Application of matrix in vector

Koh Meng Hock

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1 Representing vectors in matrix form

Let assume two arbitrary vectors such that

$$v_1 = (5, 4, 1) = 5\hat{\imath} + 4\hat{\jmath} + \hat{k}$$

$$v_2 = (-1, 2, 3) = -\hat{\imath} + 2\hat{\jmath} + 3\hat{k}.$$

In matrix form, the vectors are represented by a column matrix i.e.

$$v_1 = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

2 Vector operations in matrix form

The addition and substraction of vectors in matrix form is rather straightforward [1]. Supposed we want to describe the vector v_1 to v_2. In vector forms, we write this as

$$v_{12} = v_2 - v1 = (-1, 2, 3) - (5, 4, 1) = (-6, -2, 2)$$

In matrix form, we just have to simply perform the substraction

$$v_2 - v_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 2 \end{pmatrix}$$

We have looked for the length of a vector, or the projection of a vector on another vector. To perform such operations, we need to convert one of the column matrix to

[1]Here it is not obvious why one needs to learn how to solve vectors problem in matrix form. But you will see as you progress in your study that some problems are easier to handle using matrices. One example is in quantum mechanics where the state of a system consisting of N numbers of particles is represented by combinations of N wave functions (vectors).

row matrix, and then to multiply this with the other column matrix. Take for example, we want to normalize the vector v_1. We need to first find the norm of the vector (i.e. projection of the vector with itself).

$$v_1 \cdot v_1 = \begin{pmatrix} 5 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} = 42$$

The normalized vector is then

$$v_1^{(norm)} = \frac{1}{\sqrt{42}} \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}.$$

Similarly, to find the projection of v_1 on v_2 we perform the operation

$$\begin{pmatrix} 5 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 6.$$

One can easily show that the same answer is obtained if reverting the order i.e. $(v_2 \cdot v_1)$ instead of $(v_1 \cdot v_2)$ as done above.

Next is to find a vector perpendicular to two vectors. We are very familiar with this operation since we usually do this in vector analysis. Take the example of v_1 and v_2 above. To find a vector perpendicular to them, we take the cross product of v_1 and v_2. The resulting vector is written in the form of a determinant:

$$\begin{aligned} v_3 = v_1 \times v_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 4 & 1 \\ -1 & 2 & 3 \end{vmatrix} \\ &= 10\hat{i} - 16\hat{j} + 14\hat{k} \\ &= \begin{pmatrix} 10 \\ -16 \\ 14 \end{pmatrix} \end{aligned}$$

Notice that the components of the vector v_1 and v_2 in the determinants are arranged in rows instead of columns.

3 Finding the equation of plane parallel to two (or more) points

Let us consider two points given by $A(3, 2, 0)$ and $B(1, 1, 0)$, where in matrix form they are written as:

$$A = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1}$$

Assume that we want to find an equation of plane parallel and through A and B. We write the components of both vectors in row 2 and 3 of the determinant respectively such that

$$\begin{vmatrix} x & y & z \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0.$$

Solving it we obtained

$$z(3 - 2) = z = 0.$$

This is rather obvious since both points lies at the $z = 0$. The equation of plane is marked with green colour in Figure 1.

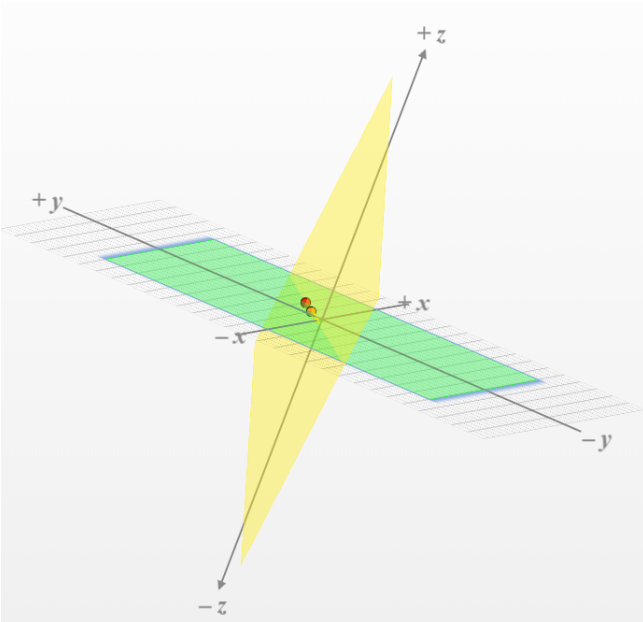

Figure 1 Equation of plane parallel (green) and going through $A(3, 2, 0)$ and $B(1, 10)$ and another which is perpendicular (yellow) to the vector AB.

4 Finding equation of plane perpendicular to two points

To illustrate this, we shall make use of the two points $A(3,2,0)$ and $B(1,1,0)$ above. First, we find the vector that connects point A and point B. We have

$$BA = A - B = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

This is one vector that we will use later. Next we find the vector perpendicular to A and B by calculating the cross product of the two vectors. This gives us another vector which we shall denote as C where

$$C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now we can use the idea in the previous section to find the equation of plane along the vector AB and C – this will give us a plane which is perpendicular to A and B. We write the determinant:

$$\begin{pmatrix} x & y & z \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x - 2y + 1$$

and since the determinant is equal to zero, we have

$$x - 2y + 1 = 0.$$

This is the plane shown in yellow colour in Figure 1.

5 Perpendicular plane – a proper approach

If we attempt to replicate the approach given above for points in 3D space (i.e. when $z \neq 0$), we will see that the method fails to find the correct equation of plane. What we could do is to actually consider three points i.e. the point $A(3,2,0)$, $B(1,1,0)$ and point $C(0,0,1)$. Point C is obtained from the cross product of point A and B. We then write the determinant as:

$$\begin{pmatrix} x & y & z & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 0$$

We have a 4×4 matrix where solving it

$$x \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} - y \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} + z \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

yields

$$-x + 2y + z - 1 = 0$$

or equivalently

$$x - 2y - z + 1 = 0.$$

This plane is plotted in red color in Figure 2.

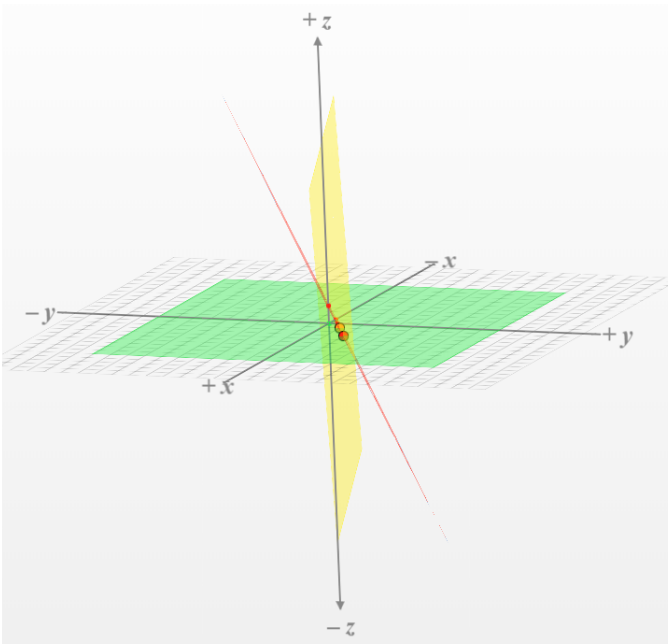

Figure 2 Similar to Figure 1 but with a different equation of plane (red colour).

6 Equation of plane – another example

Consider now two points whch are not located on the $z = 0$ plane, for e.g. $P(3, 2, 1)$ and $Q(1, 1, 3)$. We first find the equation of plane parallel to PQ and going through both points using

$$\begin{pmatrix} x & y & z \\ 3 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = 0 \tag{2}$$

which gives us

$$5x - 8y + z = 0.$$

This is the plane in green colour in Figure 3.

Let us now find the equation of plane which is perpendicular to vector PQ and going through the two points P and Q. As a start, we find the vector perpendicular to vector P and Q through cross products of these two vectors. Here we get a vector given by $5\hat{\imath} - 8\hat{\jmath} + \hat{k}$.

Now that we have three points, we can find a plane which connects all of them. We arrange these points in a determinant:

$$\begin{pmatrix} x & y & z & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 5 & -8 & 1 & 1 \end{pmatrix} = 0.$$

I will let you deal with the calculations yourselves, after which you should get the equation of plane given by:

$$20x + 4y + 22z + 90 = 0$$

shown in purple in Figure 3.

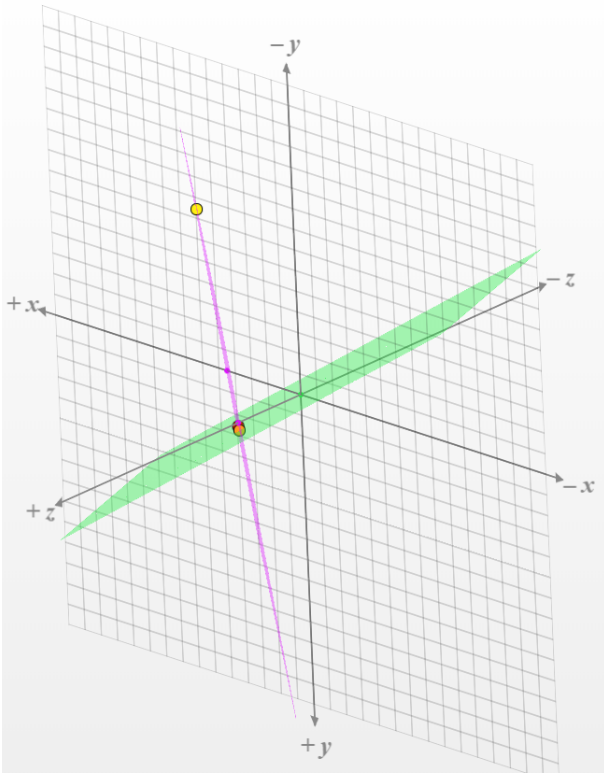

Figure 3 Equation of plane parallel to the vector connecting $P(3, 2, 1)$ and $Q(1, 1, 3)$ in green, and another which is perpendicular to this vector PQ in purple. The point in yellow is the resultant cross product between P and Q.

7 Equation of plane connecting three points

As a sum up, let me just show again how to obtain the equation of plane connecting three points. Let us consider the points $A(3, 2, 1)$, $B(1, 1, 3)$ and $C(6, 4, 3)$.

To find the equation of plane, we arrange the components of this points in a determinant such that:

$$\begin{pmatrix} x & y & z & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 6 & 4 & 3 & 1 \end{pmatrix} = 0. \tag{3}$$

Note that the elements in the first three columns are the components of the points above. And you just need to add a row of ones at the end of the determinant. Solving this would yield

$$6x - 10y + z = -1$$

which is plotted in Figure 4.

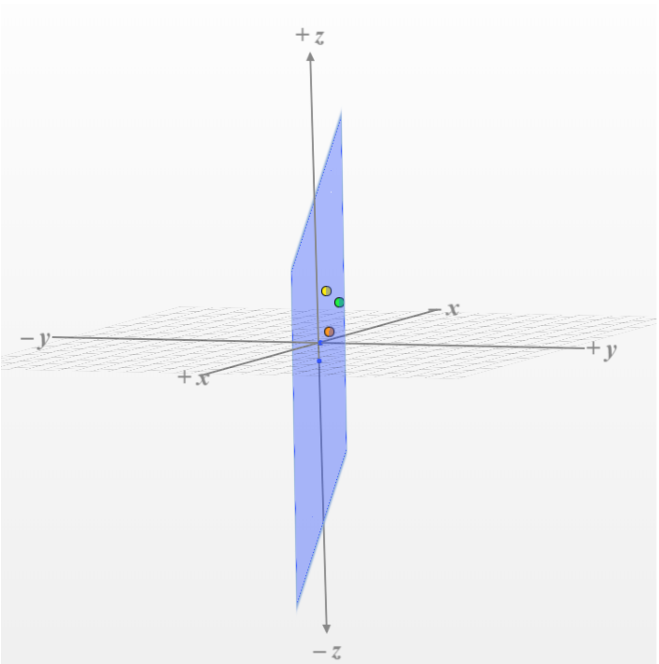

Figure 4 Equation of plane connecting three points $A(3, 2, 1)$, $B(1, 1, 3)$ and $C(6, 4, 3)$.