Matrix – Revision

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1 Some basics terminologies

1. Find the following items with respect to the matrix A.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 6 & 5 & 3 \\ 2 & -1 & 0 \end{pmatrix}$$

   (a) **Element** A_{12}, A_{32}, A_{21}.

   (b) **Minor** of A_{22}.

   (c) **Transpose** matrix.

   (d) **Conjugate** of A.

   (e) **Adjugate** (or sometimes called as adjoint or adjunct) of matrix A.

   (f) **Inverse** of the matrix.

   (g) **Trace** of the matrix.

   (h) Is A a **Hermitian** matrix?

   (i) Is A **unitary**?

   (j) Is A a **singular** matrix?

   (k) **Determinant** of A.

2. Do the same as in Question 1 for the matrix B where

$$B = \begin{pmatrix} 5 & 17 & 3 \\ 2 & 4 & -3 \\ 11 & 0 & 2 \end{pmatrix}$$

3. Find the commutator of matrix A and B. The commutator of two matrices are given as

$$[A, B] = AB - BA.$$

   The matrices A and B commutes if $[A, B] = 0$.

4. Consider another matrix C given by

$$C = \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

   (a) Is C a **skew-symmetric** matrix?
   (b) Is C a **singular** matrix?
   (c) Find its conjugate.

2 Sets of linear equations, homogenous equation, row reduction and rank of matrix

One can solve linear equations by transforming it into matrix form. Take for example a set of linear equations:

$$\begin{aligned} 2x \qquad\quad - z &= 2 \\ 6x + 5y + 3z &= 7 \\ 2x - y \qquad\quad &= 4 \end{aligned}$$

Changing it into matrix form, we get:

$$\begin{pmatrix} 2 & 0 & -1 \\ 6 & 5 & 3 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} \tag{1}$$

For the clarity sake, let us write this in the form of

$$M\,v = k$$

where M is the square matrix and k is the column matrix with elements 2, 7 and 4. In order to find x, y and z, we just need to solve

$$v = M^{-1}\,k \tag{2}$$

Let us go to the tasks and questions that we need to answer.

(a) First, decide if this is a **homogenous equation**.

(b) Solve for x, y and z using equation (2).

(c) Supposed now that we want to solve the linear equations using **row reduction** method.

i. Write the augmented matrix for the set of linear equations.

ii. Now row reduce the augmented matrix as much as possible. What is the **rank** of the augmented matrix? [1]

iii. Consider now the matrix M which is basically the first three columns of the augmented matrix above. What is the rank of M?

Why bother with the rank of matrix?!

- If rank of $M < A$ – equations are inconsistent & there is **NO** solution.
- If rank of $M = A = n$ unknowns – Yeah! There is only **ONE** solution (this is the case for homogeneous equation).
- Rank $M = A = \mathcal{R} < n$ – Uhh, still not too bad.

  We can write the $\mathcal{R}$ unknowns in terms of $n - \mathcal{R}$ unknowns.

One more thing to remember for homogeneous equation...
A set of homogeneous equation has a solution other than a **trivial solution** if its determinant is equal to zero!

Let us now try to solve for the following two sets of linear equations.

(a)
$$\begin{aligned} 2x + 3y &= 1 \\ x + 2y &= 2 \\ x + 3y &= 5 \end{aligned}$$

(b)
$$\begin{aligned} \text{x - y + 2z} &= 5 \\ \text{2x + 3y - z} &= 4 \\ \text{2x - 2y + 4z} &= 6 \end{aligned}$$

Interesting fact on determinant...
In the case whereby two columns (or rows) in a determinant are multiplicative factor of the other, the determinant is exactly zero.
One example is the matrix below where the 3rd row can be obtained from the 1st row by multiplying it with a factor of -2.

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & -3 & 2 \\ -4 & -2 & -6 \end{pmatrix}$$

[1]For example a matrix given by $\begin{pmatrix} 1 & 3 & 2 & 2 \\ -2 & 4 & 0 & 1 \\ 6 & -1 & 2 & 4 \end{pmatrix}$ is rank 3 while $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 6 & -1 & 2 & 4 \end{pmatrix}$ is of rank 2.

3 Diagonalization of matrix – non-degenerate case

Consider the following matrices:

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

Work on the following items for the matrices given above:

(a) Find all the eigenvalues. Are the eigenvalues **degenerate**?

(b) Find the eigenvector corresponding to each eigenvalue.

(c) Write the matrix which diagonalizes the matrix above.

(d) Diagonalize the matrix above.

On eigenvalues and eigenvectors ...

When we found the eigenvector (denoted e.g. by $\mathbf{v}$) for a given eigenvalue ϵ of a matrix M, we have the eigenvalue equation:

$$M\,\mathbf{v} = \epsilon\,\mathbf{v}$$

i.e. to say that if you multiply the matrix M by the eigenvector v, then we will always get back the same eigenvector multiplied by a number ϵ.
This has interesting application in quantum mechanics as you will see later.

Diagonalization of matrix ...

In order to diagonalize a matrix, we combined all the eigenvectors corresponding to all eigenvalues to form a matrix which we shall referred to as C. Next, we multiply the initial matrix M with C and its inverse such that

$$C^{-1}\,M\,C \;=\; D$$

where D is now a **diagonal matrix**.
Refer page 151 of Boas to understand the meaning of matrix C and D.

Diagonalization of a Hermitian matrix ...

For a Hermitian matrix, we can diagonalize it through

$$U^{\dagger}\,M\,U \;=\; D.$$

4 Diagonalization of matrix – degenerate case

Consider the matrix M to be diagonalized such that

$$M = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Go through the same questions in the previous section. See the tips below to find the eigenvector for degenerate case.

Find eigenvector for degenerate case – using the Gram-Schmidt method.
Take for example a 3×3 matrix where one of the eigenvalue is degenerate. This means that there are two eigenvectors which are orthogonal to one another but sharing the same eigenvalue. To find these orthogonal eigenvectors, we follow the Gram-Schmidt method.

- Find two eigenvectors for the degenerate eigenvalue (as we normally do). Let's denote this as v_1 and v_2.
- Normalize v_1 and this will be one of the eigenvector i.e. $e_1 = \frac{v_1}{|v_1|}$.
- Find the second eigenvector using:

$$e_2 = v_2 - (e_1 \cdot v_2)e_1$$

Let us try to diagonalize another two matrices given by:

$$\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$