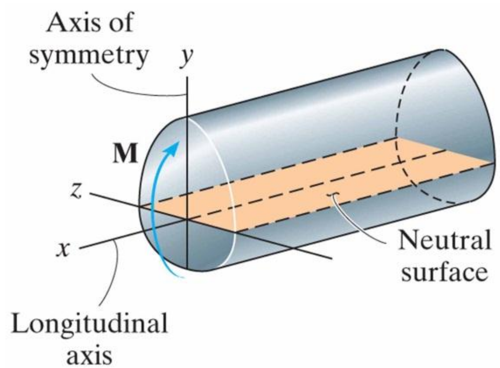
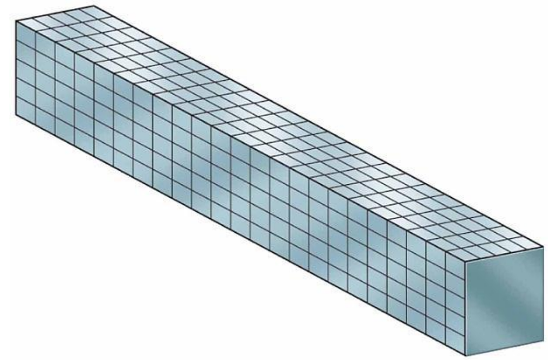
6.3 BENDING DEFORMATION OF A STRAIGHT MEMBER

In this section we will discuss the deformations that occur when a straight beam, made of a homogeneous material, is subjected to bending. The discussion will be limited to beams having a cross-sectional area that is symmetrical with respect to an axis, and the bending moment is applied about an axis perpendicular to this axis of symmetry as shown in figure.



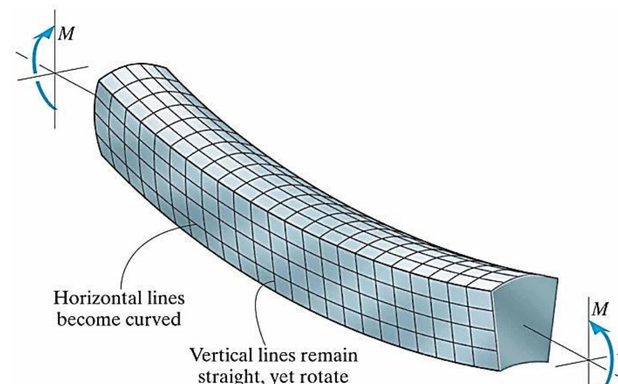


Before deformation

(a)

By using a highly deformable material such as rubber, we can physically illustrate to a bending moment. Consider, for example, the undeformed bar in figure (a) which has a square cross section and is marked with longitudinal and transverse grid lines. When a bending moment is applied, it tends to distort these lines into the pattern shown in figure (b). When a bending moment is applied, it can be seen that the longitudinal lines become curved and the vertical transverse lines remain straight and yet undergo a rotation.

The behavior of any deformable bar subjected to a bending moment causes the material within the bottom portion of the bar to stretch and the material within the top portion to compress. Consequently, between these two region there must be a surface, called the *neutral surface*, in which longitudinal fibers of the material will not undergo a change in length.



After deformation

(b)

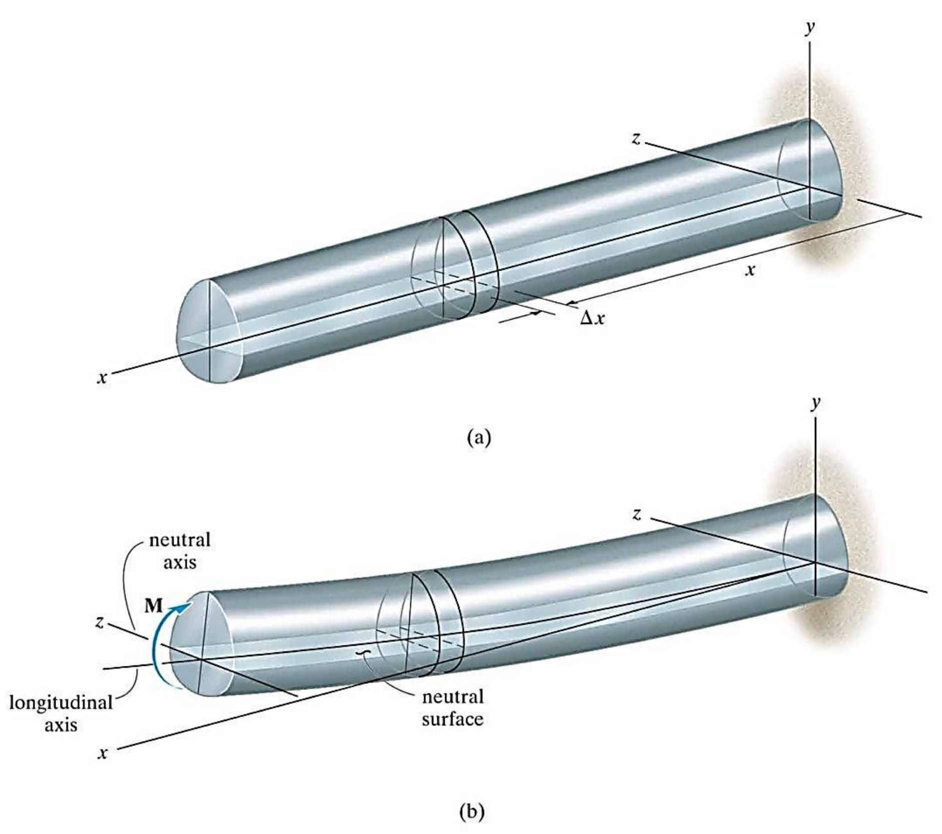
From these observations we will make the following three assumptions regarding the way the stress deforms the material:

1- The longitudinal axis *x*, which lies within the neutral surface, figure (a), does not experience any change in length. Rather the moment will tend to deform the beam so that this line becomes a curve that lies in the *x*-*y* plane symmetry, figure (b).

2- All cross sections of the beam remain plane and perpendicular to the longitudinal axis during the deformation.

3- Any deformation of the cross section within its own plane will be neglected. In particular, the *z* axis, lying in the plane of the cross section and about which the cross section rotates, is called the neutral axis.

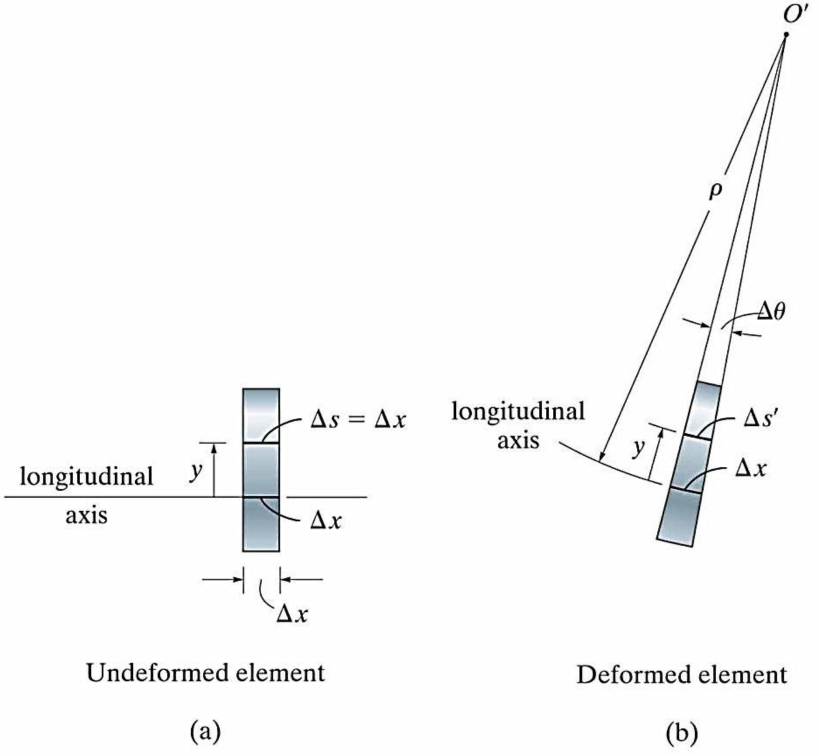
In order to show how this distortion will strain the material, we will isolate a segment of the beam that is located a distance *x* along the beam's length and has an undeformed thickness, Δx, figure (a)



The element, taken from the beam, is shown in profile view in the undeformed positions in the figure. Notice that any line segment Δx, located on the neutral surface, does not change its length, whereas any line segment Δs will contract and become $\Delta s'$ after deformation. By definition, the normal strain along Δs is determined from

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

We will now represent this strain in terms of the location y of the segment and the radius of curvature ρ of the longitudinal axis of the element. After deformation Δx has a radius of curvature ρ. Since $\Delta\theta$ defines the angle between the cross-sectional sides of the element, $\Delta x = \Delta s = \rho\ \Delta\theta$. In the same manner, the deformed length of Δs becomes $\Delta s' = (\rho - y)\ \Delta\theta$. Substituting into the above equation, we get



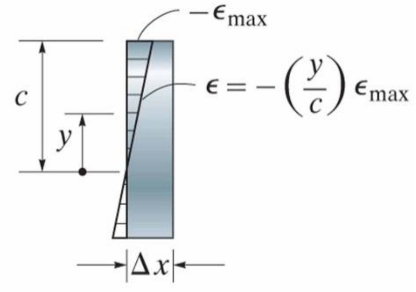
$$\epsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\ \Delta\theta - \rho\ \Delta\theta}{\rho\ \Delta\theta}$$

or

$$\epsilon = -\frac{y}{\rho}$$

This important result indicates that the longitudinal normal strain of any element within the beam depends on its location y on the cross section and the radius of curvature of the beam's longitudinal axis at the point. In other words, for any specific cross section, the longitudinal normal strain will vary linearly with y from the neutral axis.

A contraction (- ϵ) will occur in fibers located above the neutral axis, whereas elongation (+ ϵ) will occur in fibers located below the axis. This variation in strain over the cross section is shown in figure. Here the maximum strain occurs at the outermost fiber, located a distance *c* from the neutral axis. Using the previous formula;



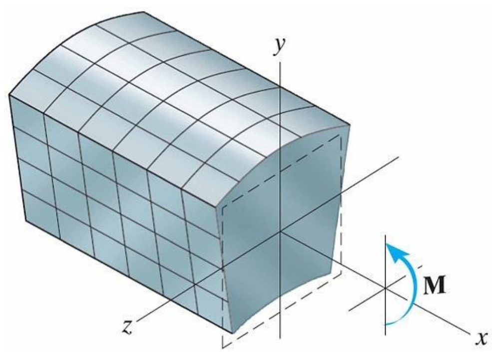
Normal strain distribution

$$\frac{\epsilon}{\epsilon_{max}} = \frac{-y/\rho}{c/\rho}$$

So that

$$\epsilon = -\left(\frac{y}{c}\right)\epsilon_{max}$$

This normal strain depends only on the assumptions made with regards to the deformation. Provided only a moment is applied to the beam, then it is reasonable to further assume that this moment causes a normal stress only in the longitudinal or *x* axis. All the other components of normal and shear stress are zero. It is uniaxial state of stress that causes the material to have the longitudinal normal strain component. Furthermore by poisson's ratio, there must be also be associated strain component, which deform the plane of the cross sectional area, although here we have neglected these deformations. Such deformations will, however, cause the cross-sectional dimensions to become smaller below the neutral axis and larger above the neutral ais. For example, if the beam has a square cross section, it will actually deform as shown in the figure.

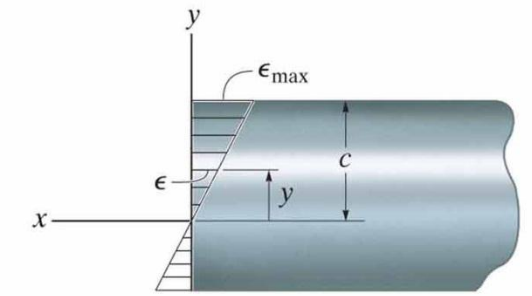


6.4 FLEXURE FORMULA

In this section we will develop an equation that relates the longitudinal stress distribution in a beam to the internal resultant bending moment acting on the beam's cross section. To do this we will assume that the material behaves in a linear-elastic manner so that Hooke's law applies. A linear variation of normal strain, figure(a), must then be the consequence of a linear variation in normal stress, figure(b). Like the strain variation, stress will vary from zero at the member's neutral axis to a maximum value, a distance *c* farthest from the neutral axis. Because of the proportionality of triangles, figure(b), or using Hooke's law, we can write

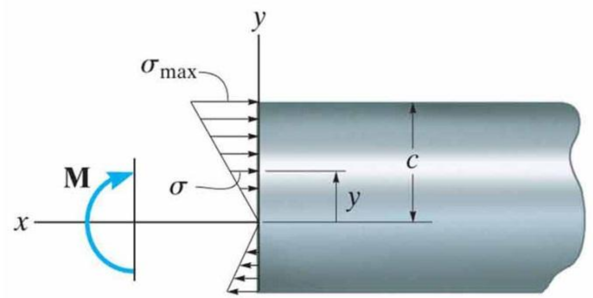
$$\sigma = -\left(\frac{y}{c}\right)\sigma_{\max}$$

This equation represents the stress distribution over the cross-sectional area. The sign convention established here is significant. For positive **M**, which acts in the +*z* direction, positive values of *y* give negative values of σ, that is, a compressive stress since it acts in the negative *x* direction. Similarly, negative *y* values will give positive or tensile values for σ. If a volume element of material is selected at a specific point on the cross section, only these tensile or compressive normal stresses will act on it.



Normal strain variation
(profile view)

(a)



Bending stress variation
(profile view)

(b)

We can locate the position of the neutral axis on the cross section by satisfying the condition that the resultant force produced by the stress distribution over the cross-sectional area must be equal to zero. Noting that the force $dF = \sigma\, dA$ acts on the arbitrary element dA in figure(c), we require

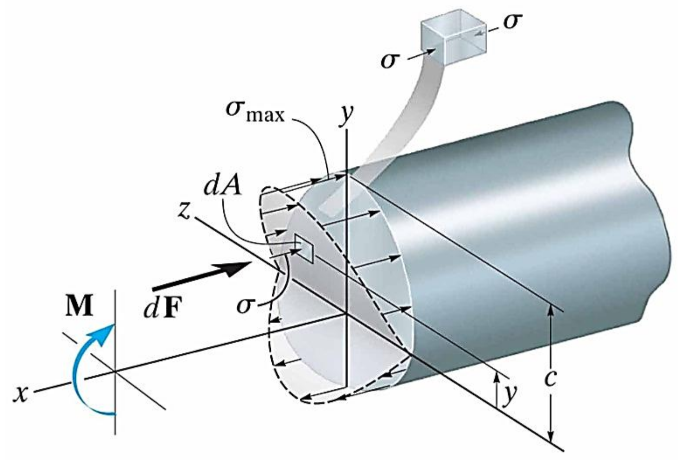
$$F_R = \Sigma F_x; \qquad 0 = \int_A dF = \int_A \sigma\, dA$$

$$= \int_A -\left(\frac{y}{c}\right)\sigma_{\max}\, dA$$

$$= \frac{-\sigma_{\max}}{c}\int_A y\, dA$$

Since $\sigma_{\max}/c$ is not equal to zero, then

$$\int_A y\, dA = 0$$

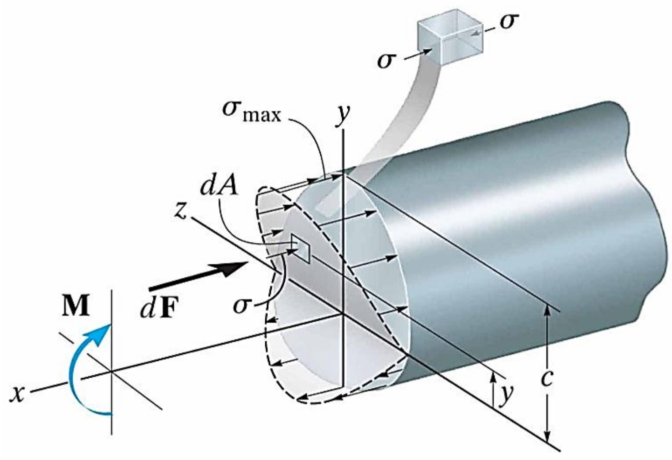


Bending stress variation

(c)

In other words, the first moment of the member's cross-sectional area about the neutral axis must be zero. This condition can only be satisfied if the neutral axis is also the horizontal centroidal axis for the cross section. Consequently, once the centroid for the member's cross-sectional area is determined, the location of the neutral axis is known.

We can determine the stress in the beam from the requirement that the resultant internal moment *M* must be equal to the moment produced by the stress distribution about the neutral axis. The moment of *dF* in figure(*c*) about the neutral axis $dM = y\, dF$. This moment is positive since, by the right-hand rule, the thumb is directed along the positive z axis when the fingers are curled with the sence of rotation caused by *dM*. Since $dF = \sigma\, dA$, we have for the entire cross-section,



Bending stress variation

(c)

$$(M_R)_z = \Sigma M_z;$$

$$M = \int_A y\, dF = \int_A y\,(-\sigma\, dA) = \int_A y\left(\frac{y}{c}\sigma_{\max}\right) dA$$

$$M = \frac{\sigma_{\max}}{c}\int_A y^2\, dA$$

Here the integral represents the moment of inertia of the beam's cross-sectional area, computed about the neutral axis. We sybolize its value as *I*. Hence the equation can be solved for stress and written in general form as

$$\boxed{\sigma_{\max} = \frac{Mc}{I}}$$

$$\boxed{\sigma_{max} = \frac{Mc}{I}}$$

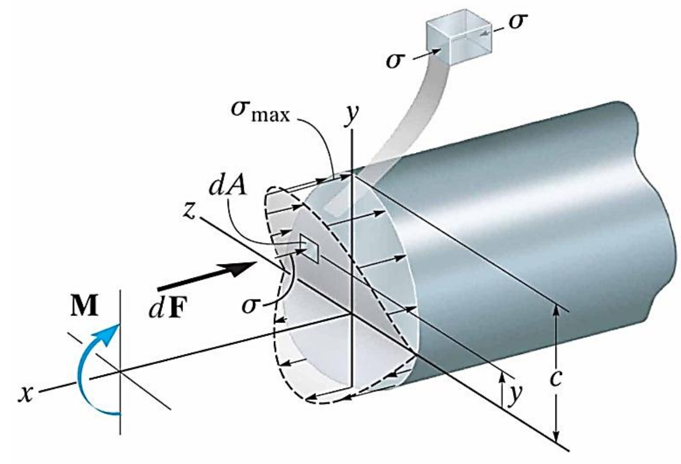
Here

σ_{max} = the maximum normal stress in the member, which occurs at a point on the cross-sectional area *farthest away* from the neutral axis

M = the resultant internal moment, determined from the method of sections and the equations of equilibrium, and computed about the neutral axis of the cross section

I = the moment of inertia of the cross-sectional area computed about the neutral axis

c = the perpendicular distance from the neutral axis to a point farthest away from the neutral axis, where σ_{max} acts



Bending stress variation

(c)

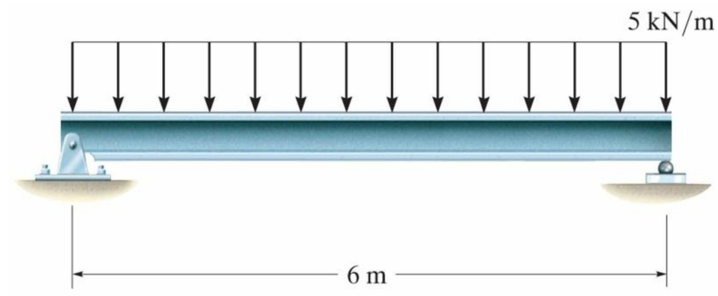
The normal stress at the intermediate *y* can be determined from an equation similar to equation above.

$$\boxed{\sigma = -\frac{My}{I}}$$

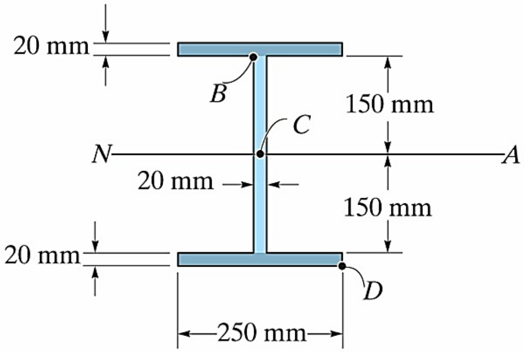
Either of the above two equations is often referred as <u>flexure formula</u>. It is used to determine the normal stress in a straight member, having a cross section that is symmetrical with respect to an axis, and the moment is applied perpendicular to this axis.

Example 6.1 (Hibbeler)

The simply supported beam in the figure(a) has the cross-sectional area shown in figure(b). Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.



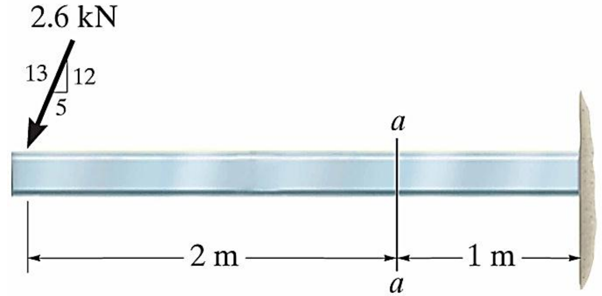
(a)



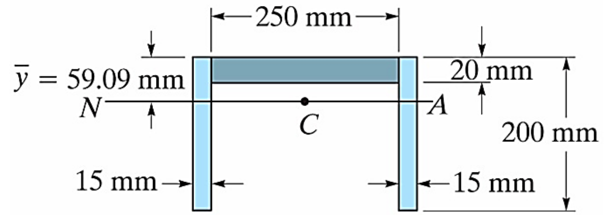
(b)

Example 6.2 (Hibbeler)

The beam shown in figure(a) has a cross sectional area in the shape of a channel, figure(b). Determine the maximum bending stress that occurs in the beam at section *a-a*.



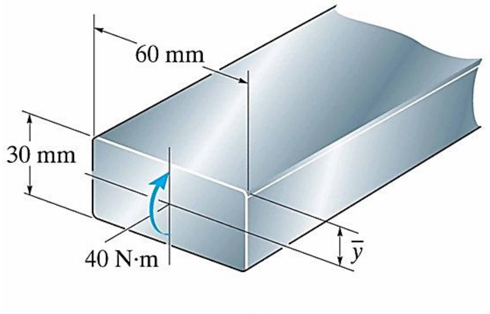
(a)



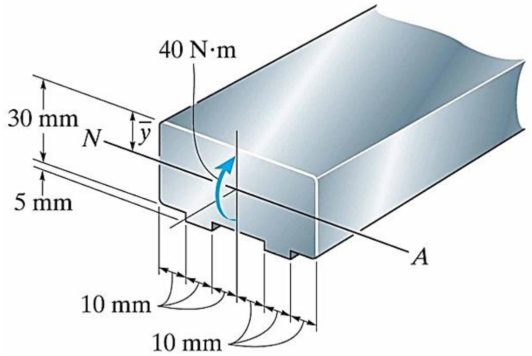
(b)

Example 6.3 (Hibbeler)

The member having a rectangular cross section, figure(a), is designed to resist a moment of 40 N.m. In order to increase its strength and rigidity, it is proposed that two small ribs be added at its bottom, figure(b). Determine the maximum normal stress in the member for both cases.



(a)



(b)

Example 6.4 (Hibbeler)

A beam is constructed from four pieces of wood, glues together as shown. If the moment acting on the cross section is M = 450 N.m, determine the resultant force the bending stress produces on the top board A and on the side board B.