Engineering Economics

Interest Rate and Equivalence

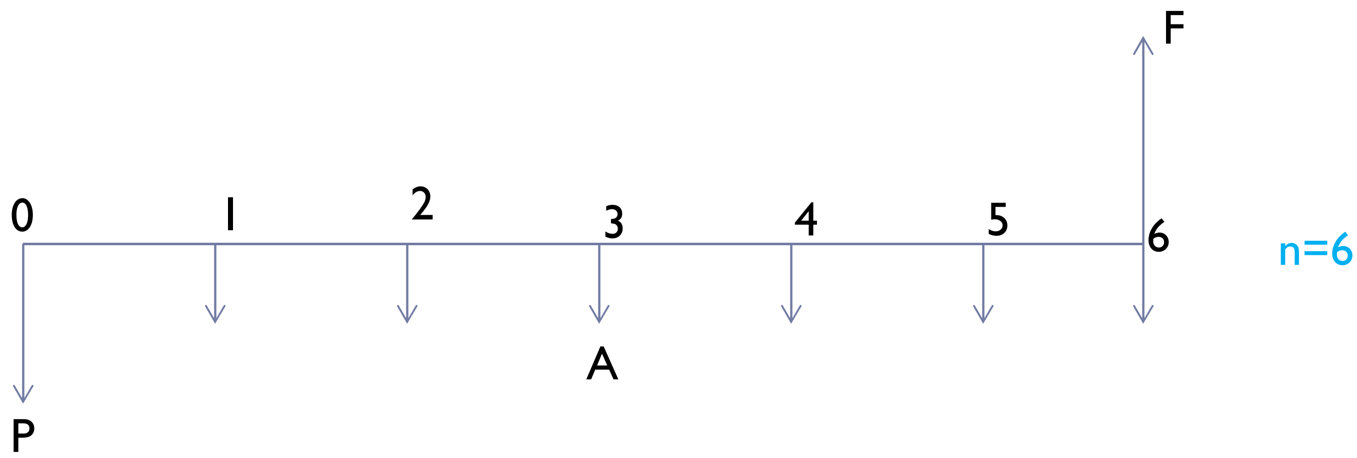
Outcome of Today's Lecture

- **After completing this lecture…**

- The students should be able to:
  - Understand interest and rate of return
  - Define and provide examples of the time values of money
  - Distinguish between simple and compound interest, and use compound interest in engineering economic analysis

Terminology and Symbols

- P= value or amount of money at present , Also referred as present worth (PW), present value (PV), net present value , discounted cash flow and Capital Cost
- F=Value or amount of money at future time. Also F is called future worth (FW) and future value (FV)
- A= Series of consecutives, equal, end of period amounts of money (Receipts/disbursement)
- n= Number of interest period; years, months or days
- i= interest rate per time period; percent per year
- t=time, stated in periods; years, months or days



Interest

- **1. Simple interest**
- Simple interest is computed only on original sum (principal), not on prior interest earned and left in the account.
- A bank account, for example, may have its simple interest every year: in this case, an account with $1000 initial principal and 20% interest per year would have a balance of $1200 at the end of the first year, $1400 at the end of the second year, and so on.

- **2. Compound Interest**
- **Compound interest** arises when interest is added to the principal of a deposit or loan, so that, from that moment on, the interest that has been added also earns interest. This addition of interest to the principal is called *compounding*.
- A bank account, for example, may have its interest compounded every year: in this case, an account with $1000 initial principal and 20% interest per year would have a balance of $1200 at the end of the first year, $1440 at the end of the second year, and so on.

Simple Interest Rate

- Interest is paid when a person/organisation borrowed money and repays a larger amount over time

Interest =Amount to be returned – Principle (original amount)

Interest =F-P

- interest rate on borrowed fund is determined using the original amount (called Principal) as

$$\text{Interest Rate}(\%) = \frac{\text{interest incurred per unit time}}{\text{Prinicipal}} \times 100$$

- Time unit of interest paid is called interest period.

Simple Interest Rate

- If the interest rate, i, is given then;

$$\text{interest} = \text{P} \times \text{i} \times \text{n}$$

- And at the end of n years the total amount of money due, F, would equal the amount of the loan, P, plus the total interest, *P.i.n*, as given by;

$$F = P + P(i)(n)$$

Simple Interest Rate

- **Example1.3:** An employee at Laserkinetics.com borrows $10,000 on May 1 and must repay a total of $10,700 exactly 1 year later. Determine the interest amount and the interest rate paid.

- **Solution:**
- Amount to be paid= $10,700
- Original amount=$10,000

- Interest=Amount to be paid-Original amount=10700-10000=$700

$$\text{Interest Rate}(\%) = \frac{\text{interest incurred per unit time}}{\text{Prinicipal}} \times 100$$

$$\text{Interest Rate}(\%) = \frac{700}{10000} \times 100 = 7\% / \, year$$

Ref. Engineering Economy By Leland Blank & Anthony Tarquin

Simple Interest Rate

- **Example1.4:** Stereographic, Inc., plans to borrow \$20000 from a bank for 1 year at 9% interest for new recording equipment.
- Compute the interest and total amount due after 1 year.

- **Solution:**
- Original (Principal) amount=\$20,000
- Interest rate=9% annual

$$9 = \frac{\text{interest incurred per year}}{20000} \times 100$$

$$\text{Interest} = \$1800$$

OR

$$\text{interest} = 20000 \times 0.09 \times 1$$
$$= 1800$$

- Total due amount after a year=20000+1800=\$21800

Ref. Engineering Economy By Leland Blank & Anthony Tarquin

Simple Interest Rate

- **Example 1.5:** Calculate the amount deposited 1 year ago to have $1000 now at an interest rate of 5% per year.
- Calculate the amount of interest earned during this period.
- **Solution:**

Interest=amount owned now-original deposit
Interest + original deposit=amount owned now

Interest rate (original deposit) no. of interest period + original deposit=amount owned now
(Interest rate x no. of interest period+1) original deposit = 1000
Original deposit=1000/(1.05)=$952.38

$$I = F - P$$
$$I + P = F$$
$$Pin + P = F$$
$$F = P(in + 1)$$
$$1000 = P(0.05 + 1)$$
$$P = 952.38$$

- Thus
- Interest = 1000-952.38= $47.62

Ref. Engineering Economy By Leland Blank & Anthony Tarquin

Example: 3.3 (Simple interest)

- You have agreed to loan a friend $5000 for 5 years at a simple interest rate of 8% per year. How much interest will you receive from the loan. How much will your friend pay you at the end of 5 years.
- **Solution**

Sr. #	Principal at which interest is computed	Interest owed at end of year n	Due at the end of year n
1	5000	400	5400
2	5000	400	5800
3	5000	400	6200
4	5000	400	6600
5	5000	400	7000

OR

$$\text{Total interest} = P \times i \times n$$

$$\text{Total interest} = 5000 \times \frac{8}{100} \times 5 = 2000$$

$$\text{Total amount due at end of loan} = 5000 + 2000 = 7000$$

Compound Interest Rate

- **Compound interest** arises when interest is added to the principal of a deposit or loan, so that, from that moment on, the interest that has been added also earns interest.
- Using notation, P, F, n, & I, compound interest *calculations assuming single payment at the end of loan period* are given by

Year	Amount at Beginning of Interest Period	+ Interest for Period	= Amount at End of Interest Period
1	P	$+iP$	$=P(1+i)$
2	$P(1+i)$	$+iP(1+i)$	$=P(1+i)^2$
3	$P(1+i)^2$	$+iP(1+i)^2$	$=P(1+i)^3$
n	$P(1+i)^{n-1}$	$+iP(1+i)^{n-1}$	$=P(1+i)^n$

In other words, a present sum P increases in n periods to $P(1+i)^n$. We therefore have a relationship between a present sum P and its equivalent future sum, F.

$$\text{Future sum} = (\text{Present sum})\ (1+i)^n$$

$$F = P(1+i)^n$$

Single payment compound interest formula

- **Compound interest** arises when interest is added to the principal of a deposit or loan, so that, from that moment on, the interest that has been added also earns interest.

- Future sum, F, using compound interest *with single payment at the end of loan period* thus becomes as;

$$F = P(1+i)^n$$

This is called single payment compound interest formula.

We will learn more about it in next class

Example: 3.4 (Compound interest)

- You have agreed to loan a friend $5000 for 5 years at a compound interest rate of 8% per year. How much interest will you receive from the loan. How much will your friend pay you at the end of 5 years.
- **Solution**

Sr. #	Principal at which interest is computed	Interest owed at end of year n	Due at the end of year n
1	5000	5000x0.08=400	5000+400=5400
2	5400	5400x0.08=432	5400+432=5832
3	5832	5832x0.08=467	5832+467=6299
4	6299	504	6803
5	6803	544	7347

$$\text{Total amount due at end of loan} = \$7347$$

Recall: In case of simple interest total amount due at the end of 5 year was $7000

Repaying a Debt

- To better understand the mechanics of interest, let say that €5000 is owed and is to be repaid in years together with 8% annual interest.

- Lets use four specific plans to repay

  - Plan 1: At end of each year pay €1000 principle plus interest due
  - Plan 2: Pay interest at end of each year and principal at end of 5 years
  - Plan 3: Pay in five equal end of year payments
  - Plan 4: Pay principal and interest in one payment at end of 5 years

Repaying a Debt

Four Plans for Repayment of €5000 in 5 Years with Interest at 8%

(a) Year	(b) Amount Owed at Beginning of Year	(c) Interest Owed for That Year, 8% × (b)	(d) Total Owed at End of Year, (b) + (c)	(e) Principal Payment	(f) Total End-of-Year Payment
Plan 1: At end of each year pay €1000 principal *plus* interest due.					
1	€5000	€ 400	€5400	€1000	€1400
		€1200		€5000	€6200

Repaying a Debt

Four Plans for Repayment of €5000 in 5 Years with Interest at 8%

(a) Year	(b) Amount Owed at Beginning of Year	(c) Interest Owed for That Year, 8% × (b)	(d) Total Owed at End of Year, (b) + (c)	(e) Principal Payment	(f) Total End-of-Year Payment
Plan 2: Pay interest due at end of each year and principal at end of 5 years.					
1	€5000	€ 400	€5400	€ 0	€ 400
2	5000	400	5400	0	400
3	5000	400	5400	0	400
4	5000	400	5400	0	400
5	5000	400	5400	5000	5400
		€2000		€5000	€7000

Repaying a Debt

Four Plans for Repayment of €5000 in 5 Years with Interest at 8%

(a) Year	(b) Amount Owed at Beginning of Year	(c) Interest Owed for That Year, 8% × (b)	(d) Total Owed at End of Year, (b) + (c)	(e) Principal Payment	(f) Total End-of-Year Payment
Plan 3: Pay in five equal end-of-year payments.					
1	€5000	€ 400	€5400	€ 852	€1252*
2	4148	331	4479	921	1252
3	3227	258	3485	994	1252
4	2233	178	2411	1074	1252
5	1159	93	1252	1159	1252
		€1260		€5000	€6260

Repaying a Debt

Four Plans for Repayment of €5000 in 5 Years with Interest at 8%

(a) Year	(b) Amount Owed at Beginning of Year	(c) Interest Owed for That Year, 8% × (b)	(d) Total Owed at End of Year, (b) + (c)	(e) Principal Payment	(f) Total End-of-Year Payment
Plan 4: Pay principal and interest in one payment at end of 5 years.					
1	€5000	€ 400	€5400	€ 0	€ 0
2	5400	432	5832	0	0
3	5832	467	6299	0	0
4	6299	504	6803	0	0
5	6803	544	7347	5000	7347
		€2347		€5000	€7347

Economic Equivalence

- Economic equivalence is a combination of interest rate and time value of money to determine the different amounts of money at different points in time that are equal in economic value.

Illustration:

At 6% interest rate, $100 today (present time) is equivalent to $106 one year from today

And $100 now is equivalent to 100/1.06=$94.34 one year ago

Ref. Engineering Economy By Leland Blank & Anthony Tarquin

Equivalence

- ***Lets recall example of repaying of debt***
- To better understand the mechanics of interest, let say that €5000 is owed and is to be repaid in 5 years together with 8% annual interest..

- Lets use four specific plans to repay
  - Plan 1: At end of each year pay €1000 principle plus interest due
  - Plan 2: Pay interest at end of each year and principal at end of 5 years
  - Plan 3: Pay in five equal end of year payments
  - Plan 4: Pay principal and interest in one payment at end of 5 years

- *Are all payment plans are equivalent to each other and to €5000 now at 8% interest rate ??*

Technique of equivalence

- We can determine an equivalent value at some point in time for any plan, based on a selected interest rate not from cash flow.

- We can use concept of time value of money and computer money year i.e., euro-year,

Plan	Total Interest Paid	Area Under Curve (euro-years)	Ratio of Total Interest Paid to Area Under Curve
1	€1200	15,000	0.08
2	2000	25,000	0.08
3	1260	15,767	0.08
4	2347	29,334	0.08

Ratio under the curve is constant and equal at 8% which indicate that repayment plans are actually equivalent

Single payment compound interest formula

- **Compound interest** arises when interest is added to the principal of a deposit or loan, so that, from that moment on, the interest that has been added also earns interest.

- Compound interest is computed with following formula;

$$\text{interest} = P \times (i + 1)^{n}$$

- The future sum, F, thus become as;

$$F = P(1 + i)^{n}$$

This is called single payment compound interest formula

Single payment compound interest formula

- The single payment formula in functional form can be written as

$$\mathrm{F} = P(F/P, i, n)$$

- The notation in parenthesis can be read as follows: "To find a future sum F, given a present sum, P, at an interest rate i per interest period and n interest periods hence" OR simply Find F, given P, at I, over n

- Similarly functional form of determining present value, P, from future sum, F at interest rate, i, over interest period, n, becomes

$$P = F(P/F, i, n)$$

$$\because \mathrm{F}(1+i)^{-n} = P$$

Example: 3-5

- If €500 were deposited in a bank saving account, how much would be in the account 3 years hence if the bank paid 6% interest compounded annually?

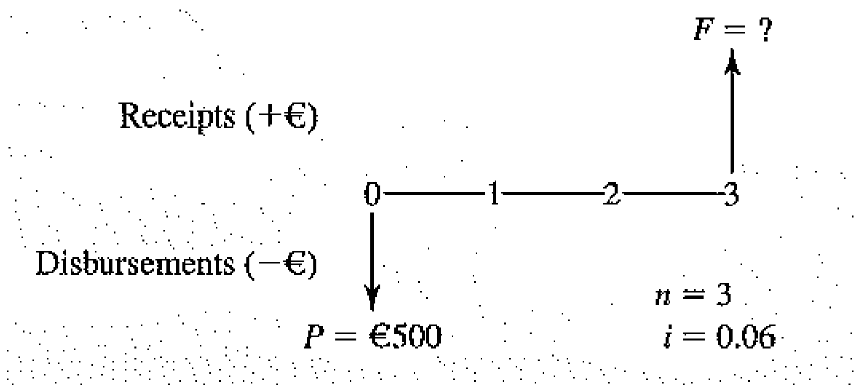
- **Solution:**

- P= € 500,
- i=6%=0.06
- n=3

$$F = P(1+i)^n$$

$$F = 500(1+0.06)^3$$

$$=595.50$$



Cash Flow Diagram

Example: 3-5

- Alternate Solution:
- P= €500,
- i=6%=0.06
- n=3

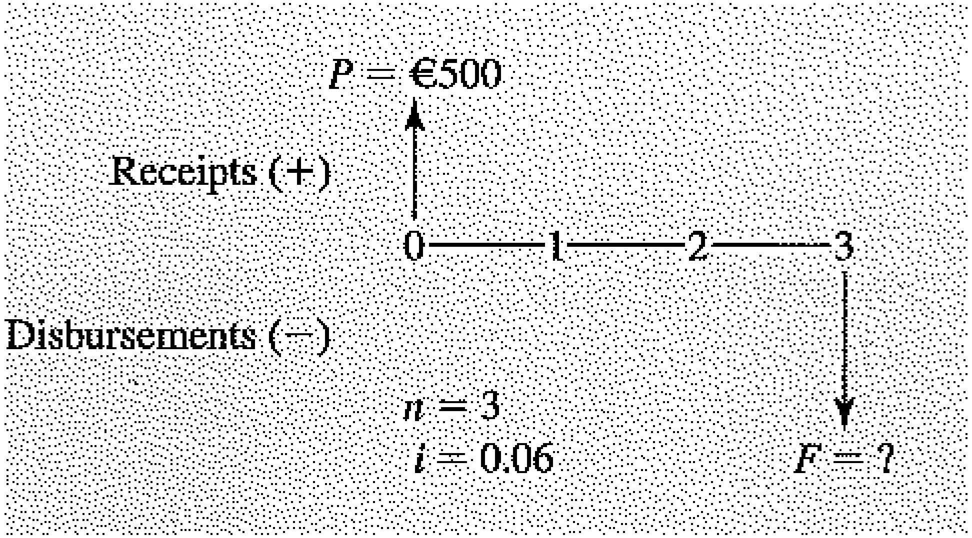
$$F = P(F/P, i, n)$$

$$F = 500(F/P, 6\%, 3)$$

Lets use Appendix B, to find F given P, look in the first column, which is headed "single payment", compound amount factor of F/P for n=3 we find = 1.191

$$F = 500(1.191) = 595.50$$

Lets plot now cash flow diagram from Bank's Point of view

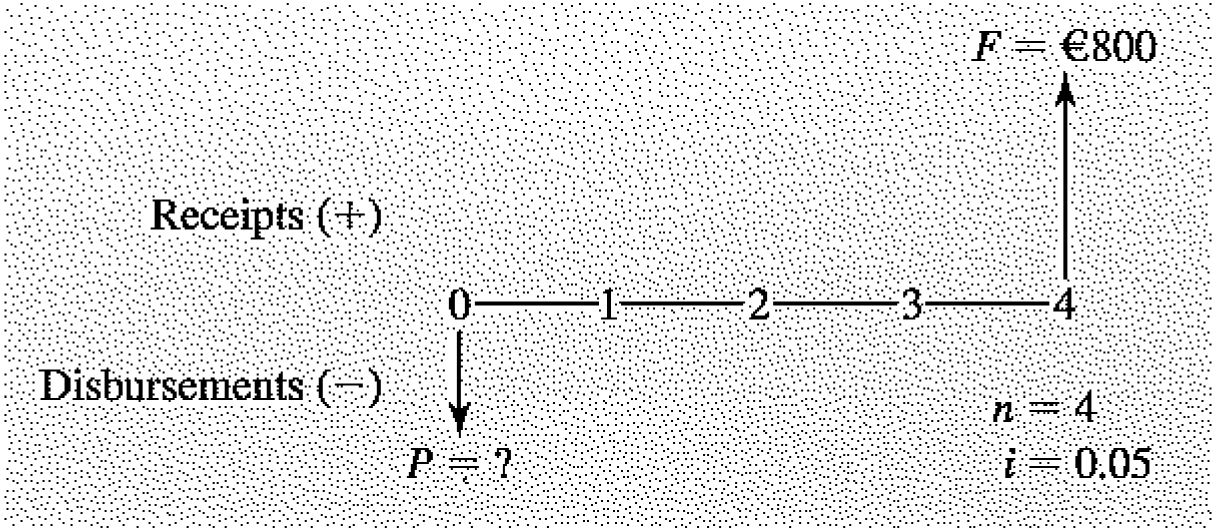


Example: 3-6

- If you wish to have € 800 in a saving account at the end of 4 years and 5% interest will be paid annually, how much should you put into saving account now?
- **Solution**

$F = €800 \quad i = 0.05 \quad n = 4 \quad P = \text{unknown}$

$$P = F(1+i)^{-n} = 800(1+0.05)^{-4} = 800(0.8227) = €658.16$$



Cash Flow Diagram

Example: 3-6

- **Alternate Solution**

$$P = F(P/F, i, n) = €800(P/F, 5\%, 4)$$

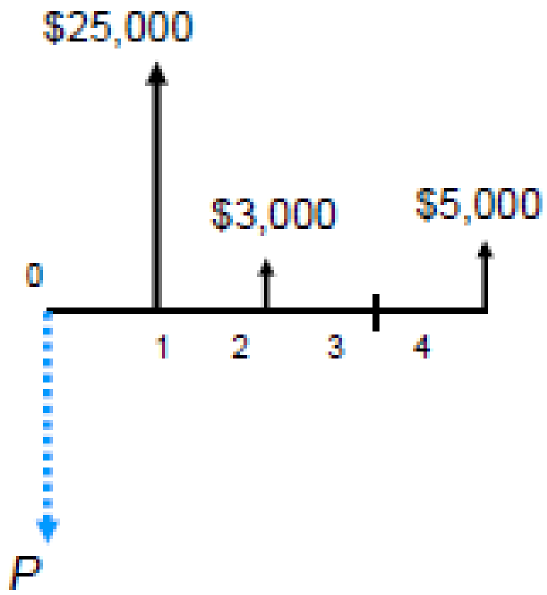
From compound interest table

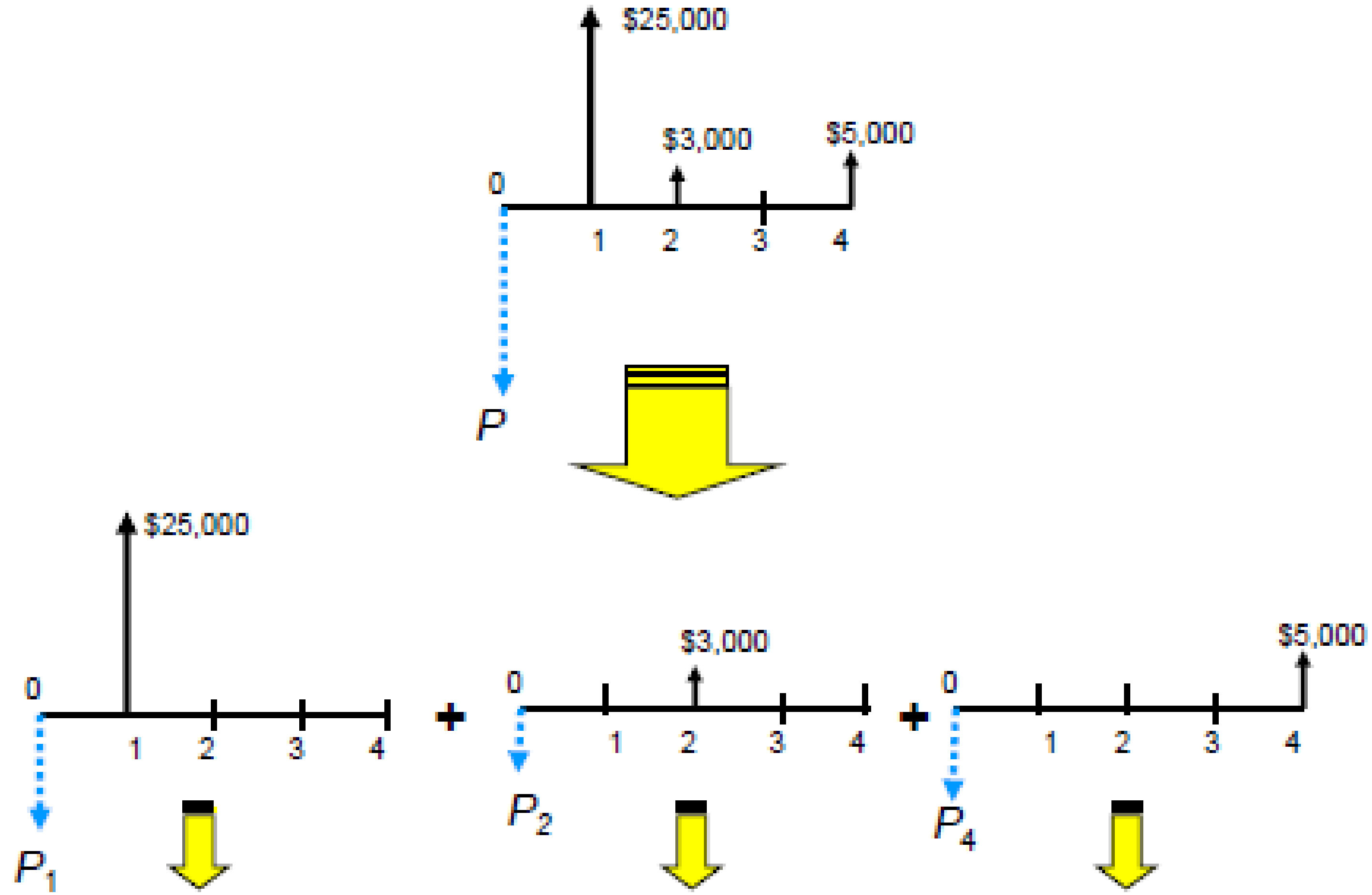
$$(P/F, 5\%, 4) = 0.8227$$

$$P = €800(0.8227) = €658.16$$

Example

- How much do you need to deposit today to withdraw $25,000 after 1 year, $3,000 after 2 yrs, and $5,000 after 4 yrs, if your account earns 10% annual interest?





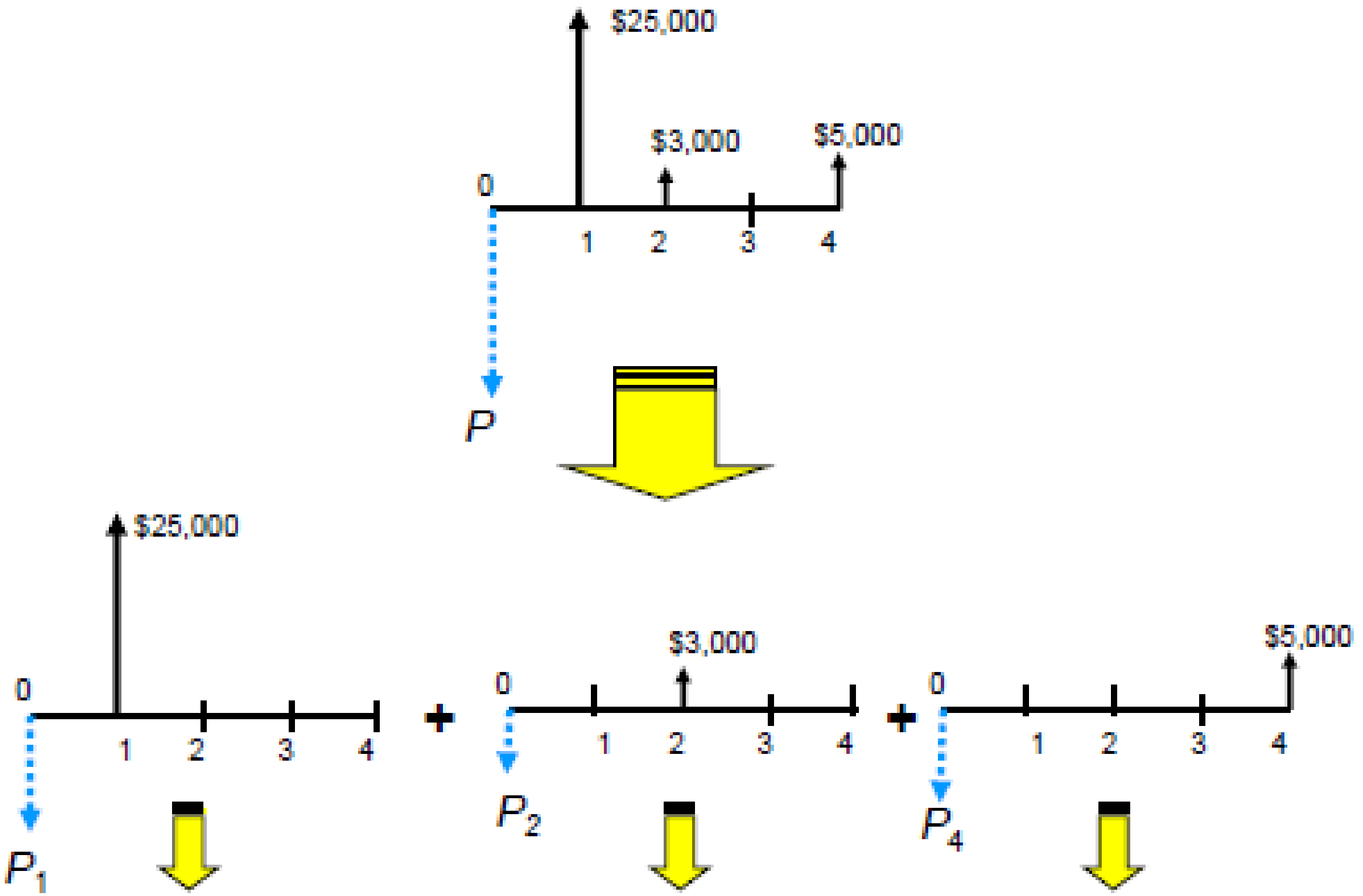
$$\because P = F(1+i)^{-n}$$
$$= 25000(1+0.1)^{-1}$$
$$= 22727.27$$

$$\because P = F(1+i)^{-n}$$
$$= 3000(1+0.1)^{-2}$$
$$= 2479.34$$

$$\because P = F(1+i)^{-n}$$
$$= 5000(1+0.1)^{-4}$$
$$= 3415.07$$

$$P = P_1 + P_2 + P_4 = \$28,622$$

Example



$$P_1 = \$25,000(P/F, 10\%, 1) = \$22,727$$

$$P_2 = \$3,000(P/F, 10\%, 2) = \$2,479$$

$$P_4 = \$5,000(P/F, 10\%, 4) = \$3,415$$

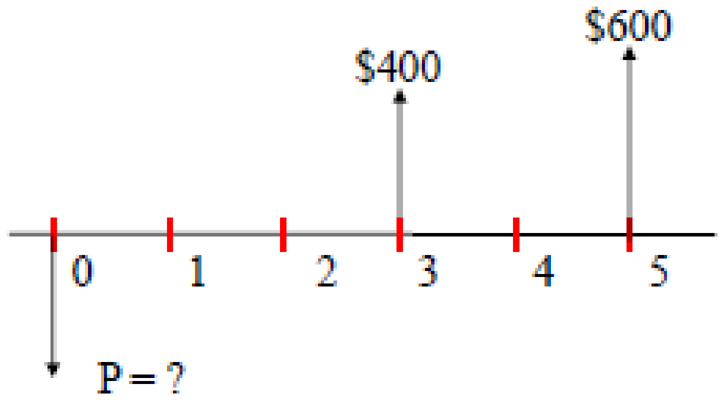
$$P = P_1 + P_2 + P_4 = \$28,622$$

Example

- In 3 years, you need $400 to pay a debt. In two more years, you need $600 more to pay a second debt. How much should you put in the bank today to meet these two needs if the bank pays 12% per year?

Interest is compounded yearly

P = 400(P/F,12%,3) + 600(P/F,12%,5)
= 400 (0.7118) + 600 (0.5674)
= 284.72 + 340.44 = $625.16



Also

$$P = 400(1+12/100)^{-3} + 600(1+12/100)^{-5}$$
$$P = 625.16$$

Appendix B

6%   Compound Interest Factors   6%

	Single Payment		Uniform Payment Series				Arithmetic Gradient		
	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor	Gradient Uniform Series	Gradient Present Worth	
n	Find *F* Given *P* *F/P*	Find *P* Given *F* *P/F*	Find *A* Given *F* *A/F*	Find *A* Given *P* *A/P*	Find *F* Given *A* *F/A*	Find *P* Given *A* *P/A*	Find *A* Given *G* *A/G*	Find *P* Given *G* *P/G*	*n*
1	1.060	.9434	1.0000	1.0600	1.000	0.943	0	0	1
2	1.124	.8900	.4854	.5454	2.060	1.833	0.485	0.890	2
3	1.191	.8396	.3141	.3741	3.184	2.673	0.961	2.569	3
4	1.262	.7921	.2286	.2886	4.375	3.465	1.427	4.945	4
5	1.338	.7473	.1774	.2374	5.637	4.212	1.884	7.934	5
6	1.419	.7050	.1434	.2034	6.975	4.917	2.330	11.459	6
7	1.504	.6651	.1191	.1791	8.394	5.582	2.768	15.450	7
8	1.594	.6274	.1010	.1610	9.897	6.210	3.195	19.841	8
9	1.689	.5919	.0870	.1470	11.491	6.802	3.613	24.577	9
10	1.791	.5584	.0759	.1359	13.181	7.360	4.022	29.602	10