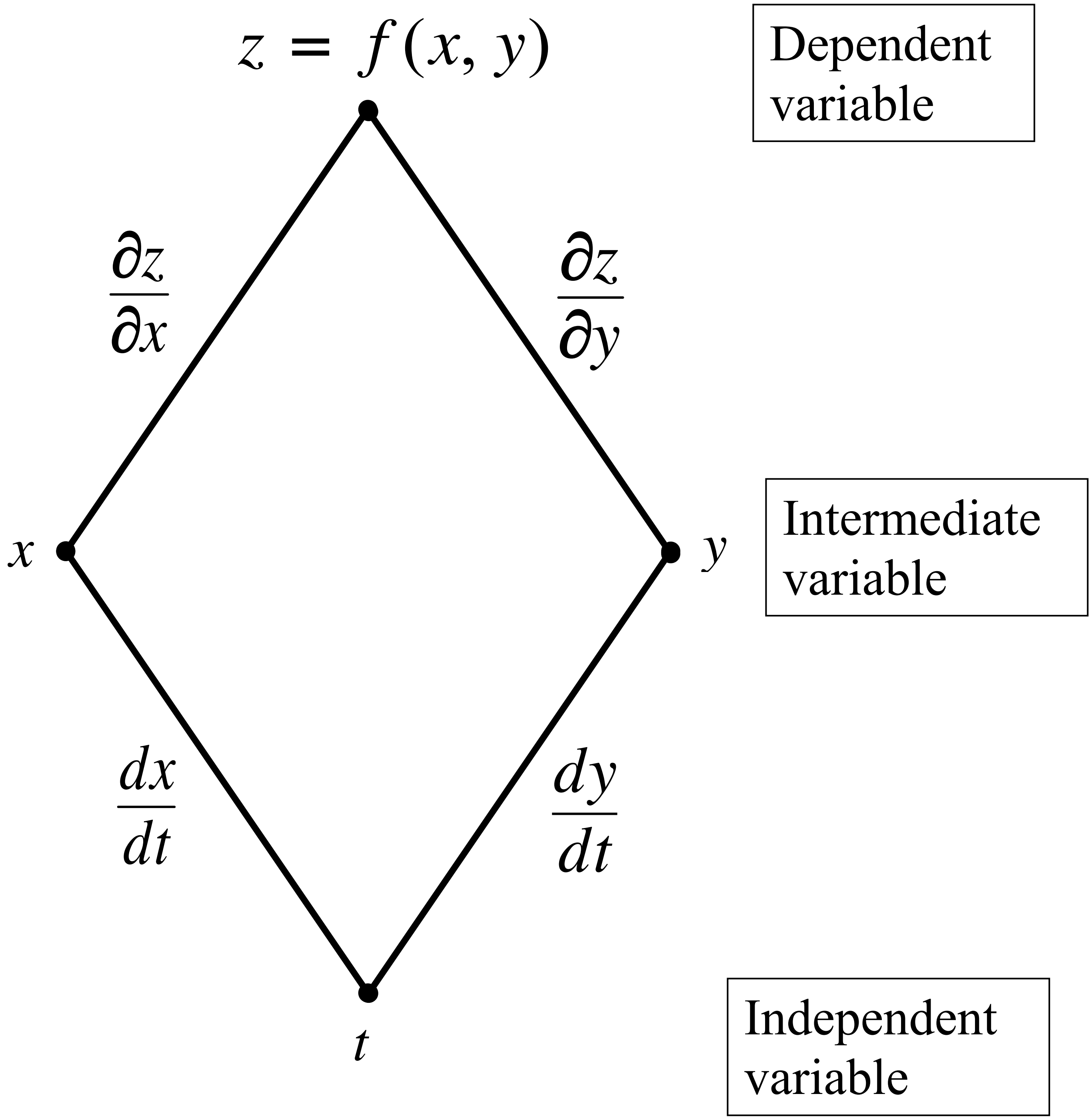
Thus we can calculate the derivative $\dfrac{dz}{dt}$ and its relationship to the derivatives $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ is given by the following theorem.

Theorem 2.1

If $z = f(x, y)$ is differentiable and x and y are differentiable functions of t, then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Chain Rule – one parameter

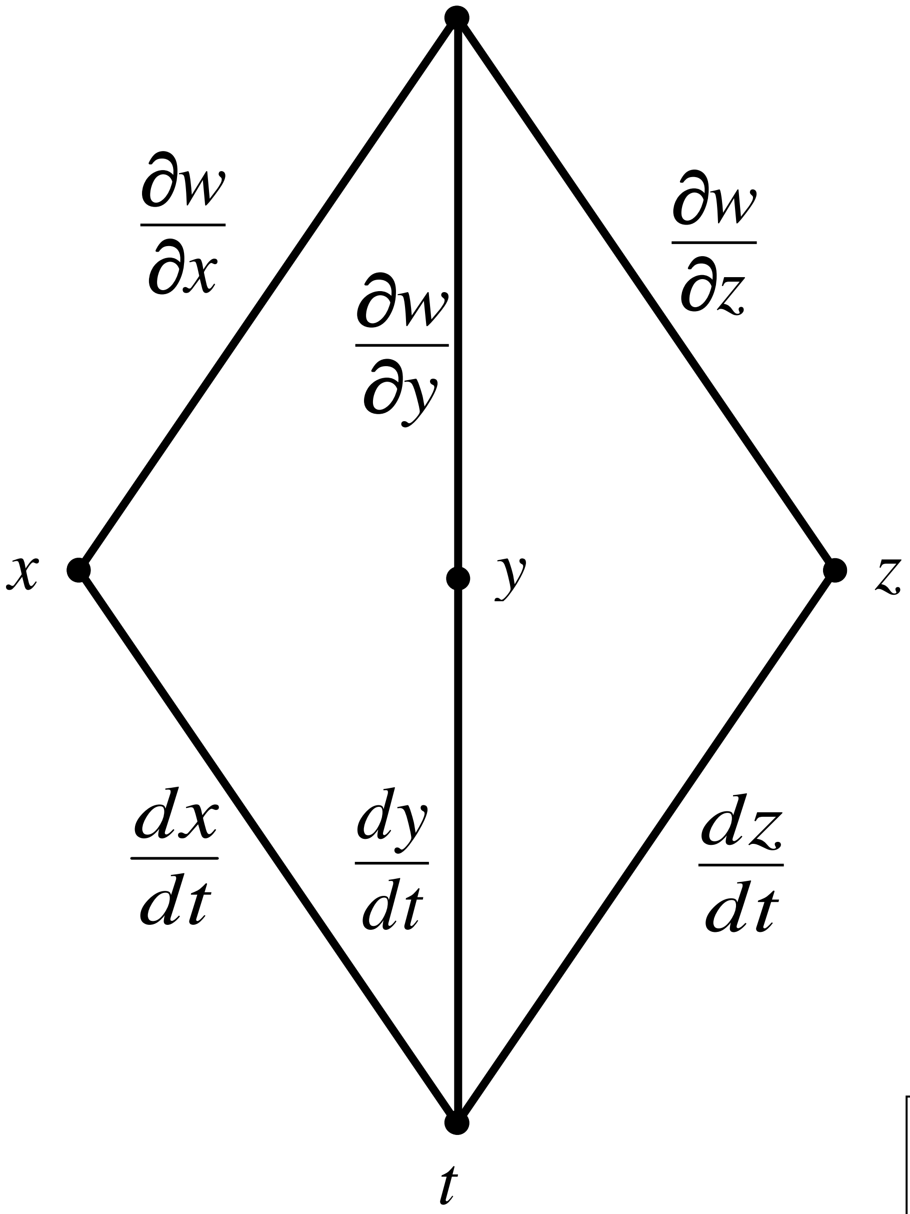


$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Chain Rule – one parameter

$$w = f(x, y, z)$$

Dependent variable



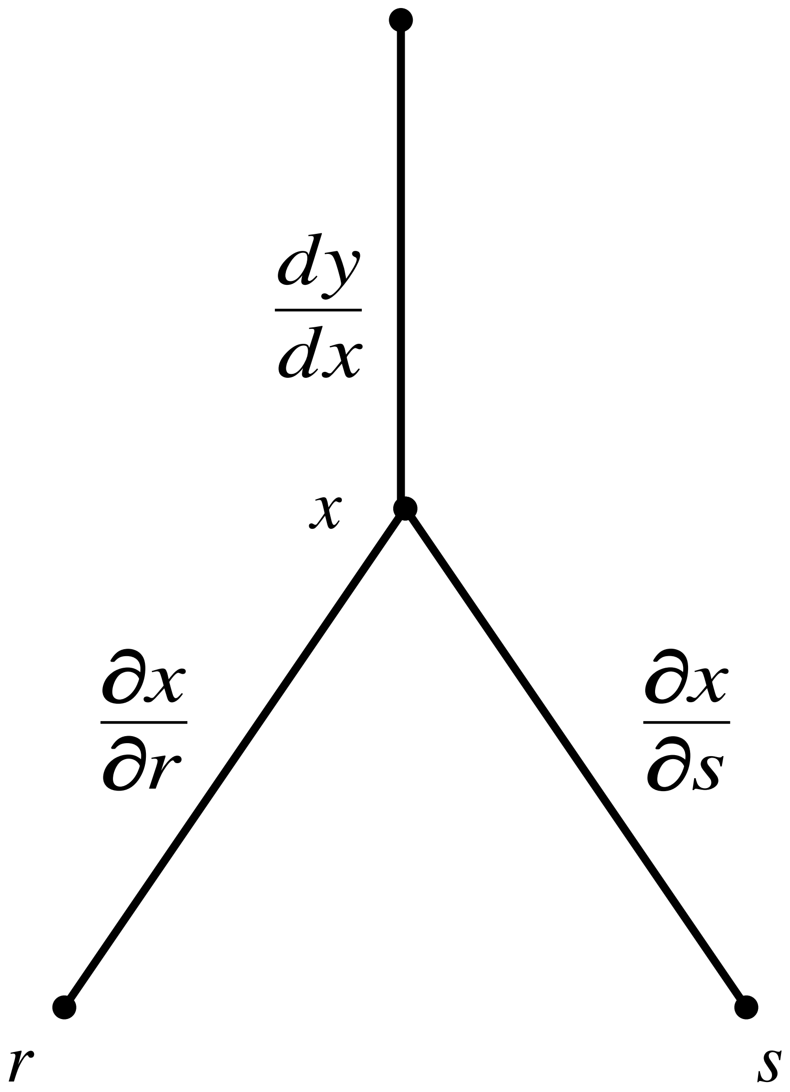
Intermediate variable

Independent variable

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

Chain Rule – two parameters

$$y = f(x)$$



$$\frac{\partial y}{\partial r} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial r}, \quad \frac{\partial y}{\partial s} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial s}$$

Theorem 2.2

Let $x = x(r, s)$ and $y = y(r, s)$ have partial derivatives at r and s and let $z = f(x, y)$ be differentiable at (x, y). Then $z = f(x(r, s), y(r, s))$ has first derivatives given by

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Example 2.14

Suppose that $z = x^3 y$ where $x = 2t$ and $y = t^2$. Find $\dfrac{dz}{dt}$.

Solution

WANT: $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$

$$z = x^3 y \quad \Rightarrow \quad \frac{\partial z}{\partial x} = 3x^2 y \text{ and } \frac{\partial z}{\partial y} = x^3$$

$$x = 2t \quad \Rightarrow \quad \frac{dx}{dt} = 2$$

$$y = t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 2t$$

Hence, $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$

$$= (3x^2 y)(2) + (x^3)(2t)$$

$$= 6(2t)^2(t^2) + (2t)^3(2t) = 40t^4$$

Example 2.15

Suppose that $z = \sqrt{xy + y}$ where $x = \cos\theta$ and $y = \sin\theta$. Find $\dfrac{dz}{d\theta}$ when $\theta = \dfrac{\pi}{2}$

Solution

WANT: $\left.\dfrac{dz}{d\theta}\right|_{\theta=\pi/2}$

From the chain rule with θ in place of t,

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\cdot\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\cdot\frac{dy}{d\theta}$$

we obtain

$$\frac{dz}{d\theta} = \frac{1}{2}(xy + y)^{-1/2}(y)(-\sin\theta)$$

$$+\frac{1}{2}(xy + y)^{-1/2}(x + 1)(\cos\theta)$$

When $\theta = \dfrac{\pi}{2}$, we have

$$x = \cos\frac{\pi}{2} = 0 \text{ and } y = \sin\frac{\pi}{2} = 1$$

Substituting $x = 0$, $y = 1$, $\theta = \dfrac{\pi}{2}$ in the formula for $\dfrac{dz}{dt}$ yields

$$\left.\frac{dz}{d\theta}\right|_{\theta=\pi/2} = \frac{1}{2}(1)(1)(-1) + \frac{1}{2}(1)(1)(0) = -\frac{1}{2}$$

Example 2.16

Let $z = 4x - y^2$ where $x = uv^2$ and $y = u^3v$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Example 2.16a

Suppose that $w = xy + yz$ where $y = \sin x$ and $z = e^x$. Use an appropriate form of the chain rule to find $\dfrac{dw}{dx}$.

Example 2.17

Find $\dfrac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$ where

$x = e^{rs^2}$, $y = \ln \dfrac{r+s}{t}$ and $z = rst^2$.

2.3.2 Partial Derivatives of Implicit Functions

The chain rule can be applied to implicit relationships of the form $F(x, y) = 0$.

Differentiating $F(x, y) = 0$ with respect to x gives

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

In other words, $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{dx} = 0$

Hence, $$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

In summary, we have the following results.

Theorem 2.3

If $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

Theorem 2.3 has a natural extension to functions $z = f(x, y)$, of two variables.

Theorem 2.4

If $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Example 2.18

If y is a differentiable function of x such that

$$x^3 + 4x^2 y - 3xy + y^2 = 0$$

find $\dfrac{dy}{dx}$.

Solution

KNOW: $\dfrac{dy}{dx} = \dfrac{-F_x}{F_y}$

Let $F(x, y) = x^3 + 4x^2y - 3xy + y^2$. Then

$$F_x = 3x^2 + 8xy - 3y$$

and $\quad F_y = 4x^2 - 3x + 2y$

$$\therefore \quad \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(3x^2 + 8xy - 3y)}{4x^2 - 3x + 2y}$$

Alternatively, differentiating the given function implicitly yields

$$3x^2 + \left(8xy + 4x^2\frac{dy}{dx}\right) - \left(3y + 3x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(3x^2 + 8xy - 3y)}{4x^2 - 3x + 2y}$$

which agrees with the result obtained by Theorem 2.3.

Example 2.19a

If $\sin(x + y) + \cos(x - y) = y$ determine $\dfrac{dy}{dx}$.

Example 2.19b

If $z^2xy + zy^2x + x^2 + y^2 = 5$ determine expressions for $\dfrac{\partial z}{dx}$ and $\dfrac{\partial z}{dy}$.

2.5 Local Extrema

Focus of Attention

- What is the relative extremum of a function of two variables?
- What does a saddle point mean?
- What is a critical point of a function of two variables?
- What derivative tests could be used to determine the nature of critical points?

In this section we will see how to use partial derivatives to locate maxima and minima of functions of two variables.

First we will start out by formally defining local maximum and minimum:

Definition 2.5

A function of two variables has a **local maximum** at (a, b) if $f(x,y) \le f(a,b)$ when (x, y) is near (a, b). The number $f(a,b)$ is called a **local maximum value**.

If $f(x,y) \ge f(a,b)$ when (x, y) is near (a,b), then $f(a,b)$ is a **local minimum value**.

Note

- The points (x, y) is in some disk with center (a, b).
- Collectively, local maximum and minimum are called **local extremum**.
- Local extremum is also known as **relative extremum**.

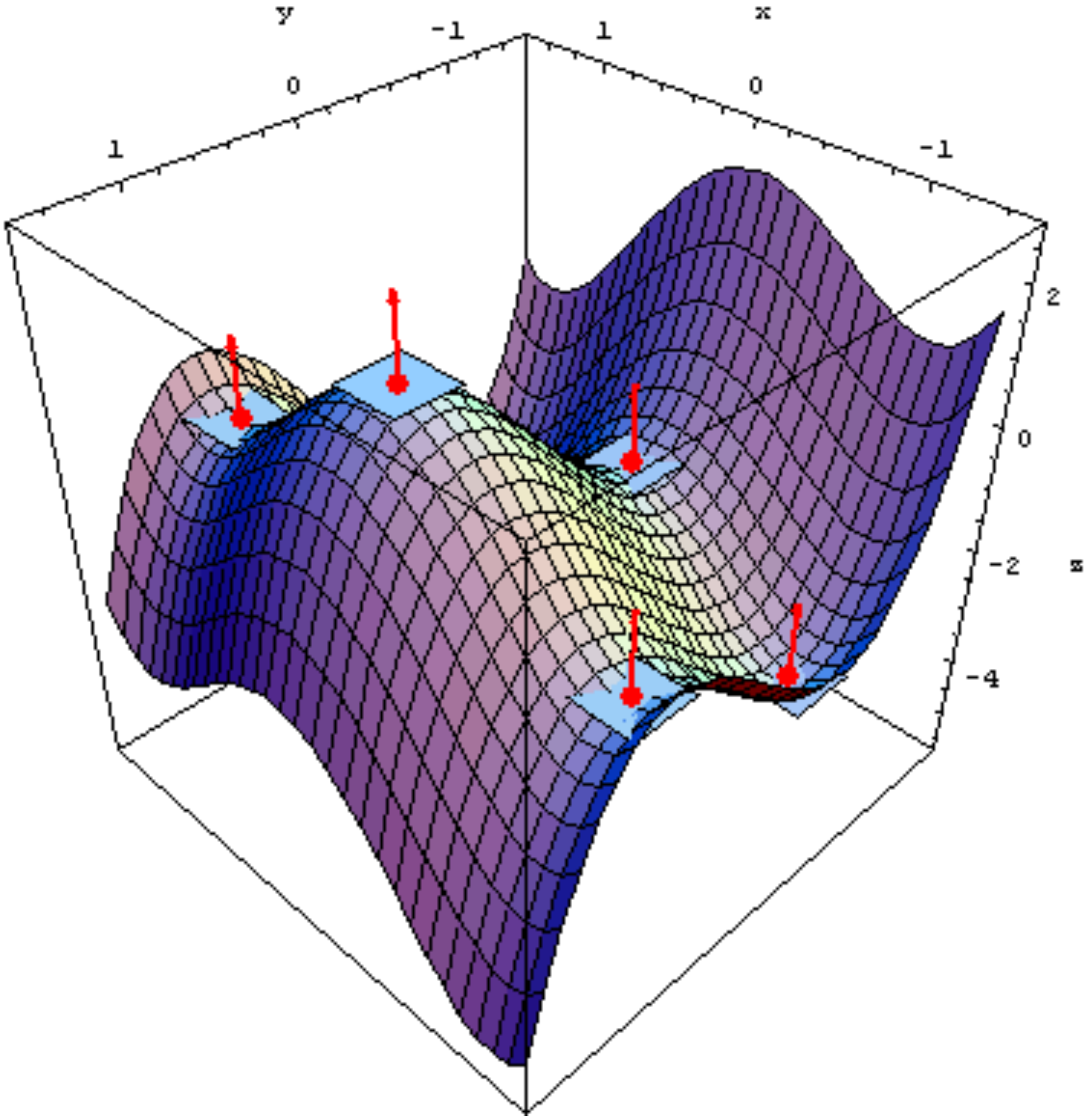
The process for finding the maxima and minima points is similar to the one variable process, just set the derivative equal to zero. However, using two variables, one needs to use a system of equations. This process is given below in the following theorem:

Theorem 2.5

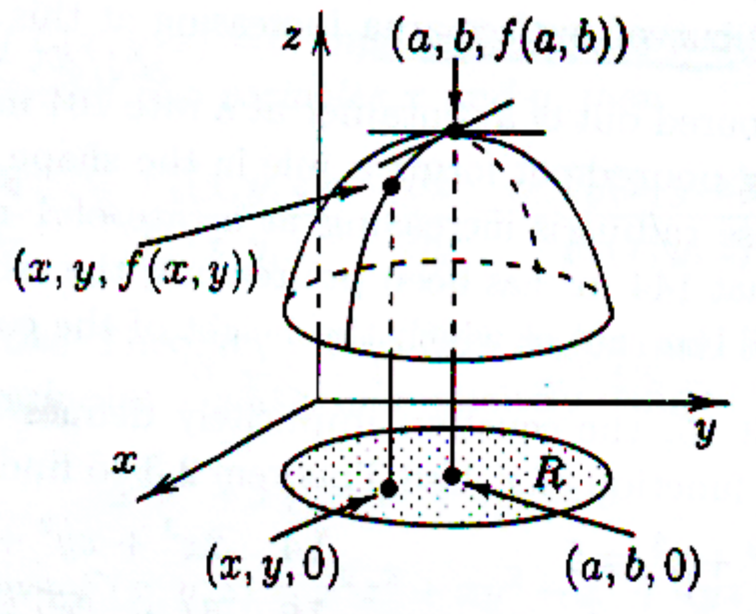
If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist at this point, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Definition 2.6

A point (a, b) is called a **critical point** of the function $z = f(x,y)$ if $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or if one or both partial derivatives do not exist at (a, b).

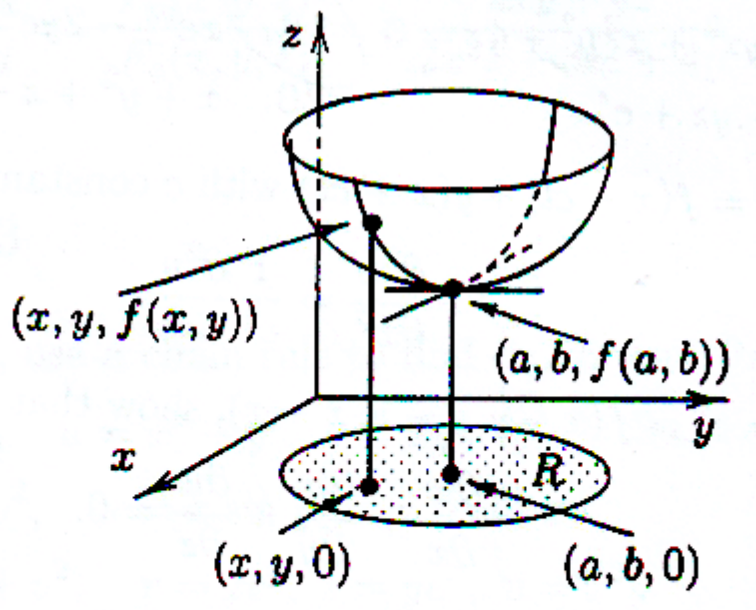


Relative Max



Point $(a, b, f(a, b))$ is a local maximum

Relative Min.

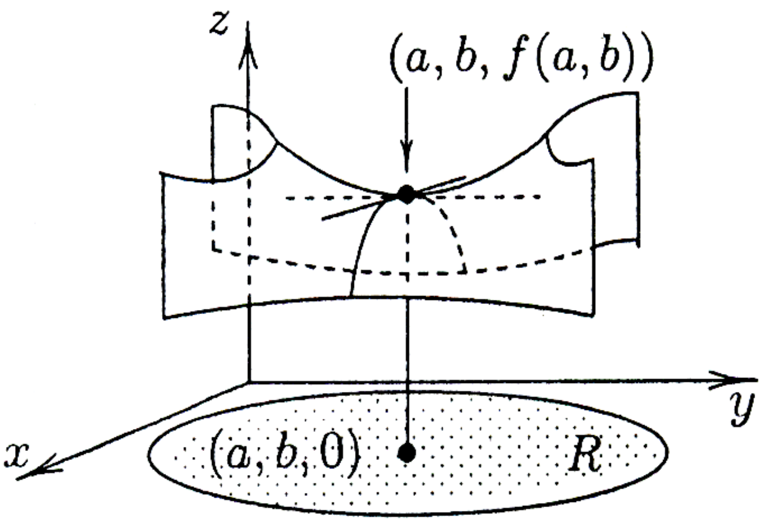


Point $(a, b, f(a, b))$ is a local minimum

Saddle Point

Point $(a, b, f(a, b))$ is a local minimum

Saddle Point



Point $(a, b, f(a, b))$ is a saddle point

Remark

The values of z at the local maxima and local minima of the function $z = f(x,y)$ may also be called the extreme values of the function, $f(x,y)$.

Theorem 2 : Second-Partials Test

Let $f(x,y)$ have a critical point at (*a*, *b*) and assume that *f* has continuous second-order partial derivatives in a disk centered at (*a*, *b*). Let

$$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(i) If *D* > 0 and $f_{xx}(a,b) > 0$, then *f* has a local minimum at (*a*, *b*).

(ii) If *D* > 0 and $f_{xx}(a,b) < 0$, then *f* has a local maximum at (*a*, *b*).

(iii) If *D* < 0, then *f* has a saddle point at (*a*, *b*).

(iv) If *D* = 0, then no conclusion can be drawn.

Remark

The expression $f_{xx}f_{yy} - {f_{xy}}^2$ is called the **discriminant** or **Hessian** of *f*. It is sometimes easier to remember it in the determinant form,

$$f_{xx}f_{yy} - {f_{xy}}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

If the discriminant is positive at the point (*a*, *b*), then the surface curves the same way in all directions:

- downwards if $f_{xx} < 0$, giving rise to a local maximum
- upwards if $f_{xx}(a,b) > 0$, giving a local minimum.

If the discriminant is negative at (*a*, *b*), then the surface curves up in some directions and down in others, so we have a saddle point.

Example 2.34

Locate all local extrema and saddle points of $f(x,y) = 1 - x^2 - y^2$.

Solution

- First determine f_x and f_y:

$$f_x(x,y) = -2x \text{ and } f_y(x,y) = -2y.$$

- Secondly, solve the equations, $f_x = 0$ and $f_y = 0$ for x and y:

$$-2x = 0 \quad \text{and} \quad -2y = 0$$

So the only critical point is at (0, 0).

- Thirdly, evaluate f_{xx} , f_{yy} and f_{xy} at the critical point.

$$f_{xx}(x,y) = -2,\ f_{xy}(x,y) = 0 \text{ and } f_{yy}(x,y) = -2$$

At the point (0, 0),

$$f_{xx}(0,0) = -2, \quad f_{xy}(0,0) = 0 \quad \text{and}$$
$$f_{yy}(0,0) = -2$$

- Compute D:

$$D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

Since $D = 4 > 0$ and $f_{xx}(0, 0) = -2 < 0$, the second partials test tell us that a local maximum occurs at (0, 0).

In other words, the point (0, 0, 1) is a local maximum, with f having a corresponding maximum value of 1.

Example 2.35

Locate all local extrema and saddle points of $f(x, y) = 8x^3 - 24xy + y^3$.

Prompts/Questions

- What are the critical points?
  - How are they calculated?
- How do you classify these points?
  - Can you use the Second Derivative Test?

Solution

$$f_x = 24x^2 - 24y, \quad f_y = -24x + 3y^2$$

♦ Find the critical points, solve

$$24x^2 - 24y = 0 \qquad (1)$$

$$-24x + 3y^2 = 0 \qquad (2)$$

From Eqn. (1), $y = x^2$. Substitute this into Eqn. (2) to find

$$-24x + 3(x^2)^2 = 0$$

$$x = 0,2$$

If $x = 0$, then $y = 0$

If $x = 2$, then $y = 4$

So the critical points are (0, 0), (2, 4).

- Find f_{xx} , f_{yy} and f_{xy} and compute D:

  $f_{xx}(x,y) = 48x$, $f_{xy}(x,y) = -24$ and $f_{yy}(x,y) = 6y$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 48x & -24 \\ -24 & 6y \end{vmatrix} = 288xy - 576$$

- Evaluate D at the critical points:

  At (0, 0), $D = -576 < 0$, so there is a saddle point at (0, 0).

  At (2, 4), $D = 288(2)(4)-576 = 1728 > 0$

and $f_{xx}(2, 4) = 48(2) = 96 > 0$. So there is a local minimum at (2, 4).

Thus f has a saddle point (0, 0, 0) and local minimum (2, 4, –64).

Example 2.36

Find the local extreme values of the function.

(i) $f(x, y) = x^2 y^4$

(ii) $h(x, y) = x^3 + y^3$

Prompts/Questions

- What are the critical points?
- Can you use the second partials test?
  - What do you do when the test fails?
- How does the function behave near the critical points?

Solution

(i) The partial derivatives of f are $f_x = 2xy^4$, $f_y = 4x^2y^3$.

Solving $f_x = 0$ and $f_y = 0$ simultaneously, we note that the critical points occurs whenever $x=0$ or $y=0$. That is every point on the $x-$ or $y-$ axis is a critical point.

So, the critical points are (x,o) and $(0,y)$.

Using the Second Derivative Test:

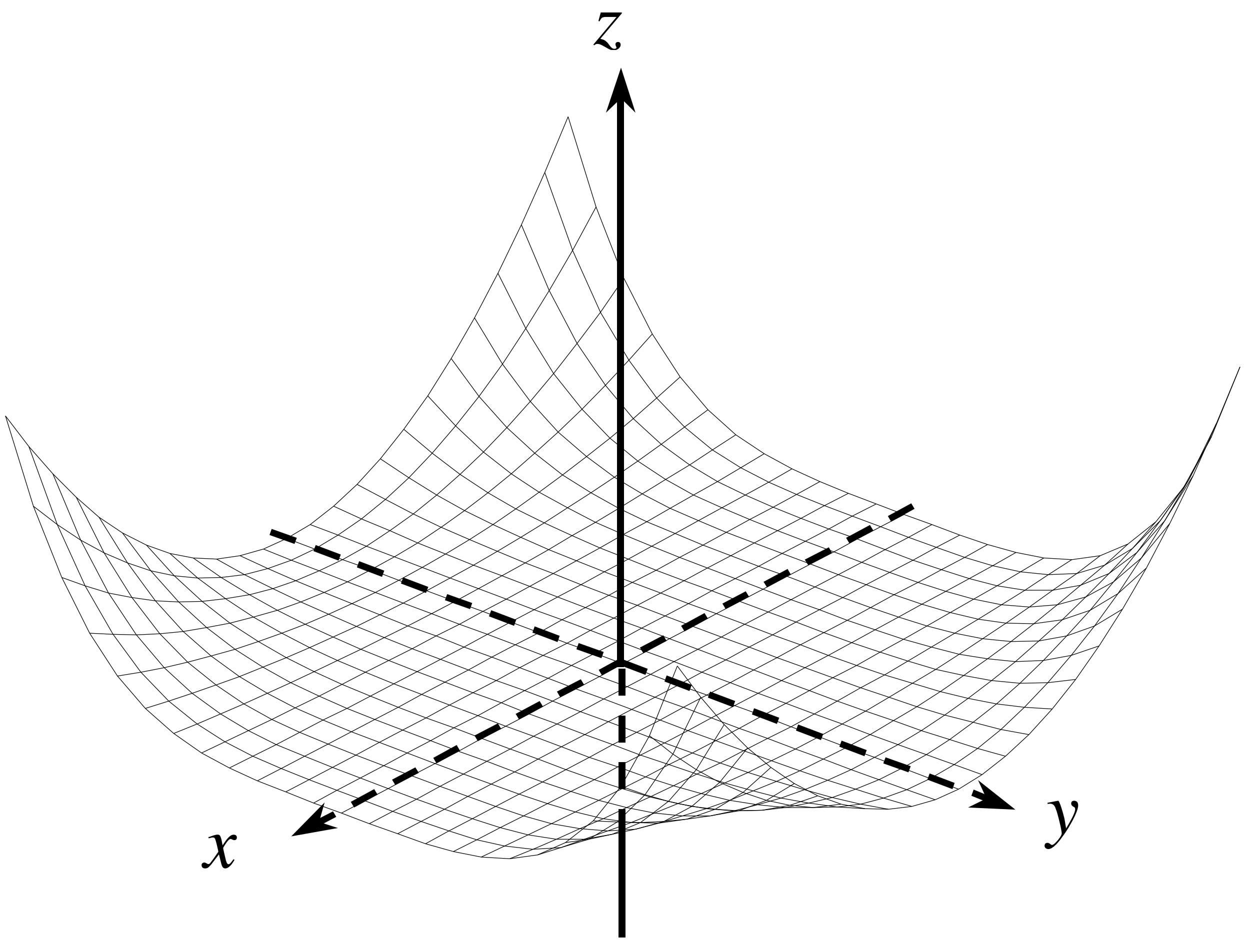
$$D = \begin{vmatrix} 2y^4 & 8xy^3 \\ 8xy^3 & 12x^2y^2 \end{vmatrix} = 24x^2y^6 - 64x^2y^6$$

$$= -40x^2y^6$$

For any critical point $(x_0,0)$ or $(0,y_0)$, the second partials test fails.

Let's analyse the function. Observed that $f(x,y)=0$ for every critical point (either $x=0$ or $y=0$ or both. Since $f(x,y)=x^2y^4>0$ when $x\neq 0$ and $y\neq 0$, it follows that each critical point must be a local minimum.

The graph of f is shown below.

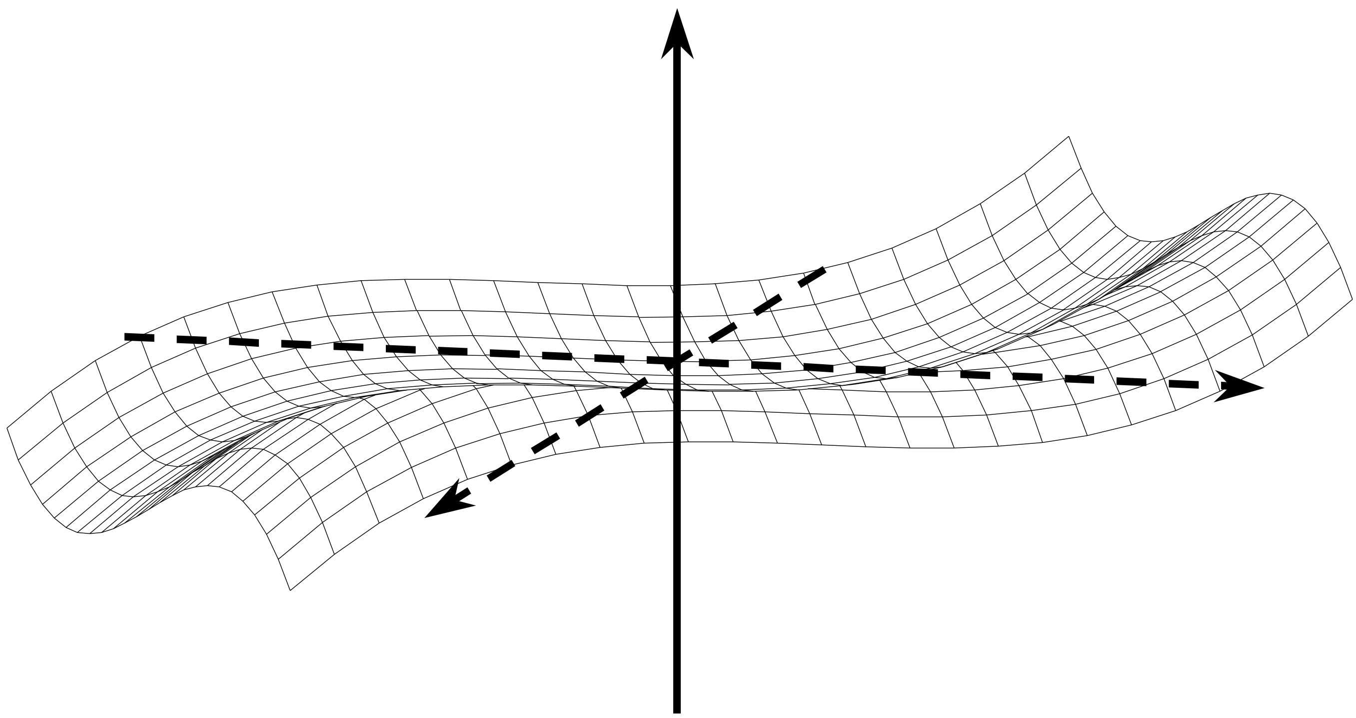


Graph of $f(x,y)=x^2y^4$

(ii) $h_x(x,y) = 3x^2, \quad h_y(x,y) = 3y^2$. Solving the equations $h_x = 0$ and $h_y = 0$ simultaneously, we obtain (0, 0) as the only critical point.

The second partials test fails here. Why?

Let us examine the traces on the coordinate planes… *finish it off*



Graph of $h(x,y) = x^3 + y^3$

$h(x,y)$ has neither kind of local extremum nor a saddle point at (0, 0).

Question

In equations 1 - 3 , find critical points of $f(x, y)$ and determine whether $f(x, y)$ at that point is a local maximum or a local minimum, or the value of the saddle point.

1. $f \;\; x, y \;\; = 3 - x^2 - y^2 + 6y$
2. $f \;\; x, y \;\; = x^2 + y^2 - 4x + 6y + 23$
3. $f \;\; x, y \;\; = x^2 - y^2 + 2x + 6y - 4$

2.6 Absolute Extrema

Focus of Attention

- Where can absolute extreme values of $f(x, y)$ occur?
- Under what circumstances does a function of two variables have both an absolute maximum and an absolute minimum?
- What is the procedure for determining absolute extrema?

The only places a function $f(x, y)$ can ever have an absolute extremum value are

- interior critical points
- boundary points of the function's domain

Theorem 2.12 Extreme-Value Theorem

If $f(x, y)$ is continuous on a closed bounded region *R*, then *f* has an **absolute extremum** on *R*.

Theorem 2.13

If $f(x, y)$ has an absolute extremum at an interior point of its domain, then this extremum occurs at a critical point.

Note

Absolute extremum is also known as **global extremum**.

Finding Absolute Extrema

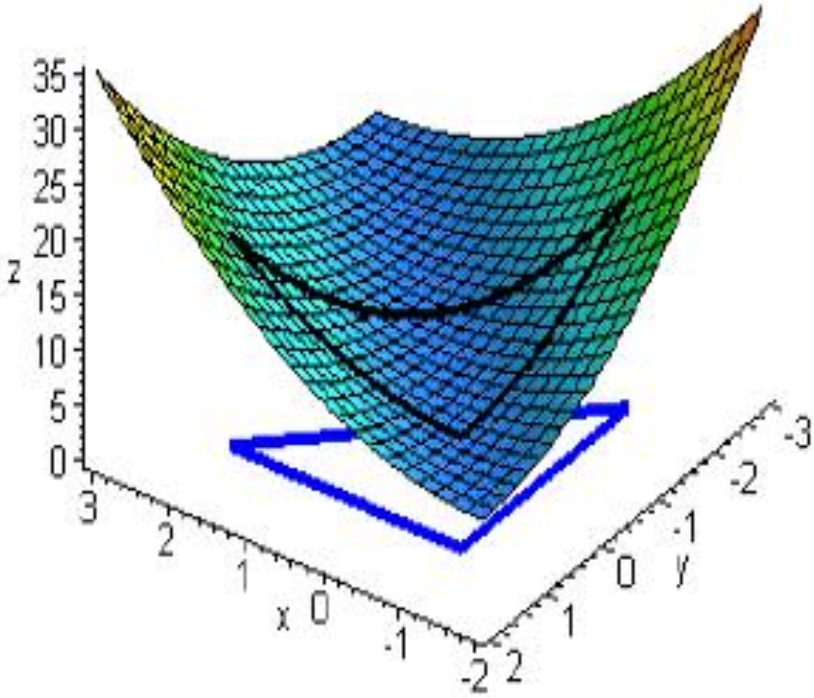
Given a function f that is continuous on a closed, bounded region R:

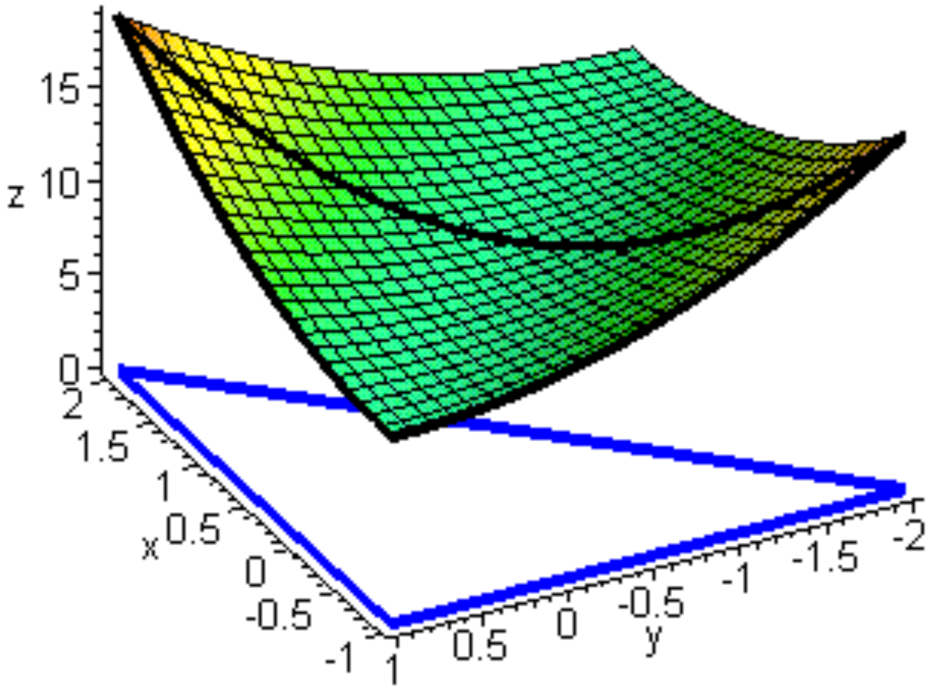
Step 1: Find all critical points of f in the interior of R.

Step 2: Find all boundary points at which the absolute extrema can occur (critical points, endpoints, etc.)

Step 3: Evaluate $f(x, y)$ at the points obtained in the preceding steps.

The largest of these values is the absolute maximum and the smallest is the absolute minimum.





Illustration

Finding absolute extrema on closed and bounded region

- Critical points & boundary points
- Absolute extreme values – smallest & largest values

Example 2.37

Find the absolute extrema of the function $f(x,y) = \sqrt{x^2 + y^2}$ over the disk $x^2 + y^2 \leq 1$.

Prompts/Questions

- Where can absolute extreme occur?
  - What are the critical points?
  - What are the boundary points?
- How do you decide there is an absolute minimum? Absolute maximum?

Solution

Step 1: $f_x = \dfrac{x}{\sqrt{x^2 + y^2}}, \; f_y = \dfrac{y}{\sqrt{x^2 + y^2}}$

$f_x \neq 0$ and $f_y \neq 0$ for all (x, y). But f_x and f_y do not exist at (0, 0). Thus (0, 0) is the only critical point of *f* and it is inside the region.

Step 2: Examine the values of f on the boundary curve $x^2 + y^2 = 1$. Because $y^2 = 1 - x^2$ on the boundary curve, we find that

$$f(x, y) = \sqrt{x^2 + (1 - x^2)} = 1$$

That is, for every point on the boundary circle, the value of f is 1.

Step 3: Evaluating the value of f at each of the points we have found:

Critical point: $f(0, 0) = 0$

Boundary points: $f(x, y) = 1$

We conclude that the absolute minimum value of f on R is 0 and the absolute maximum value is 1.

Example 2.38

Find the absolute extrema of the function $f(x,y)=3xy-6x-3y+7$ on the closed triangular region in the first quadrant bounded by the lines $x=0$, $y=0$,

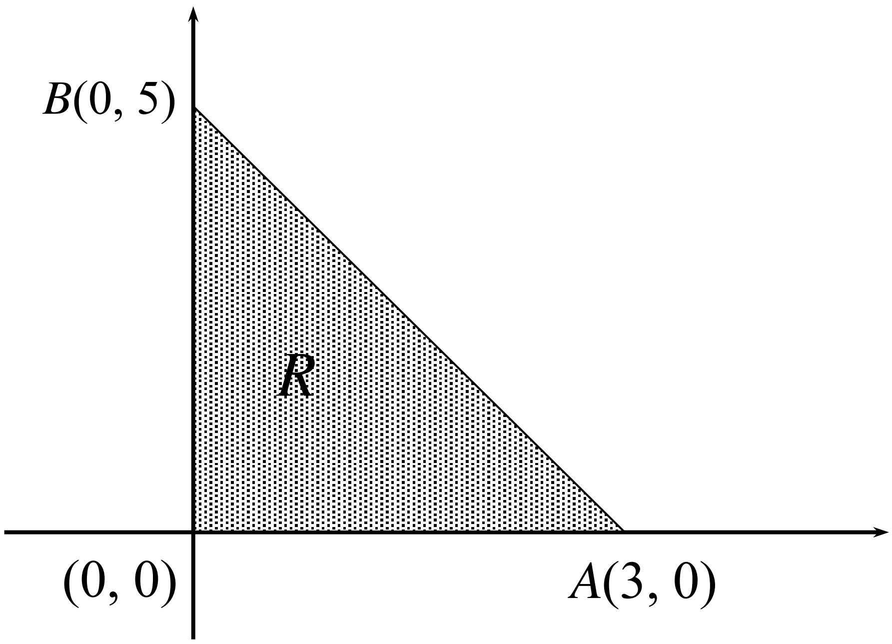
$$y=-\frac{5}{3}x+5.$$

Prompts/Questions

- Where can absolute extreme occur?
  - Can you find the points?
- How do you determine the absolute maximum? Absolute minimum?

Solution

The region is shown in the figure.



Critical points:

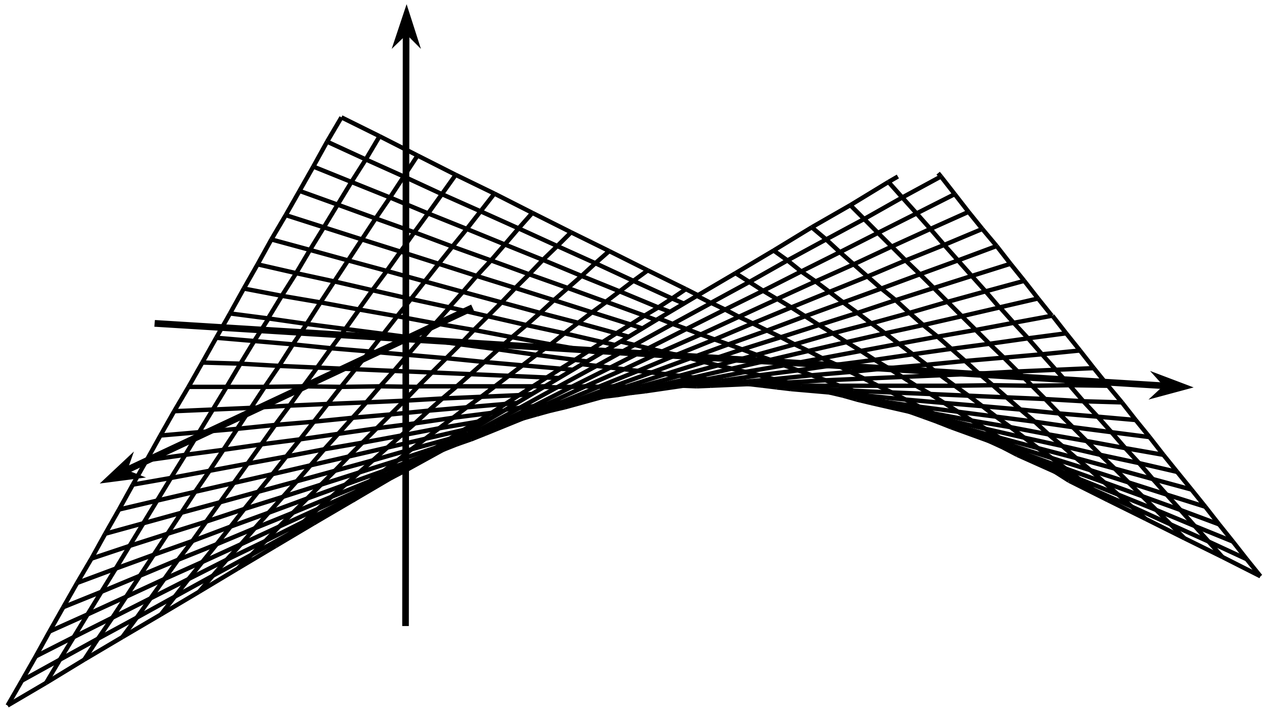
$$f_x = 3y - 6 = 0, \quad f_y = 3x - 3 = 0$$

(1, 2) is the only critical point in the interior of *R*.

Boundary points:

The boundary of *R* consists of three line segments. We take one side at a time.

Graph of $f(x, y) = 3xy - 6x - 3y + 7$



- On the segment OA, $y = 0$.

The function $f(x, y)$ simplifies to a function of single variable x

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \le x \le 3$$

This function has no critical numbers because $u'(x) = -6$ is nonzero for all x. Thus the extreme values occur at the endpoints (0, 0) and (3, 0) of R.

- On the segment OB, $x = 0$.

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical numbers because $v'(y) = -3$ is nonzero for all y. Thus the extreme values occur at the endpoints (0, 0) and (0, 5) of R.

- Segment AB: we already accounted the endpoints of AB, so we look at the interior points of AB.

With $y = -\frac{5}{3}x + 5$, we have

$$w(x) = 3x\left(-\frac{5}{3}x + 5\right) - 6x - 3\left(-\frac{5}{3}x + 5\right) + 7$$

$$= -5x^2 + 14x - 8, \quad 0 \le x \le 3$$

Setting $w'(x) = -10x + 14 = 0$ gives $x = 7/5$. The critical number is $(7/5, 8/3)$.

Evaluating the value of f for the points we have found:

$(0, 0)$	$f(0, 0) = 7$
$(3, 0)$	$f(3, 0) = -11$
$(0, 5)$	$f(0, 5) = -8$
$(7/5, 8/3)$	$f(7/5, 8/3) = 9/5$
$(1, 2)$	$f(1, 2) = 1$

We conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$.

Example 2.39

Find the shortest distance from the point (0, 3, 4) to the plane $x + 2y + z = 5$.

Solution

KNOW: the distance from a point (x, y, z) to (0, 3, 4) is

$$d = \sqrt{(x-0)^2 + (y-3)^2 + (z-4)^2}$$

WANT: to minimise d

Let (x, y, z) be a point on the plane $x + 2y + z = 5$. We know

$$z = 5 - x - 2y$$

So $d = \sqrt{x^2 + (y-3)^2 + (5 - x - 2y - 4)^2}$

Instead of d, we can minimize the expression

$$d^2 = f(x, y) = x^2 + (y-3)^2 + (1 - x - 2y)^2$$

Find the critical values:

$$f_x = 2x - 2(1 - x - 2y) = 4x + 4y - 2 = 0$$

$$f_y = 2(y - 3) - 4(1 - x - 2y) = 4x + 10y - 10 = 0$$

The only critical point is (–5/6, 4/3). Also $f_{xx} = 4,\ f_{yy} = 10,\ f_{xy} = 4$, so $D > 0$ which means there is a local minimum at (–5/6, 4/3).

This local minimum must also be the absolute minimum because there must be only one point on the plane that is closest to the given point. The shortest distance is,

$$d = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{4}{3} - 3\right)^2 + \left[1 + \frac{5}{6} - 2\left(\frac{4}{3}\right)\right]^2} = \frac{5}{\sqrt{6}}$$

Note

In general it can be difficult to show that a local extremum is also an absolute extremum. In practice, the determination is made using physical or geometrical considerations.

Example 2.40

Suppose we wish to construct a rectangular box with volume 32 ft^3. Three different materials will be used in the construction. The material for the sides cost RM1 per square foot, the material for the bottom costs RM3 per square foot, and the material for the top costs RM5 per square foot. What are the dimensions of the least expensive such box?

Reflection

- Where do absolute extreme values of $f(x, y)$ occur?

………………………………………………………

………………………………………………………

- What are the conditions that guarantee a $f(x, y)$ has an absolute maximum and an absolute minimum?

………………………………………………………

………………………………………………………

- How do you find the absolute maximum or minimum value of a function on a closed and bounded domain? On an open or unbounded region?

. . . .

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Lagrange Multipliers

We want to optimize (*i.e.* find the minimum and maximum value of) a function,

$$f(x, y, z)$$

subject to the constraint

$$g(x, y, z) = k$$

The constraint may be the equation that describes the boundary of a region or it may not be.

Method of Lagrange Multipliers

STEP 1: Solve the following system of equation

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$

Where we have to solve these 4 equations simultaneously:

$$\langle f_x, f_y, f_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle \qquad g(x, y, z) = k,$$

or

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x, y, z) &= k \end{aligned}$$

STEP 2: Plug in all solutions, (x, y, z) from the first step into $f(x, y, z)$ and identify the minimum and maximum values, provided they exist.

The constant, λ, is called the **Lagrange Multiplier**.

Example 1: Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution: We want to find the largest volume and so the function that we want to optimize is given by,

$$f(x, y, z) = xyz$$

Next we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides, so the constraint is given by,

$2xy + 2xz + 2yz = 64 \longrightarrow xy + xz + yz = 32$

Thus $g(x, y, z) = xy + xz + yz$.

Here are the four equations that we need to solve.

$$yz = \lambda(y+z) \qquad (f_x = \lambda g_x) \qquad \textbf{(1)}$$
$$xz = \lambda(x+z) \qquad (f_y = \lambda g_y) \qquad \textbf{(2)}$$
$$xy = \lambda(x+y) \qquad (f_z = \lambda g_z) \qquad \textbf{(3)}$$
$$xy + xz + yz = 32 \qquad (g(x,y,z) = 32) \qquad \textbf{(4)}$$

Hence, $(x, y, z) = (3.266,3.266,3.266)$.

Example 2: Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Example 3: Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2$.