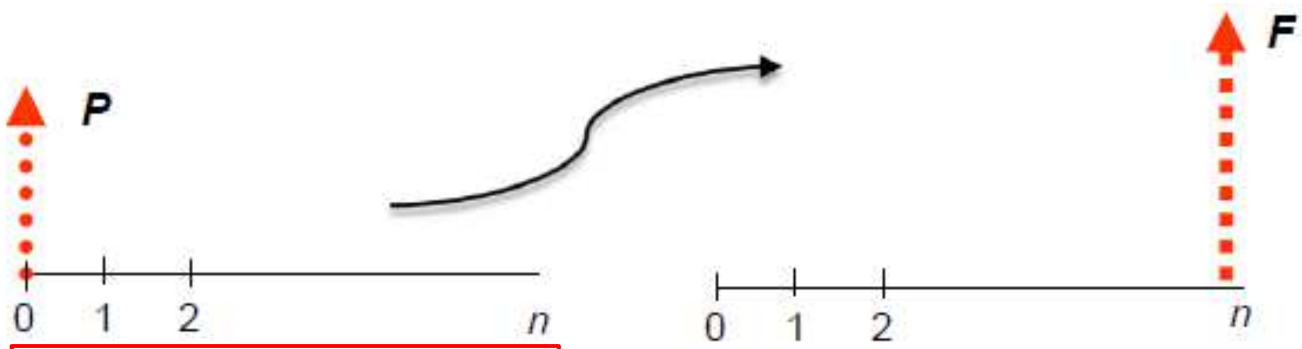
Engineering Economics

More Interest Formulas

Uniform Series

- In chapter 3 (i.e., interest and equivalence), we dealt with single payments compound interest formula:



$$F = P\,(1+i)^{n}$$

$$P = F\,(1+i)^{-n}$$

Algebraic Equivalent

$$F = P\,(F/P,i\%,n)$$

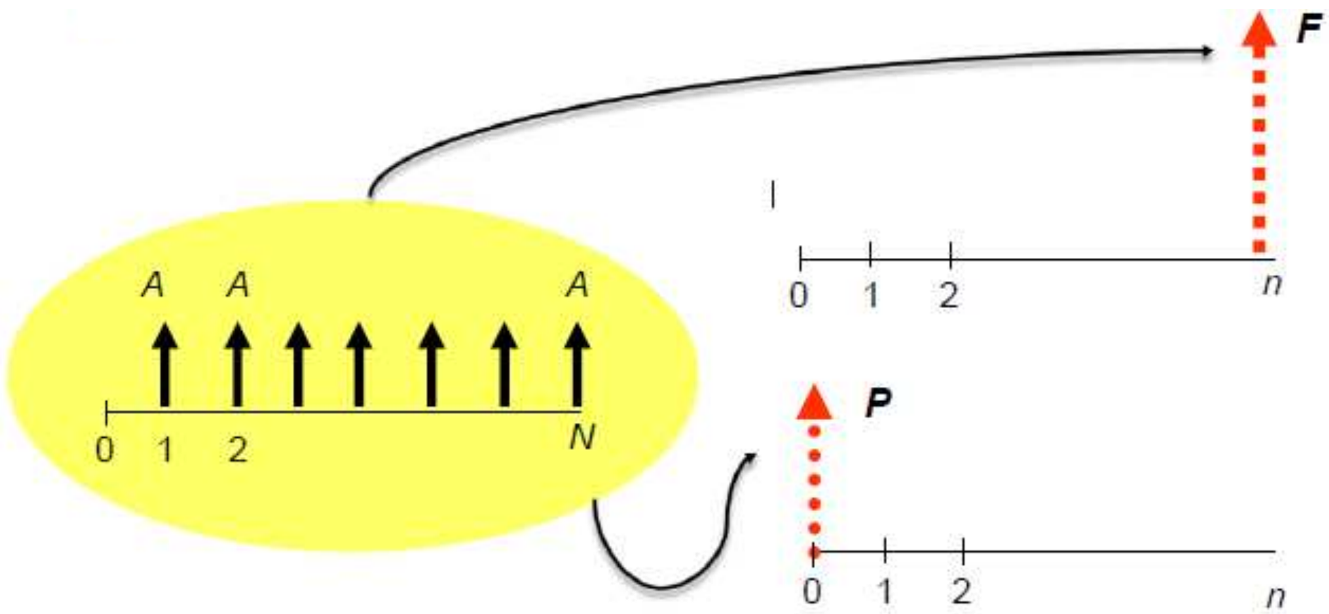
$$P = F\,(P/F,i\%,n)$$

Functional Notation

- Examples:
- ____
____
____

Uniform Series

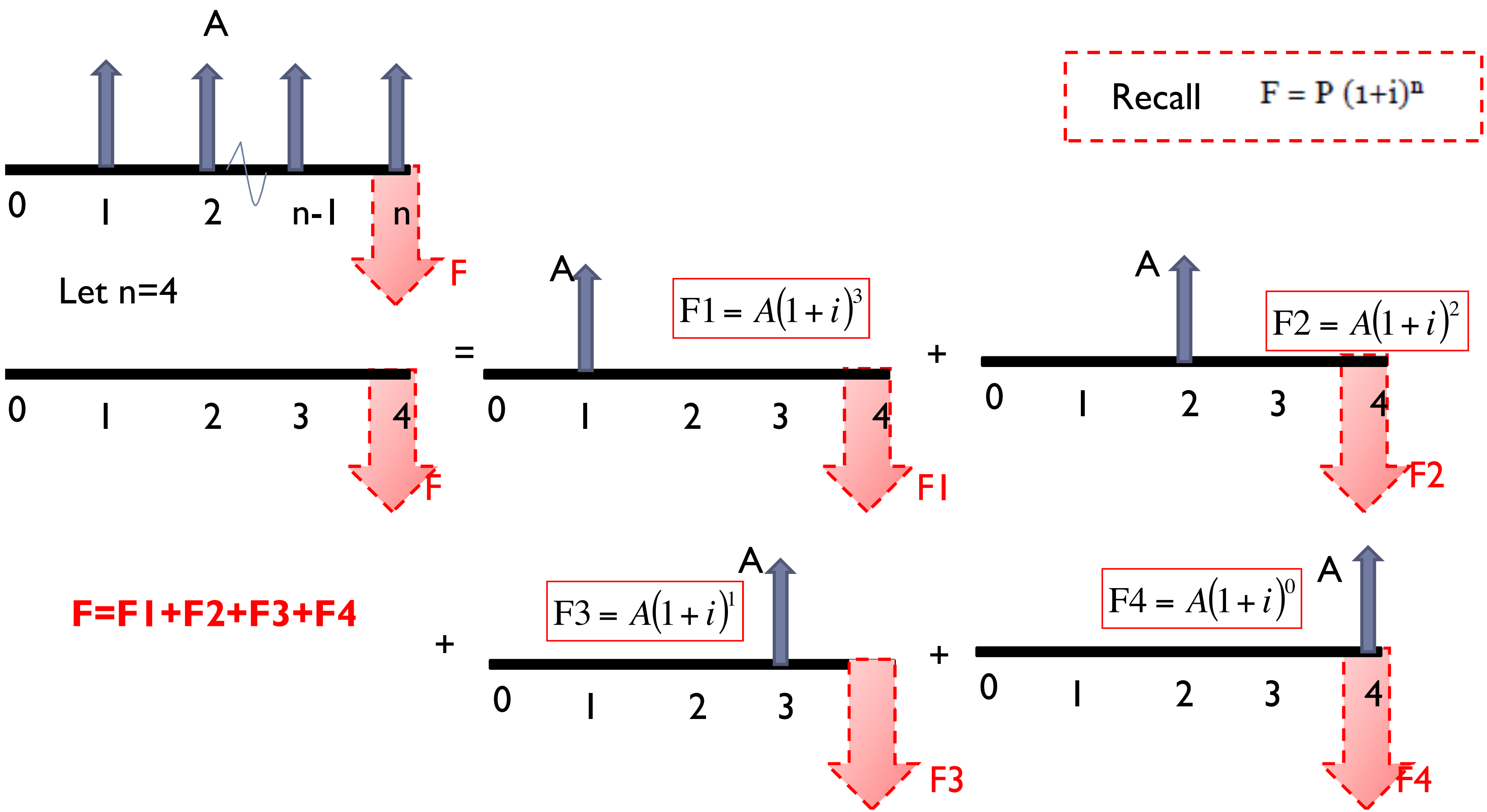
- Quite often we have to deal with uniform (equidistant and equal-valued) cash flows during a period of time:



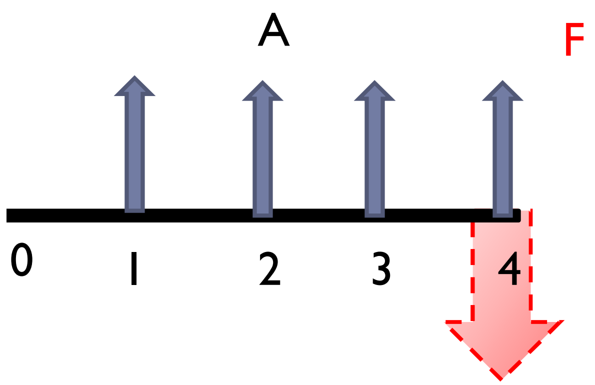
- *Remember: A= Series of consecutives, equal, end of period amounts of money (Receipts/disbursement)*
- Examples: ____
- ____

Deriving Uniform Series Formula

- Let's compute Future Worth, *F*, of a stream of equal, end-of-period cash flows, *A*, *at interest rate*, *i*, *over interest period*, *n*



Deriving Uniform Series Formula



F=F1+F2+F3+F4

$$F = \begin{matrix} F1 = A(1+i)^3 + F2 = A(1+i)^2 + \\ F3 = A(1+i)^1 + F4 = A(1+i)^0 \end{matrix}$$

$$F = A(1+i)^3 + A(1+i)^2 + A(1+i)^1 + A$$

For general case, we can write that

$$F = A(1+i)^{n-1} + A(1+i)^{n-2} + A(1+i)^{n-3} + \ldots + A$$

$$F = A\left[(1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \ldots + 1\right] \quad \text{Eq. (1)}$$

Multiplying both sides with (1+i)

$$F(1+i) = A(1+i)^{n} + A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots + A(1+i)$$

$$F(1+i) = A\left[(1+i)^{n} + (1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)\right] \quad \text{Eq. (2)}$$

Deriving Uniform Series Formula

- Eq. (2)-Eq. (1)

$$F(1+i) = A\left[(1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)\right] \qquad \text{Eq. (2)}$$

$$- \quad F = A\left[(1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \ldots + 1\right] \qquad \text{Eq. (1)}$$

$$iF = A\left[(1+i)^n - 1\right] \qquad \text{Eq. (3)}$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A\left[F/A, i\%, n\right] \qquad \text{Eq. (4)}$$

Where $\left[\frac{(1+i)^n - 1}{i}\right]$ is called uniform series compound amount factor and has notation $\left[F/A, i\%, n\right]$

Deriving Uniform Series Formula

- Eq. (4) can also be written as

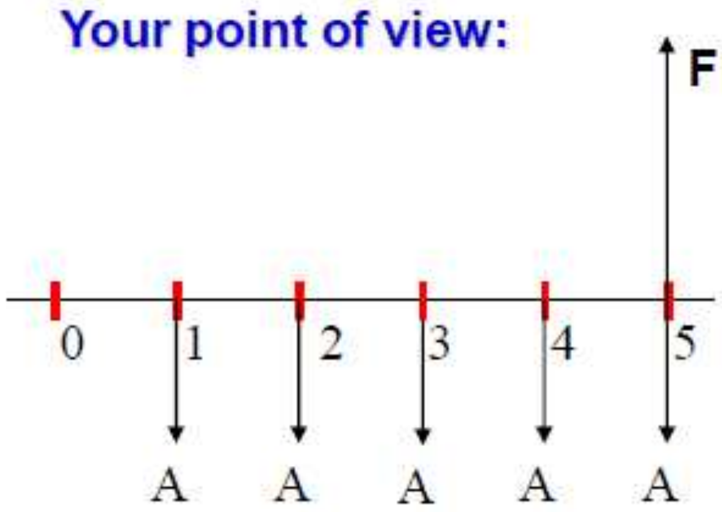
$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F[A/F, i\%, n] \qquad \text{Eq. (5)}$$

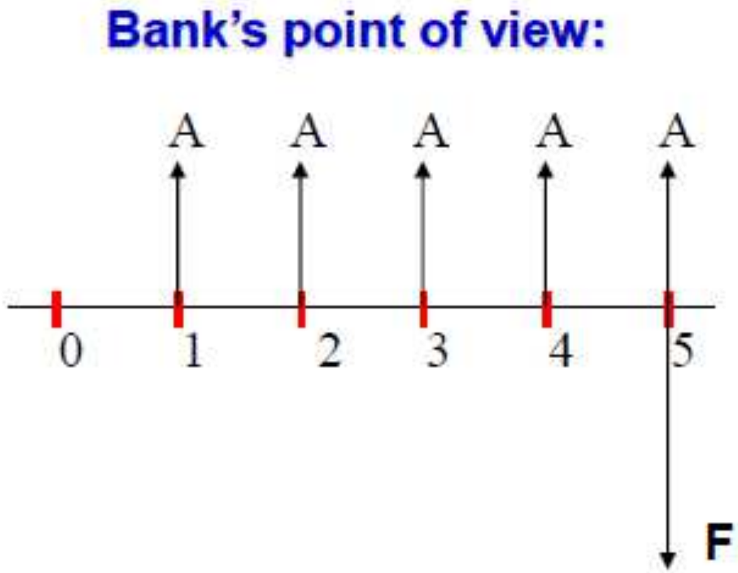
Where $\left[\frac{i}{(1+i)^n - 1}\right]$ is called uniform series sinking fund factor and has notation $[A/F, i\%, n]$

$$\left[\frac{Find}{given}, i\%, n\right]$$

Example 4-1

Example 4-1: You deposit \$500 in a bank at the end of each year for five years. The bank pays 5% interest, compounded annually. At the end of five years, immediately following your fifth deposit, how much will you have in this account?





Solution:

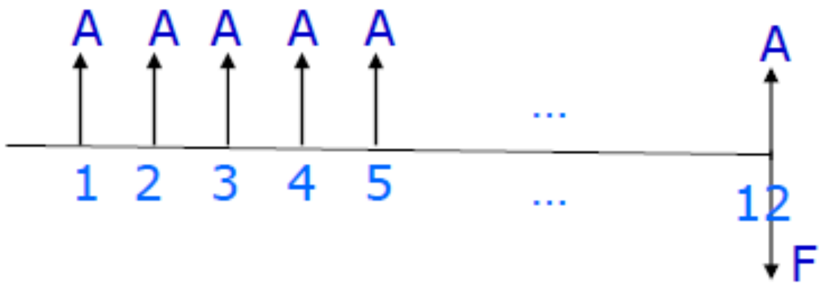
$F = A\,(F/A, i\%, n) = A\,[(1+i)^n - 1]/i$

$= \$500[(1.05)^5 - 1]/(0.05) = \$500\,(5.5256) = \$2{,}762.82 \approx \$2{,}763$

Example 4-2

Example 4-2: How much money do you put in bank every month to have $1,000 at the end of the year. Assume you will put the same amount in the bank each month and the bank pays ½ % interest monthly?

Solution:



A = 1000 (A/F, ½%,12) = 1000 (0.0811) = $81.10/month

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] =$$

Example 4-2

1/2%			Compound Interest Factors						1/2%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor	Gradient Uniform Series	Gradient Present Worth	
	Find *F* Given *P*	Find *P* Given *F*	Find *A* Given *F*	Find *A* Given *P*	Find *F* Given *A*	Find *P* Given *A*	Find *A* Given *G*	Find *P* Given *G*	
n	*F/P*	*P/F*	*A/F*	*A/P*	*F/A*	*P/A*	*A/G*	*P/G*	*n*
1	1.005	.9950	1.0000	1.0050	1.000	0.995	0	0	1
2	1.010	.9901	.4988	.5038	2.005	1.985	0.499	0.991	2
3	1.015	.9851	.3317	.3367	3.015	2.970	0.996	2.959	3
4	1.020	.9802	.2481	.2531	4.030	3.951	1.494	5.903	4
5	1.025	.9754	.1980	.2030	5.050	4.926	1.990	9.803	5
6	1.030	.9705	.1646	.1696	6.076	5.896	2.486	14.660	6
7	1.036	.9657	.1407	.1457	7.106	6.862	2.980	20.448	7
8	1.041	.9609	.1228	.1278	8.141	7.823	3.474	27.178	8
9	1.046	.9561	.1089	.1139	9.182	8.779	3.967	34.825	9
10	1.051	.9513	.0978	.1028	10.228	9.730	4.459	43.389	10
11	1.056	.9466	.0887	.0937	11.279	10.677	4.950	52.855	11
12	1.062	.9419	.0811	.0861	12.336	11.619	5.441	63.218	12
13	1.067	.9372	.0746	.0796	13.397	12.556	5.931	74.465	13
14	1.072	.9326	.0691	.0741	14.464	13.489	6.419	86.590	14
15	1.078	.9279	.0644	.0694	15.537	14.417	6.907	99.574	15

Deriving Uniform Series Formula

- If we use the sinking fund formula (Eq. 5) and substitute the single payment compound amount formula, we obtain

$$\mathrm{A} = F\left[\frac{i}{(1+i)^n - 1}\right] = P(1+i)^n\left[\frac{i}{(1+i)^n - 1}\right] \qquad \because F = P(1+i)^n$$

$$\mathrm{A} = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i\%, n) \qquad \text{Eq. (6)}$$

- It means we can determine the values of A when the present sum P is known

Where $\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$ is called uniform series capital recovery factor and has notation $P(A/P, i\%, n)$

Deriving Uniform Series Formula

- Eq. (6) can be rewritten as

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i\%, n)$$ Eq. (7)

- It means we can determine present sum P when the value of A is known

Where $\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$ is called uniform series present worth factor and has notation $A(P/A, i\%, n)$

Example 4-3

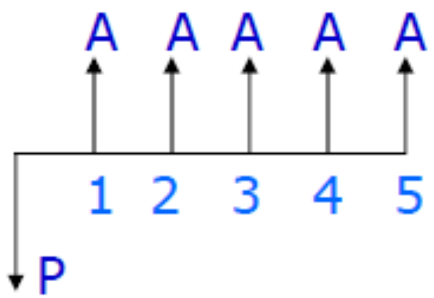
<u>Example 4-3:</u> Suppose on January 1 you deposit $5,000 in a bank paying 8% interest, compounded annually. You want to withdraw all the money in five equal end-of year sums, beginning December 31st of the first year.

<u>Solution</u>:

Given: P = $5000 n = 5 i = 8% A = unknown

$A = P\,(A/P,8\%,5) = P\{[i\,(1+i)^n]/[(1+i)^n - 1]\}$

$= 5000\,(0.2504564545) = \$1{,}252.28$



The withdrawal amount is about $1,252

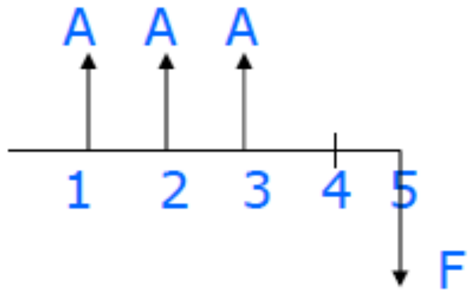
Note: This is the source of the $1,252 in Plan C from Chapter 3

Example 4-6

Compute the value of the following cash flows at the end of year 5 given i = 15%.

Year	Cash flow
1	+ 100
2	+ 100
3	+ 100
4	0
5	- F

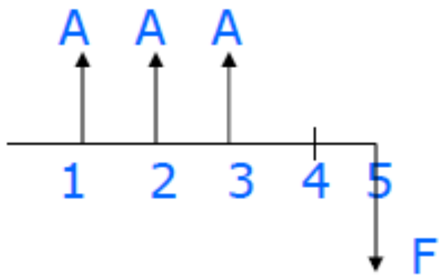
Note the zero



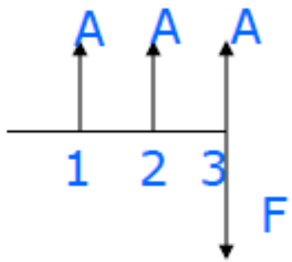
The Sinking Fund Factor diagram is based on the assumption the withdrawal coincides with the last deposit. This does not happen in this example.

Example 6

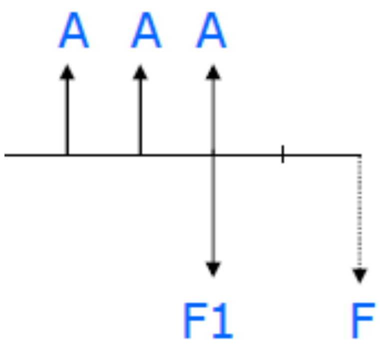
What we have is:



The standard approach is:



<u>First Approach:</u>



Use the “standard” approach to compute F_1.
Then compute the future value of F_1 to get F.

$F_1 = 100\ (F/A,15\%,3) = 100\ (3.472)$

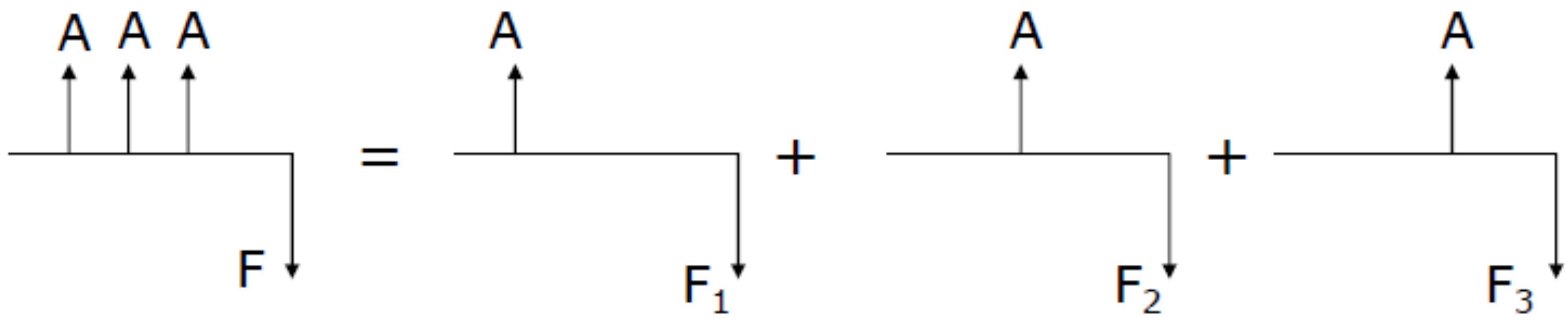
$= \$347.20$

$F = F_1\ (F/P,15\%,2) = 347.20\ (1.322)$

$= \$459.00$

Example 6

<u>Second Approach</u>: Compute the future values of each deposit then add them.



$F = F_1 + F_2 + F_3$

$= 100(F/P,15\%,4) + 100(F/P,15\%,3) + 100(F/P,15\%,2)$

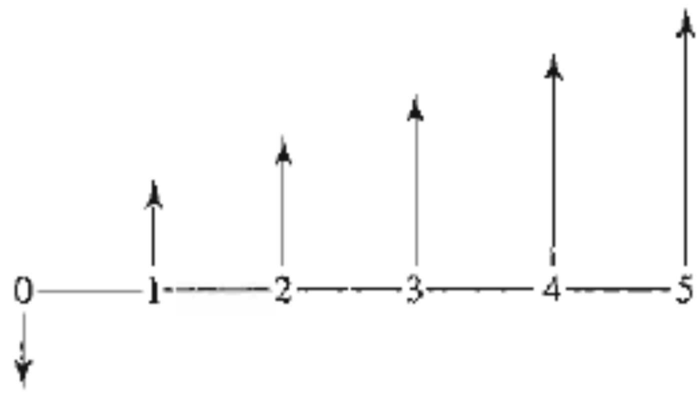
$= 100\ (1.749) + 100\ (1.521) + 100\ (1.322) = \$459.20$

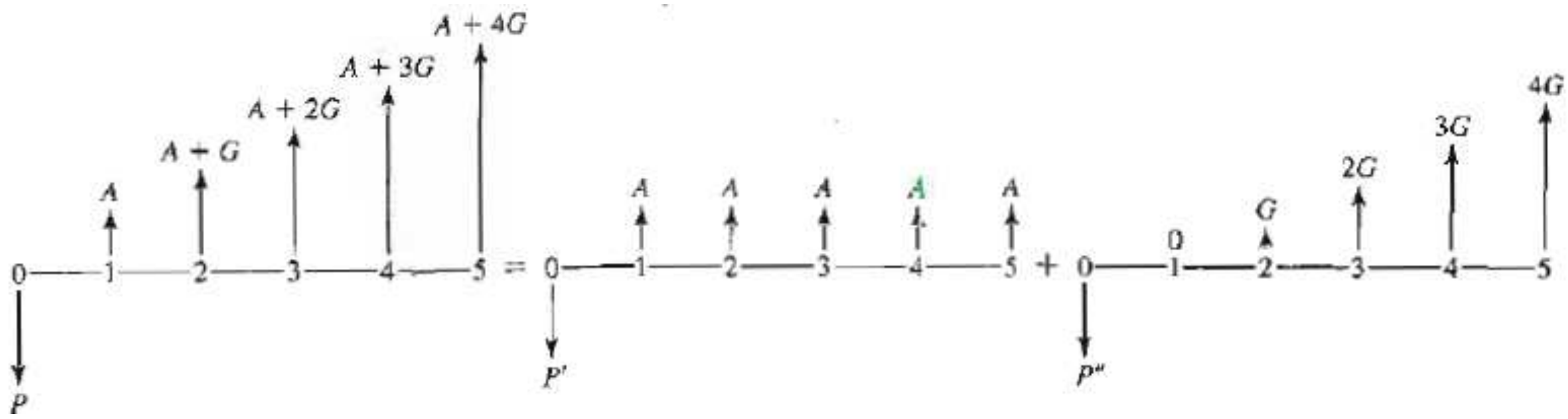
More interest Formulas

- Uniform Series
- Arithmetic Gradient
- Geometric Gradient
- Nominal and Effective Interest
- Continuous Compounding

Arithmetic Gradient Series

- It's frequently happen that the cash flow series is not constant amount.
- It probably is because of operating costs, construction costs, and revenues to increase of decrease from period to period by a constant percentage

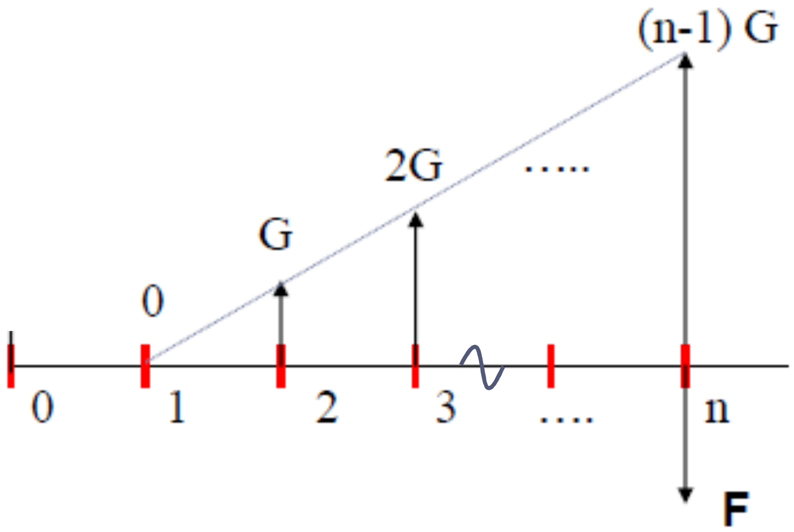




$$P = P' + P'' = A(P/A, i, n) + G(P/G, i, n)$$

Arithmetic Gradient Series

- Let the cash flows increase/decrease by a uniform fixed amount *G* every subsequent period



Recall

$F = P\,(1+i)^n$

Write a future worth value for each period individually, and add them

$$F = G(1+i)^{n-2} + 2G(1+i)^{n-3} + \ldots + (n-2)G(1+i)^{1} + (n-1)G(1+i)^{0}$$

$$F = G\left[(1+i)^{n-2} + 2(1+i)^{n-3} + \ldots + (n-2)(1+i)^{1} + (n-1)\right] \quad \text{Eq. (1)}$$

Arithmetic Gradient Series

- Multiplying Eq. (1) with (1+i), we get

$$(1+\mathrm{i})\mathrm{F} = G\left[(1+i)^{n-1} + 2(1+i)^{n-2} + \ldots + (n-2)(1+i)^{2} + (n-1)(1+i)^{1}\right] \qquad \text{Eq. (2)}$$

- Eq. (2)-Eq. (1)

$$(1+\mathrm{i})\mathrm{F} = G\left[(1+i)^{n-1} + 2(1+i)^{n-2} + \ldots + (n-2)(1+i)^{2} + (n-1)(1+i)^{1}\right]$$

$$- \quad \mathrm{F} = G\left[(1+i)^{n-2} + 2(1+i)^{n-3} + \ldots + (n-2)(1+i)^{1} + (n-1)\right]$$

$$\mathrm{iF} = G\left[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^{2} + (1+i)^{1} - n + 1\right]$$

$$\mathrm{iF} = G\left[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^{2} + (1+i)^{1} + 1\right] - nG \qquad \text{Eq. (3)}$$

$$\mathrm{iF} = G\left[\frac{(1+i)^{n} - 1}{i}\right] - nG$$

$$F = \frac{G}{i}\left[\frac{(1+i)^{n} - 1}{i} - n\right] = G\left[\frac{(1+i)^{n} - 1 - ni}{i^{2}}\right]$$

$$F = G\left[\frac{(1+i)^{n} - in - 1}{i^{2}}\right] = G\left[F/G, i\%, n\right] \qquad \text{Eq. (4)}$$

Arithmetic gradient future worth factor

$$\text{iF}(1+i) = G\left[(1+i)^{n} + (1+i)^{n-1} + \ldots + (1+i)^{3} + (1+i)^{2} + (1+i)\right] - nG(1+i)$$

$$- \quad \text{iF} = G\left[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^{2} + (1+i)^{1} + 1\right] - nG$$

$$iiF = G\left[(1+i)^{n} - 1\right] - nGi$$

$$iF = G\left[\frac{(1+i)^{n} - 1}{i}\right] - nG$$

Arithmetic Gradient Series

- Substituting F from single payment compound formula, we can write Eq.(4) as

Recall

F = P (1+i)^n

$$P = G\left[\frac{(1+i)^n - in - 1}{(1+i)^n i^2}\right] = G[P/G, i\%, n]$$

Eq. (5)

- (P/G ,i%, n) is known as Arithmetic gradient present worth factor
- Now substituting value of F from uniform series compound amount factor, we can write Eq. (4) as

$$F = G\left[\frac{(1+i)^n - in - 1}{i^2}\right] = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\because F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$A = G\left[\frac{i\left((1+i)^n - in - 1\right)}{\left((1+i)^n - 1\right)i^2}\right]$$

$$A = G(A/G, i\%, n)$$

(A/G, i%, n) is known as Arithmetic gradient uniform series factor

Arithmetic Gradient Series

Arithmetic Gradient Present Worth - (P/G, i%, n):

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

Arithmetic Gradient Future Worth - (F/G, i%, n):

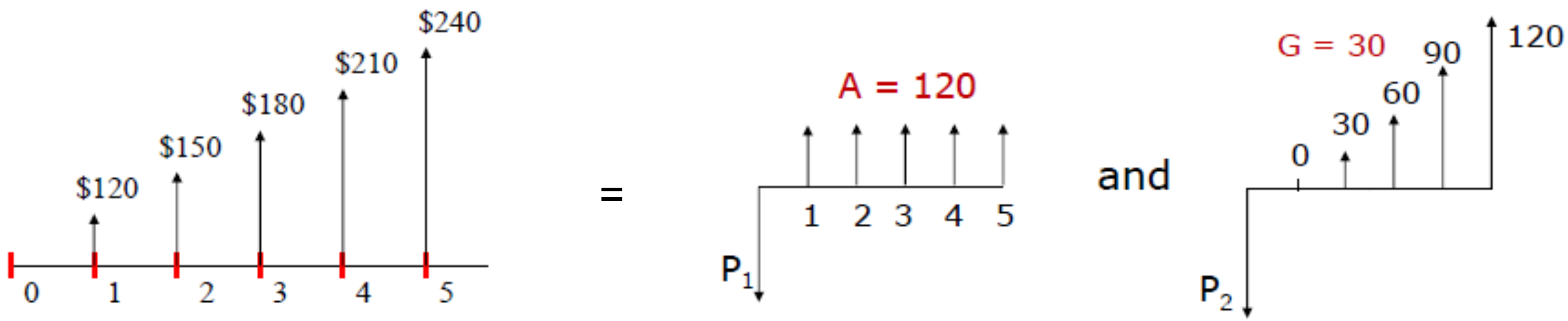
$$F = G\left[\frac{(1+i)^n - in - 1}{i^2}\right]$$

Arithmetic Gradient Uniform Series - (A/G, i%, n):

$$A = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

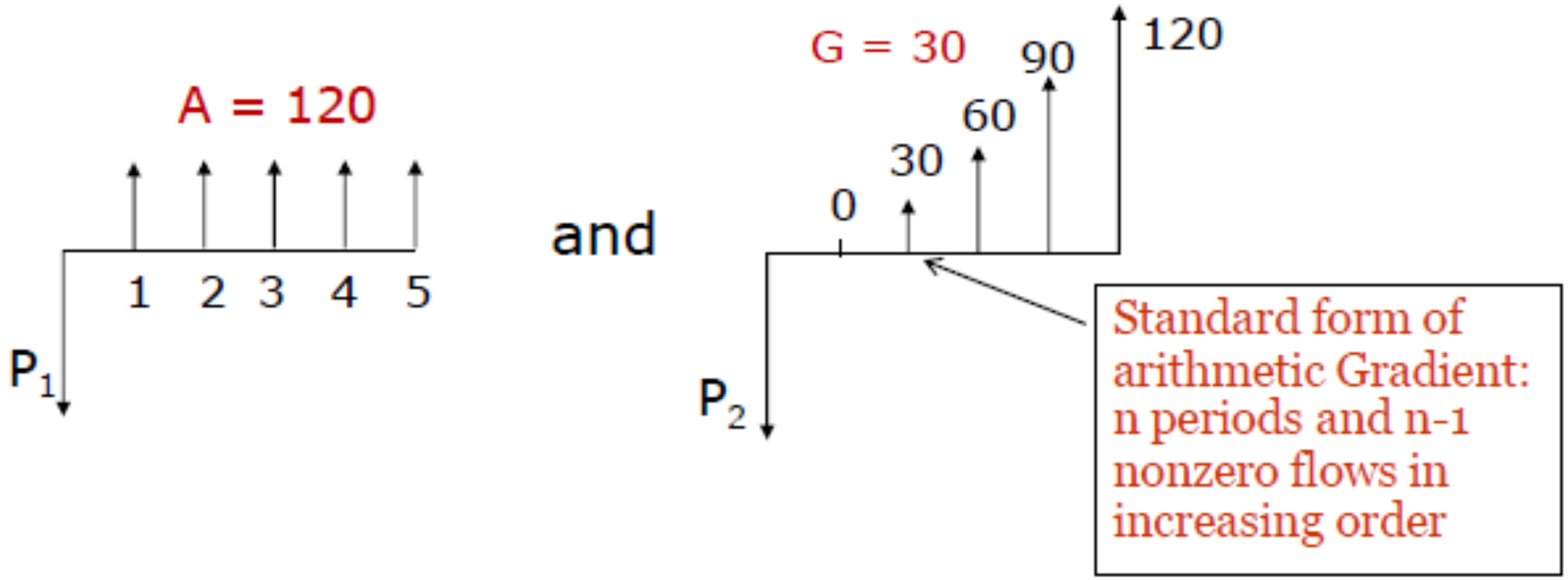
Example 8

- Suppose you buy a car. You wish to set up enough money in a bank account to pay for standard maintenance on the car for the first five years. You estimate the maintenance cost increases by G = \$30 each year. The maintenance cost for year 1 is estimated as \$120. i = 5%. Thus, estimated costs by year are \$120, \$150, \$180, \$210, \$240.



Example 4-8

We break up the cash flows into two components:

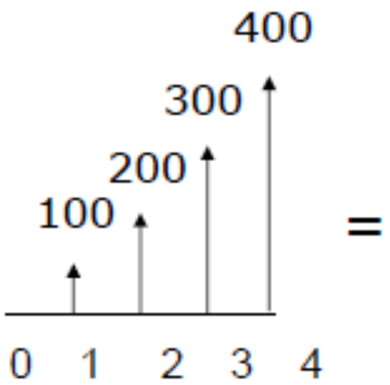


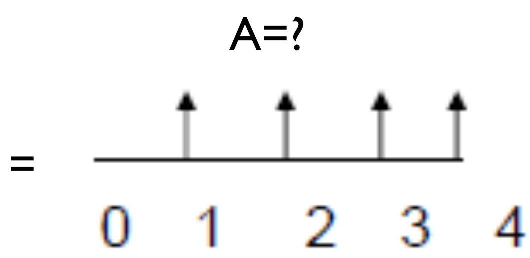
P_1 = A (P/A,5%,5) = 120 (P/A,5%,5) = 120 (4.329) = 519
P_2 = G (P/G,5%,5) = 30 (P/G,5%,5) = 30 (8.237) = 247
P = P_1 + P_2 = \$766.

Note: 5 and not 4. Using 4 is a common mistake.

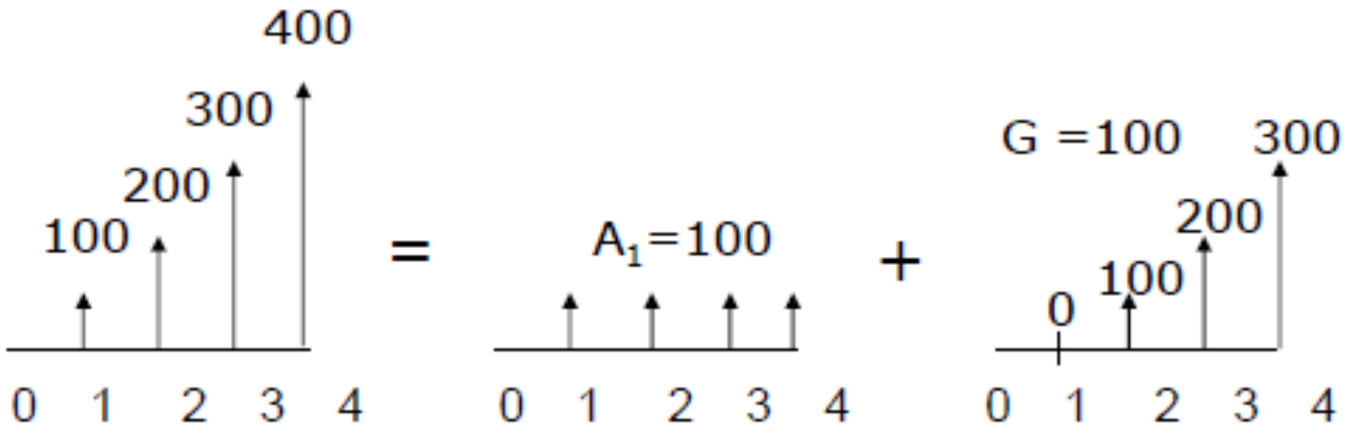
Example 9

- Maintenance costs of a machine start at $100 and go up by $100 each year for 4 years. What is the equivalent uniform annual maintenance cost for the machinery if i= 6%.





Example 9



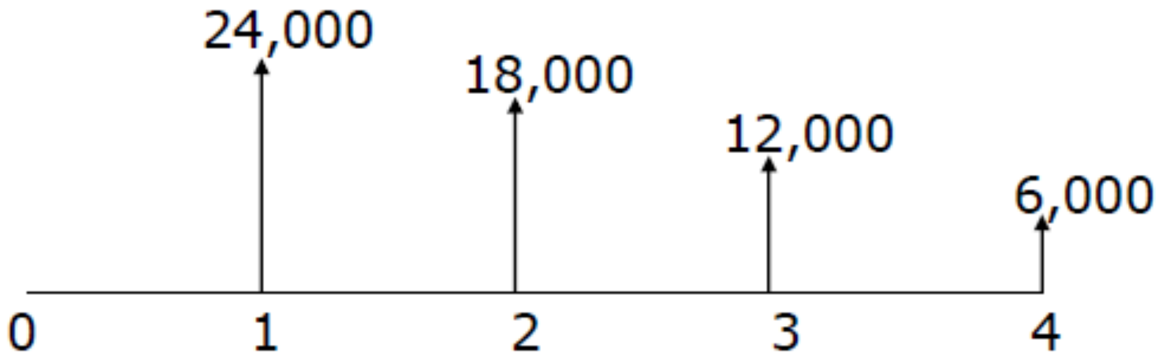
- First part is in the form of a \$100 uniform series.
- Second part is now in the standard form for the gradient equation with $n = 4$, $G = 100$

$A = A_1 + G\,(A/G, 6\%, 4) = 100 + 100\,(1.427)$

$= \$242.70$

Example 4-10

- *Example 4-10*: With i = 10%, n = 4, find an equivalent uniform payment A for the following CFD

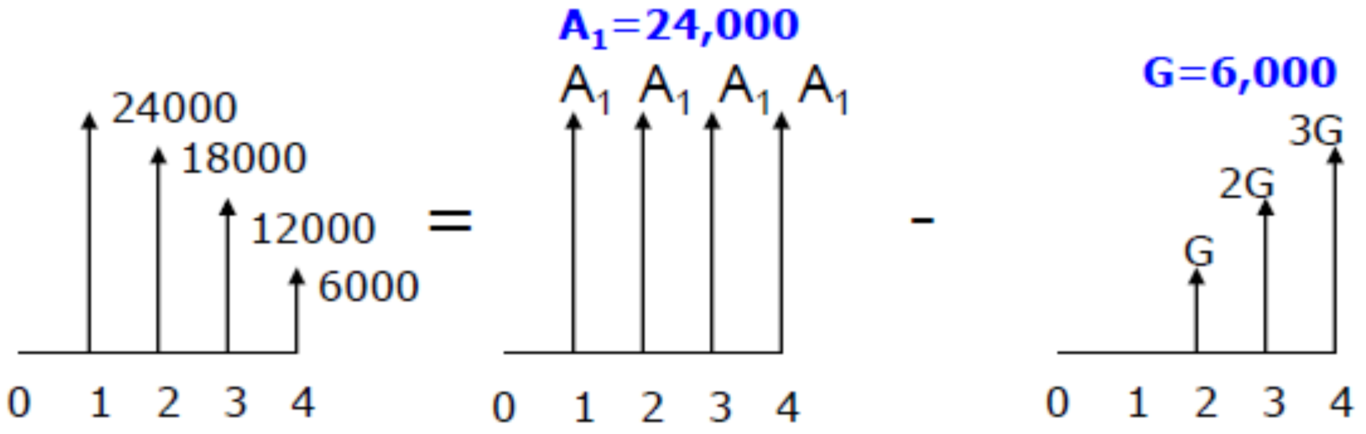


  - This is a problem with decreasing costs instead of increasing costs.

Solution:

- The cash flow can be rewritten as the DIFFERENCE of the following two diagrams: (1) the standard form we need for arithmetic gradient, and (2) a series of uniform payments.

Example 10



$$A = A_1 - G(A/G,10\%,4)$$

$$= 24{,}000 - 6{,}000\,(A/G,10\%,4)$$

$$= 24{,}000 - 6{,}000(1.381)$$

$$= 15{,}714$$