Hypothesis Testing

Null and Alternative Hypotheses and Errors in Hypothesis Testing

- Null hypothesis, H_0, is a statement of the basic proposition being tested
  - Represents the status quo and is not rejected unless there is convincing sample evidence that it is false
- Alternative hypothesis, H_a, is an alternative accepted only if there is convincing sample evidence it is true
- One-Sided, “Greater Than” $H_0: \mu \leq \mu_0$ vs. $H_a: \mu > \mu_0$
- One-Sided, “Less Than” $H_0 : \mu \geq \mu_0$ vs. $H_a : \mu < \mu_0$
- Two-Sided, “Not Equal” $H_0 : \mu = \mu_0$ vs. $H_a : \mu \neq \mu_0$

where μ_0 is a given constant value (with the appropriate units) that is a comparative value

Types of Decisions

- As a result of testing H_0 vs. H_a, will decide either of the following decisions for the null hypothesis H_0:
  - Do not reject H_0 or reject H_0
- To "test" H_0 vs. H_a, use the "test statistic"

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

- z measures the distance between μ_0 and $\bar{x}$ on the sampling distribution of the sample mean
- If the population is normal or n is large*, the test statistic z follows a normal distribution

Error Probabilities

- Type I Error: Rejecting H_0 when it is true
  - α is the probability of making a Type I error
  - $1 - \alpha$ is the probability of not making a Type I error
- Type II Error: Failing to reject H_0 when it is false
  - β is the probability of making a Type II error
  - $1 - \beta$ is the probability of not making a Type II error

	State of Nature	
Decision	H_0: $\mu \leq 50$ True	H_0: $\mu \leq 50$ False
Reject H_0: $\mu \leq 50$	Type I error	Correct decision
Do not reject H_0: $\mu \leq 50$	Correct decision	Type II error

Typical Values

- Usually set α to a low value
  - So there is a small chance of rejecting a true H_0
- Typically, $\alpha = 0.05$
  - Strong evidence is required to reject H_0
  - Usually choose α between 0.01 and 0.05
    - $\alpha = 0.01$ requires very strong evidence to reject H_0
- Tradeoff between α and β
  - For fixed sample size, the lower α, the higher β
    - And the higher α, the lower β

z Tests about a Population Mean: σ Known

- Test hypotheses about a population mean using the normal distribution
- Called z tests
- Require that the true value of the population standard deviation σ is known
  - In most real-world situations, σ is not known
    - But often is estimated from s of a single sample
    - When σ is unknown, test hypotheses about a population mean using the t distribution
  - Here, assume that we know σ

Steps in Testing a “Greater Than” Alternative

1. State the null and alternative hypotheses
2. Specify the significance level α
3. Select the test statistic
4. Determine the critical value rule for deciding whether or not to reject H_0
5. Collect the sample data and calculate the value of the test statistic
6. Decide whether to reject H_0 by using the test statistic and the rejection rule
7. Interpret the statistical results in managerial terms and assess their practical importance

Steps in Testing a "Greater Than" Alternative in Trash Bag Case #1

1. State the null and alternative hypotheses

   H_0: $\mu \leq 50$
   H_a: $\mu > 50$

2. Specify the significance level α

   - $\alpha = 0.05$

3. Select the test statistic

   - Use the test statistic

$$z = \frac{\bar{x} - 50}{\sigma_{\bar{x}}} = \frac{\bar{x} - 50}{\sigma / \sqrt{n}}$$

   - A positive value of this this test statistic results from a sample mean that is greater than 50 lbs

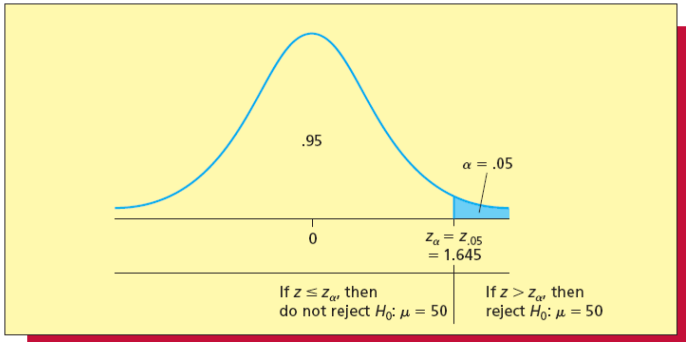
Steps in Testing a "Greater Than" Alternative in Trash Bag Case #2

4. Determine the rejection rule for deciding whether or not to reject H_0
   - To decide how large the test statistic must be to reject H_0 by setting the probability of a Type I error to α, do the following:
   - The probability α is the area in the right-hand tail of the standard normal curve
   - Use the normal table to find the point z_α (called the rejection or critical point)
   - Reject H_0 in favor of H_a if the test statistic z is greater than the rejection point z_α
   - In the trash bag case, the rejection rule is to reject H_0 if the calculated test statistic z is > 1.645

Steps in Testing a "Greater Than" Alternative in Trash Bag Case #3

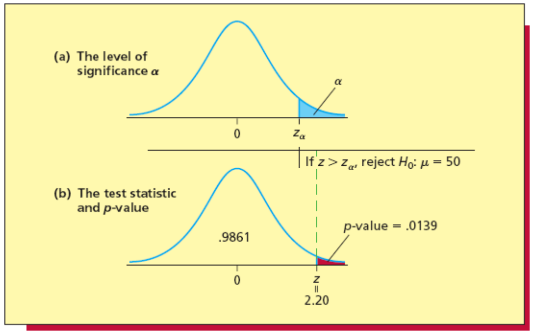
5. Collect the sample data and calculate the value of the test statistic
   - In the trash bag case, assume that σ is known and σ = 1.65 lbs
   - For a sample of n = 40, ☒ = 50.575 lbs. Then

$$z = \frac{\bar{x} - 50}{\sigma / \sqrt{n}} = \frac{50.575 - 50}{1.65 / \sqrt{40}} = 2.20$$



Steps in Testing a "Greater Than" Alternative in Trash Bag Case #4

6. Decide whether to reject H_0
   - Compare the value of the test statistic to the rejection point according to the rejection rule
   - Here, $z = 2.20$ is greater than $z_{0.05} = 1.645$
   - Therefore reject H_0: $\mu \leq 50$ in favor of H_a: $\mu > 50$ at the 0.05 significance level
7. Interpret the statistical results
   - Conclude mean breaking strength of new bag exceeds 50 lbs



Effect of α

- At $\alpha = 0.01$, the rejection point is $z_{0.01} = 2.33$
- In the trash example, the test statistic $z = 2.20$ is $< z_{0.01} = 2.33$
- Therefore, cannot reject H_0 in favor of H_a at the $\alpha = 0.01$ significance level
  - This is the opposite conclusion reached with $\alpha=0.05$
  - So, the smaller we set α, the larger is the rejection point, and the stronger is the statistical evidence that is required to reject the null hypothesis H_0

The p-Value

- The p-value or the observed level of significance is the probability of obtaining the sample results if the null hypothesis H_0 is true
  - The p-value is used to measure the weight of the evidence against the null hypothesis
- Sample results that are not likely if H_0 is true have a low p-value and are evidence that H_0 is not true
  - The p-value is the smallest value of α for which we can reject H_0
- The p-value is an alternative to testing with a z test statistic

Steps Using a p-value to Test a "Greater Than" Alternative

4. Collect the sample data and compute the value of the test statistic
   - In the trash bag case, the value of the test statistic was calculated to be z = 2.20
5. Calculate the p-value by corresponding to the test statistic value
   - In the trash bag case, the area under the standard normal curve in the right-hand tail to the right of the test statistic value z = 2.20
   - The area is 0.5 – 0.4861 = 0.0139
   - The p-value is 0.0139

Steps Using a p-value to Test a “Greater Than” Alternative Continued

5. Continued
   - If H_0 is true, the probability is 0.0139 of obtaining a sample whose mean is 50.575 lbs or higher
   - This is so low as to be evidence that H_0 is false and should be rejected
6. Reject H_0 if the p-value is less than α
   - In the trash bag case, α was set to 0.05
   - The calculated p-value of 0.0139 is $< \alpha = 0.05$
     - This implies that the test statistic $z = 2.20$ is greater than the rejection point $z_{0.05} = 1.645$
   - Therefore reject H_0 at the $\alpha = 0.05$ significance level

Steps in Testing a "Less Than" Alternative in Payment Time Case #1

1. State the null and alternative hypotheses
   - In the payment time case, H_0: $\mu \geq 19.5$ vs. H_a: $\mu < 19.5$, where μ is the mean bill payment time (in days)
2. Specify the significance level α
   - In the payment time case, set $\alpha = 0.01$
3. Select the test statistic
   - In the payment time case, use the test statistic

$$z = \frac{\bar{x} - 19.5}{\sigma_{\bar{x}}} = \frac{\bar{x} - 19.5}{\sigma / \sqrt{n}}$$

   - A negative value of this this test statistic results from a sample mean that is less than 19.5 days

Steps in Testing a "Less Than" Alternative in Payment Time Case #2

4. Determine the critical value rule
   - Decide how much less than 0 test statistic must be to reject H_0 with probability of α
   - The probability α is the area in the left-hand tail of the standard normal curve
   - Use standard normal table to find the rejection point $-z_\alpha$
   - Since $\alpha = 0.01$ in the payment time case, the rejection point is $-z_\alpha = -z_{0.01} = -2.33$
   - Reject H_0 in favor of H_a if the test statistic z is calculated to be less than the rejection point $-z_\alpha$
   - In the payment time case, reject H_0 if the test statistic $-z$ is less than -2.33

Steps in Testing a "Less Than" Alternative in Payment Time Case #3

5. Collect the sample data and calculate the value of the test statistic
   - In the payment time case, assume that σ is known and σ = 4.2 days
   - For a sample of n=65, ☒ = 18.1077 days:

$$z = \frac{\bar{x} - 19.5}{\sigma/\sqrt{n}} = \frac{18.1077 - 19.5}{4.2/\sqrt{65}} = -2.67$$

Steps in Testing a "Less Than" Alternative in Payment Time Case #4

6. Decide whether to reject H_0 by using the test statistic and the rejection rule
   - Compare the value of the test statistic to the rejection point according to the rejection rule
   - In the payment time case, $z = –2.67$ is less than $z_{0.01} = 2.33$
   - Therefore reject H_0: $\mu \geq 19.5$ in favor of
     H_a: $\mu < 19.5$ at the 0.01 significance level
7. Interpret the statistical results in managerial terms and assess their practical importance
   - Can conclude that the mean bill payment time of the new billing system is less than 19.5 days

Steps Using a p-value to Test a "Less Than" Alternative

(Steps 1–3 are the same)

4. Collect the sample data and compute the value of the test statistic
   - In the payment time case, the value of the test statistic was calculated to be $z = -2.67$
5. Calculate the p-value by corresponding to the test statistic value
   - In the payment time case, the area under the standard normal curve in the left-hand tail to the left of the test statistic $z = -2.67$
   - The area is = 0.0038
   - The p-value is 0.0038

Steps Using a p-value to Test a "Less Than" Alternative Continued

5. Continued
   - If H_0 is true, then the probability is 0.0038 of obtaining a sample whose mean is as low as 18.1077 days or lower
   - This is so low as to be evidence that H_0 is false and should be rejected
6. Reject H_0 if the p-value is less than α
   - In the payment time case, α was 0.01
   - The calculated p-value of 0.0038 is < α = 0.01
     - This implies that the test statistic $z = –2.67$ is less than the rejection point $–z_{0.01} = –2.33$
   - Therefore, reject H_0 at the α = 0.01 significance level

Steps in Testing a "Not Equal To" Alternative in Valentine Day Case #1

1. State null and alternative hypotheses
   - In case, H_0: $\mu = 330$ vs. H_a: $\mu \neq 330$
2. Specify the significance level α
   - In the case, set $\alpha = 0.05$
3. Select the test statistic

$$z = \frac{\bar{x} - 330}{\sigma_{\bar{x}}} = \frac{\bar{x} - 330}{\sigma / \sqrt{n}}$$

   - Positive value results from $\bar{X}$ greater than 330
   - Negative value results from $\bar{X}$ less than 330
   - Value close to 0 results from $\bar{X}$ nearly 330

Steps in Testing a "Not Equal To" Alternative in Valentine Day Case #2

4. Determine the critical value rule for deciding whether or not to reject H_0
   - Decide how much less than 0 test statistic must be to reject H_0 with probability of α
   - Use normal table to find the rejection points $z_{\alpha/2}$ and $z_{\alpha/2}$
     - $z_{\alpha/2}$ is the point on the horizontal axis under the standard normal curve that gives a right-hand tail area equal to $\alpha/2$
     - $-z_{\alpha/2}$ is the point on the horizontal axis under the standard normal curve that gives a left-hand tail area equal to $\alpha/2$

Steps in Testing a "Not Equal To" Alternative in Valentine Day Case #3

4. Continued
   - Because $\alpha = 0.05$, $\alpha/2=0.025$
     - The area under the standard normal to the left of the mean is 0.5 + 0.025 = 0.525
     - From normal table, the area is 0.525 for $z = 1.96$
   - Rejection points are $z_\alpha=1.96$, $-z_\alpha= -1.96$
   - Reject H_0 in favor of H_a if the test statistic z satisfies either:
     - z greater than the rejection point $z_{\alpha/2}$, or
     - $-z$ less than the rejection point $-z_{\alpha/2}$
     - This is the rejection rule

Steps in Testing a “Not Equal To” Alternative in Valentine Day Case #4

5. Collect the sample data and calculate the value of the test statistic
   - In the Valentine Day case, assume that σ is known and σ = 40
   - For a sample of n = 100, x = 326
   - Then

$$z = \frac{\bar{x} - 330}{\sigma/\sqrt{n}} = \frac{326 - 330}{40/\sqrt{100}} = -1.00$$

Steps in Testing a "Not Equal To" Alternative in Valentine Day Case #5

6. Decide whether to reject H_0 by using the test statistic and the rejection rule
   - Compare the value of the test statistic to the rejection point according to the rejection rule
   - In this case, $-z = -1.00$ is $< -z_{0.025} = -1.96$
   - Therefore cannot reject $H_0: \mu = 330$ in favor of $H_a: \mu \neq 330$ at the 0.05 significance level
7. Interpret the statistical results in managerial terms and assess their practical importance
   - Cannot conclude that the mean order quantity this year of the Valentine Day box at large retail stores will differ from 330 boxes

Steps Using a p-value to Test a "Not Equal To" Alternative

(Steps 1–3 are the same)

4. Collect the sample data and compute the value of the test statistic
   - In the Valentine Day case, the value of the test statistic was calculated to be $z = -1.00$
5. Calculate the p-value by corresponding to the test statistic value
   - In the Valentine Day case, the area under the standard normal curve in the left-hand tail to the left of the test statistic value $z = -1.00$
   - The area is 0.1587
   - The p-value is $0.1587 \cdot 2 = 0.3174$

Steps Using a p-value to Test a "Not Equal To" Alternative Continued

5. Continued
   - That is, if H_0 is true, probability is 0.3174 of obtaining a sample whose mean is at least as extreme as 326
   - This probability is not so low as to be evidence that $_{H0}$ is false and should be rejected
6. Reject H_0 if the p-value is less than a
   - In the Valentine Day case, α was 0.05
   - Calculated p-value of 0.3174 is greater than α
     - This implies that the test statistic $z = –1.00$ is greater than the rejection point $–z_{0.025} = –1.96$
   - Therefore do not reject H_0 at the $\alpha = 0.05$ significance level

t Tests about a Population Mean: σ Unknown

- Assume the population being sampled is normally distributed
- The population standard deviation σ is unknown, as is the usual situation
  - If the population standard deviation σ is unknown, then it will have to estimated from a sample standard deviations
- Under these two conditions, have to use the t distribution to test hypotheses

Defining the t Statistic: σ Unknown

- Let $\bar{x}$ be the mean of a sample of size n with standard deviation s
- Also, μ_0 is the claimed value of the population mean
- Define a new test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

- If the population being sampled is normal, and s is used to estimate σ, then …
- The sampling distribution of the t statistic is a t distribution with n – 1 degrees of freedom

t Tests about a Population Mean: σ Unknown

Alternative	**Reject H_0 if:**	**p-value**
$H_a: \mu > \mu_0$	$t > t_\alpha$	Area under t distribution to right of t
$H_a: \mu < \mu_0$	$t < -t_\alpha$	Area under t distribution to left of –t
$H_a: \mu \neq \mu_0$	$\lvert t \rvert > t_{\alpha/2}$ *	Twice area under t distribution to right of |t|

t_α, $t_{\alpha/2}$, and p-values are based on $n - 1$ degrees of freedom (for a sample of size n)

* either $t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$

z Tests about a Population Proportion

Alternative	**Reject H_0 if:**	**p-value**
H_a: $\rho > \rho_0$	$z > z_\alpha$	Area under t distribution to right of z
H_a: $\rho < \rho_0$	$z < -z_\alpha$	Area under t distribution to left of –z
H_a: $\rho \neq \rho_0$	$\|z\| > z_{\alpha/2}$ *	Twice area under t distribution to right of |z|

Where the test statistics is $$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

* either $z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$

Selecting an Appropriate Test Statistic for a Test about a Population Mean