Complex numbers – Euler's formula

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This short write-up is to explain about how one gets the Euler's formula written as

$$e^{i\theta} = \cos\theta \ + \ i\sin\theta \tag{1}$$

Lets begin by consider the exponential of x i.e. e^x. We know that this can be written as:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \tag{2}$$

Now if we substitute $x = i\theta$, we obtain

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \tag{3}$$

Recall that $i^2 = -1$ so that the expression in (3) can be written as:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \cdots \tag{4}$$

Rearranging the equation to

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) + \left(i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} \cdots\right) \tag{5}$$

and identifying that the terms in the first and second parentheses on the right-hand side are the expressions for $\cos\theta$ and $\sin\theta$, respectively, i.e.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots \tag{6}$$

$$\sin\theta = (i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} \cdots \tag{7}$$

we then have the Euler's formula in (1). This justify writing complex number in the form of

$$x + iy = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}. \tag{8}$$

Good to know!

The Euler's formula makes the multiplication and division of complex numbers easier to handle. Say we want to multiply two complex numbers $z_1 = 3e^{3i}$ and $z_2 = 2e^{2i}$.
Using the Euler's formula, we can evaluate $z_1 \cdot z_2$ as:

$$z_1 \cdot z_2 = 3e^{3i} \cdot 2e^{2i} = 6e^{(3+2)i} = 6e^{5i} \tag{9}$$

Dividing complex numbers e.g. z_1/z_2 is also equally easy:

$$z_1/z_2 = \frac{3e^{3i}}{2e^{2i}} = \frac{3}{2}e^{(3-2)i} = \frac{3}{2}e^{i}. \tag{10}$$