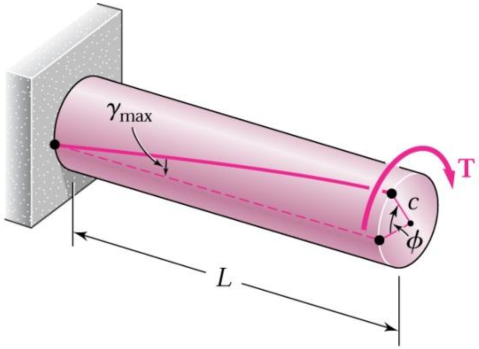
5.4 ANGLE OF TWIST

Being able to compute the angle of twist for a shaft is important when analyzing the reactions on statically indeterminate shafts.



In this section we will develop a formula for determining the angle of twist ϕ (phi) of one end of a shaft with respect to its other end. The material is assumed to be homogeneous and to behave in a linear-elastic manner when the torque is applied. Considering first the case of a shaft of length L and of uniform cross section of radius c subjected to torque T at its free end, we recall from section 5.2 that the angle of twist ϕ and the maximum shearing strain γ_{max} are related as follows:

$$\gamma_{max} = \frac{c\emptyset}{L}$$

But in the elastic range, the yield stress is not exceeded anywhere in the shaft, Hooke's law applies, and we have $\gamma_{max} = \tau_{max}/G$ or recalling the torsion formula

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

Equating the above equations and solving for ϕ, we write

$$\boxed{\phi = \frac{TL}{JG}}$$

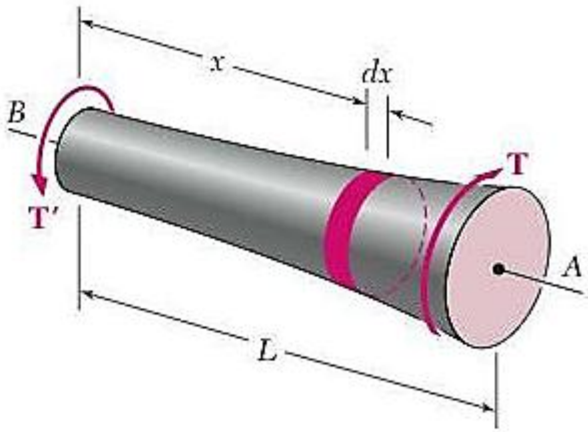
where ϕ is expressed in radians. The relation obtained shows that, within the elastic range, the angle of twist is proportional to the torque T applied to the shaft.

In the case of a shaft with a variable circular cross section, as shown in figure, formula for the angle of twist ϕ may be applied to a disk of thickness *dx*. The angle by which one face of the disk rotates with respect to the other is thus

$$d\emptyset = \frac{Tdx}{JG}$$

Where *J* is a function of *x* which may be determined. Integrating in *x* from *0* to *L*, we obtain the total angle of twist of the shaft:

$$\phi = \int_0^L \frac{T\,dx}{JG}$$



$\phi =$The angle of twist of one end of the shaft with respect to the other end, measured in radians

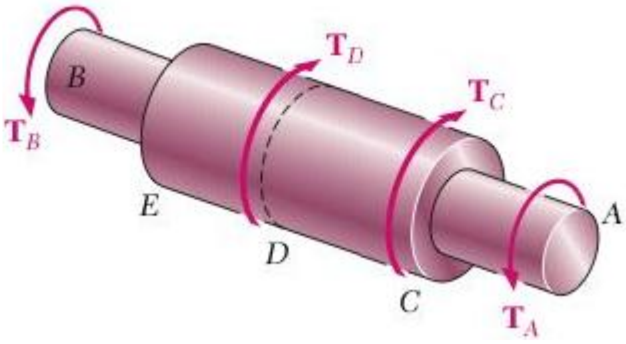
T(x) = the internal torque at the arbitrary position *x*, found from the method of sections and the equation of moment-equilibrium applied about the shaft’s axis

J(x) = the shaft’s polar moment of inertia expressed as a function of position *x*

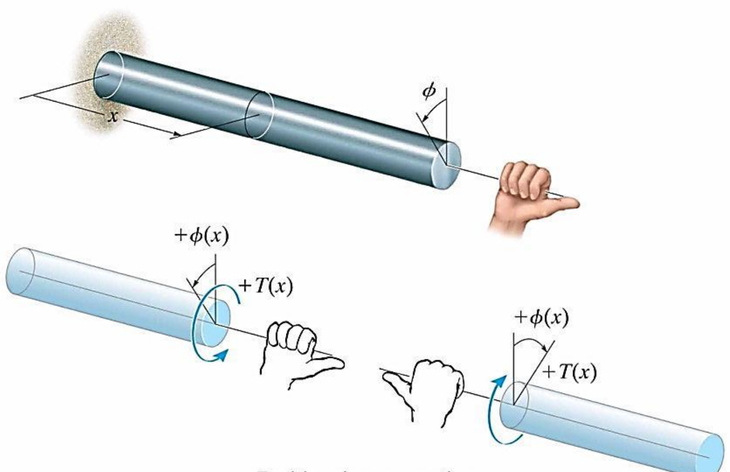
G = the shear modulus for the material

If the shaft is subjected to several different torques, or the cross-sectional area or shear modulus changes abruptly from one region of the shaft to the next, angle of twist equation can be applied to each segment of the shaft where these quantites are all constant. The angle of twist of one end of the shaft with respect to the other is then found from the vector addition of the angles of twist of each segment. For this case

$$\emptyset = \sum \frac{TL}{JG}$$

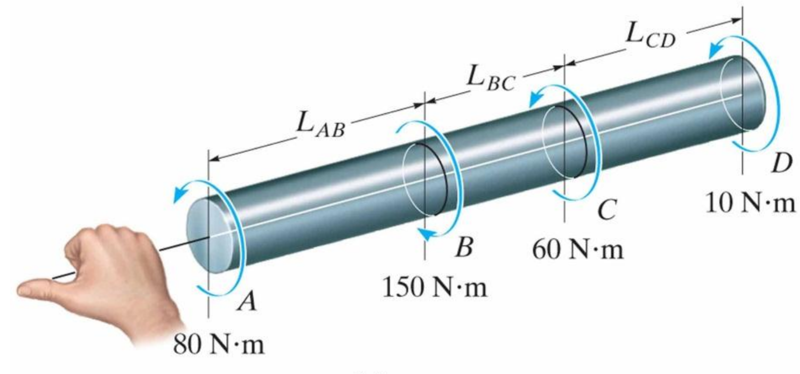


Sign convention. In order to apply the above equations, we must develop a sign convention for the internal torque and the angle of twist of one end of the shaft with respect to the other end. To do this we will use the right-hand rule, whereby both the torque and angle will be positive provided the thumb is directed outward from the shaft when the fingers curl to give the tendency for rotation.



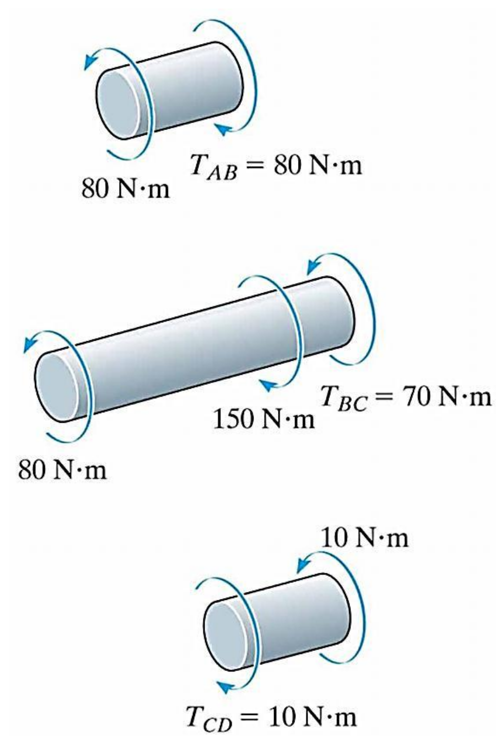
Positive sign convention for T and ϕ.

To illustrate the use of this sign convention, consider the shaft shown in figure (a), which is subjected to four torques. The angle of twist of end *A* with respect to end *D* is to be determined. For this problem, three segments of the shaft must be considered, since the internal torque changes at *B* and *C*. Using the method of sections, the internal torques are found for each segment, figure(b). By the right-hand rule, with positive torques directed away from the sectioned end of the shaft, we have $T_{AB} = +80$N.m, $T_{BC} = -70$ N.m and $T_{CD} = -10$ N.m. and we have



(a)

$$\emptyset_{A/D} = \frac{+80L_{AB}}{JG} + \frac{-70L_{BC}}{JG} + \frac{-10L_{CD}}{JG}$$

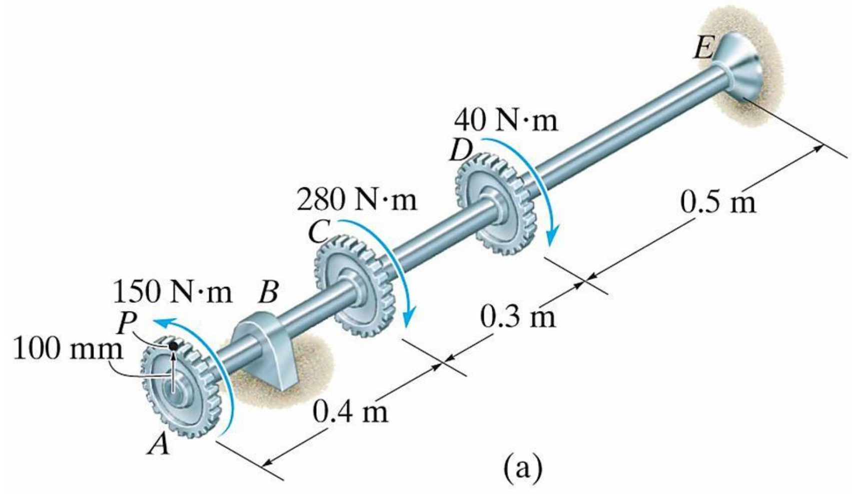


(b)

If the other data is substituted and the answer is found as a positive quantity, it means that end *A* will rotate as indicated by the curl of the right-hand fingers when the thumb is directed away from the shaft. The double subscript notation is used to indicate this relative angle of twist $(\emptyset_{A/D})$; however, if the angle of twist is to be determined relative to a fixed point, then only a single subscript will be used. For example if *D* is located at a fixed support, then the computed angle of twist will be denoted as $\emptyset_A$.

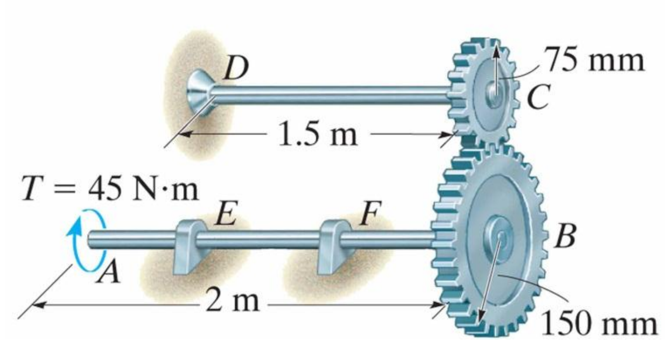
Example 5.8(Hibbeler)

The gears attached to the fixed-end steel shaft are subjected to the torques shown in figure. If the shear modulus of elasticity is $G = 80$ GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth P on gear A. The shaft turns freely within the bearing at B.



Example 5.9(Hibbeler)

The two solid steel shafts shown in figure are coupled together using the meshed gears. Determine the angle of twist of end *A* of shaft *AB* when the torque *T* = 45 N.m is applied. Take *G* = 80 GPa. Shaft *AB* is free to rotate within bearings *E* and *F*, whereas shaft *DC* is fixed at *D*. Each shaft has a diameter of 20 mm.



5.5 STATICALLY INDETERMINATE TORQUE-LOADED MEMBERS

A torsionally loaded shaft may be classified as statically indeterminate if the moment equation of equilibrium, applied about the axis of the shaft, is not adequate to determine the unknown torques acting on the shaft. An example of this situation is shown in figure *(a)*. As shown on the free-body diagram, the reactive torques at the supports *A* and *B* are unknown. We require

$$\sum M_x = 0; \qquad T - T_A - T_B = 0$$

Since only one equilibrium equation is relevant and there are two unknowns, this problem is statically indeterminate.

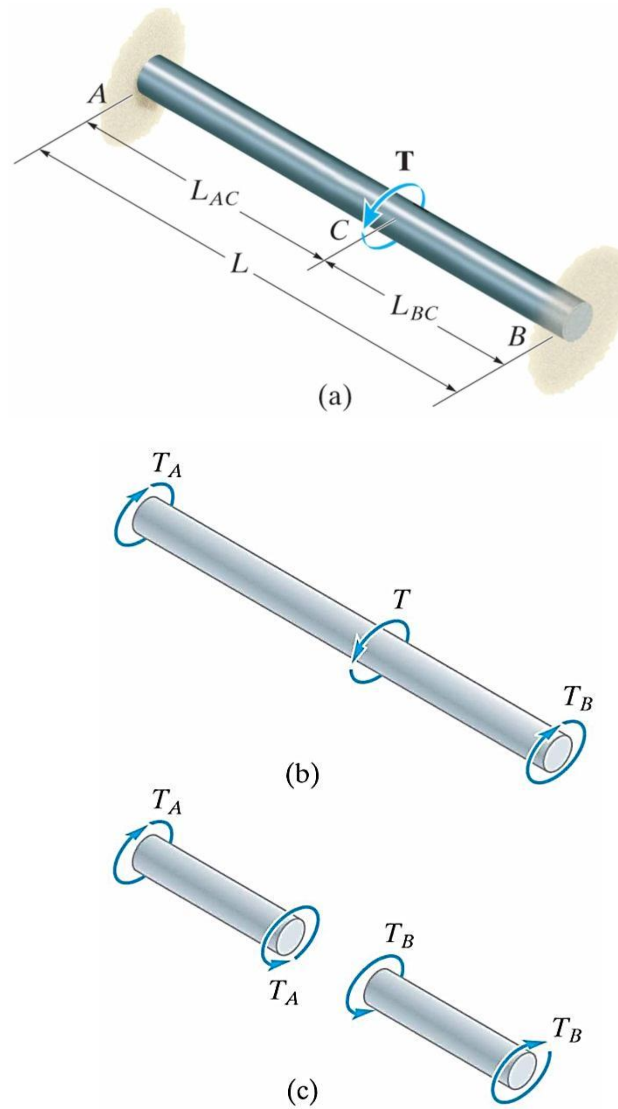
The necessary condition of compatibility, or kinematic condition, requires the angle of twist of one end of the shaft with respect to the other end to be equal to zero, since the end supports are fixed. Therefore

$$\emptyset_{A/B} = 0.$$

Realizing that the internal torque in segment *AC* is $+T_A$, and in the segment *CB* the internal torque is $-T_B$, the above equation can be written as

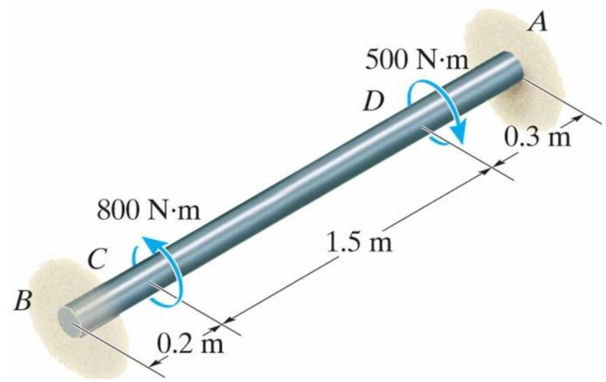
$$\frac{T_A L_{AC}}{JG} - \frac{T_B L_{BC}}{JG} = 0$$

Solving the above two equations, reactions can be obtained.



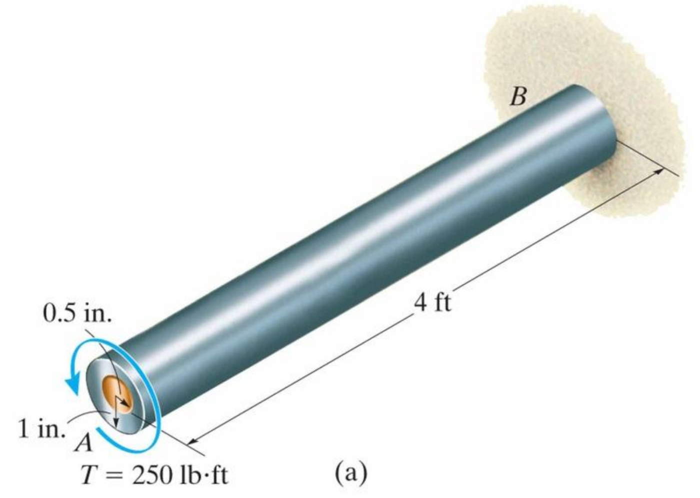
Example 5.10 (Hibbeler)

The solid steel shaft shown in figure has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.



Example 5.11(Hibbeler)

The shaft shown in figure is made from a steel tube, which is bonded to a brass core. If a torque of T = 250 lb.ft is applied at its end, plot the shear stress distribution along a radial line of its cross-sectional area. Take $G_{st} = 11.4(10^3)$ ksi, $G_{br} = 5.20(10^3)$ ksi.



(a)

Example 5.12 (Beer&Johnston)

The horizontal shaft *AD* is attached to a fixed base at *D* and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion *CD* of the shaft. Knowing that the entire shaft is made of steel for which G=77 GPa, determine the angle of twist at end *A*.