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Generalized Conjugacy Class Graph of Some Finite Non-Abelian Groups

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Abstract. The graph related to conjugacy classes is a graph whose vertices are non central conjugacy classes, where two vertices are connected by an edge if they are not coprime. In this paper, we generalize the graph related to conjugacy classes by defining a graph called generalized conjugacy classes graph. This graph is found for some groups. Besides, some graph properties are provided.

Keywords: Conjugacy class graph, graph theory, graph properties.
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INTRODUCTION

Throughout this paper, Γ denotes a simple undirected graph and G denotes a finite non-abelian group. In the following we state a brief history about graph theory followed by some fundamental concepts in graph theory.

The first appearance of graph theory was in 1736 when Leonard Euler considered Konigsberg bridge problem. Euler solved this problem by drawing a graph with points and lines. Years later, the usefulness of graph theory has been proven to a large number of devise fields.

The followings are some basic concepts of graph theory that are needed in this paper. These concepts can be found in one of the references ([1], [2]).

A graph Γ is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise, Γ is called indirected. Two vertices are adjacent if they are linked by an edge. A complete graph is a graph where each ordered pair of distinct vertices are adjacent, denoted by K_n.

A non-empty set S of $V(\Gamma)$ is called an independent set of Γ if there is no adjacent between two elements of S in Γ. Meanwhile, the independent number is the number of vertices in the maximum independent set and it is denoted by $\alpha(\Gamma)$. However, the maximum number c for which Γ is c -vertex colorable is known as chromatic number, denoted by $\chi(\Gamma)$. The diameter is the maximum distance between any two vertices of Γ, denoted by $d(\Gamma)$. Furthermore, a clique is a complete subgraph in Γ, while the clique number is the size of the largest clique in Γ and is denoted by $\omega(\Gamma)$. The dominating set $X \subseteq V(\Gamma)$ is a set where for each v outside X, there exists $x \in X$ such that v is adjacent to x. The minimum size of X is called the dominating number and it is denoted by $\gamma(\Gamma)$ ([1], [2], [3]).

Lemma 1: [1] If Γ is a simple graph with $v \geq 3$ and $\deg(v) \geq \frac{|v(\Gamma)|}{2}$, , then Γ is Hamiltonian.

In 1990, a new graph called graph related to conjugacy class was introduced by Bertram *et al.* [4]. The vertices of this graph are non central conjugacy classes i.e $|V(\Gamma)| = K(G) - |Z(G)|$, where $K(G)$ is the number of

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conjugacy class of a group and $Z(G)$ is the center of a group G. A pair of vertices of this graph are connected by an edge if their cardinalities are not coprime.

This paper is divided into three sections. The first section focuses on some background about some topics in graph theory and algebra, while the second section provides some earlier and recent publication that are related to conjugacy class graph. In the third section, we presented our results on which include generalized conjugacy class graph.

PRELIMINARIES

In this section, some works that are related to the probability that an element of a group fixes a set and graph theory are stated. We commence with brief information about the probability of a group element fixes a set, followed by some related work on graph theory more precisely to graph related to conjugacy classes.

In 2013, the probability that a group element fixes a set was introduced by Omer *et al.* [5]. Their work then has been extended to finding the probability for some finite non-abelian groups. In this paper, we connect this concept to graph theory by using the conjugacy classes that are obtained under group action on a set to graph conjugacy class.

Some related works on conjugacy class graph include.

Bianchi *et al*. [6] who studied the regularity of the graph related to conjugacy classes and provided some results. In addition, Moreto *et al*. [7] in 2005 classified the groups in which conjugacy classes sizes are not coprime for any five distinct classes. Furthermore, You *et al.* [8] also classified the groups in which conjugacy classes are not set-wise relatively prime for any four distinct classes. Ilangovan and Sarmin [9], found some graph properties of graph related to conjugacy classes of two- generator two-groups of class two. Moreover, Moradipour *et al.* [10] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups

RESULTS AND DISCUSSION

In this section, we introduce our main results, where the graph related to conjugacy classes graph is generalized by defining a graph called the generalized conjugacy class graph. We shorten the notation to $\Gamma_G^{\Omega_c}$ for convenience. The vertices of this graph are non-central conjugacy classes under group action on a set. A pair of vertices is adjacent if their cardinalities are not coprime. The definition of generalized conjugacy class graph is given in the following.

Definition 1: Let G be a finite non-abelian group and let Ω be a set of G. If G acts on Ω, the vertices of generalized conjugacy class graph are $K(\Omega)-|A|$, where $K(\Omega)$ is non-central conjugacy classes under group action on Ω and $A=\{\omega\in\Omega, g\omega=\omega g, g\in G\}$. Two vertices in $\Gamma_G^{\Omega_c}$ are connected by an edge if their cardinalities are not set-wise relatively prime.

The following proposition clarifies the generalized conjugacy class graph properties.

Proposition 1: Let G be a finite non-abelian group and Ω be a set. If G acts on Ω, then the generalized conjugacy class graph has the following properties:

$$\Gamma_G^{\Omega_c}=\begin{cases}(i) & \chi\left(\Gamma_G^{\Omega_c}\right) & = & \min\{cl(v_i), v_i\in\Omega\},\\ (ii) & \omega\left(\Gamma_G^{\Omega_c}\right) & = & \max\{cl(v_i), v_i\in\Omega\},\\ (iii) & \alpha\left(\Gamma_G^{\Omega_c}\right) & = & K(\Omega)-|A|,\\ (iv) & \gamma\left(\Gamma_G^{\Omega_c}\right) & = & K(\Omega)-|A|,\\ (v) & d\left(\Gamma_G^{\Omega_c}\right) & = & \max\{d(v_i,v_j), v_i,v_j\in\Omega\}.\end{cases}$$

Proof: The proof is clear since the chromatic number $\chi\left(\Gamma_G^{\Omega_c}\right)$ is the minimum number of coloring vertices in $\Gamma_G^{\Omega_c}$. However, the maximum number of complete subgraphs is the maximum conjugacy class. The proof is then straightforward.

Proposition 2: Let G be a finite non-abelian group and Ω be a set. Let G acts on Ω and $\Gamma_G^{\Omega_c}$ be its generalized conjugacy class graph. Then generalized conjugacy class graph $\Gamma_G^{\Omega_c}$ is not a connected graph.

Proof: Assume that $\Gamma_G^{\Omega_c}$ is a connected graph. Then there exists v_1 and $v_2 \in \Gamma_G^{\Omega_c}$ such that v_1 is adjacent to v_2 since $C(v_1) \bigcup C(v_2) \subseteq C(G)$. However, there exists $v_3 \in \Gamma_G^{\Omega_c}$ such that v_3 is not adjacent to either v_1 or v_2, a contradiction. Then, $\Gamma_G^{\Omega_c}$ is not a connected graph.

In the following, we find the generalized conjugacy class graph in which a group acts on the set of all subsets of commuting elements of size two in the form of (a,b), namely Ω. The graph is found for some finite groups, starting with the dihedral group of order $2n$. .

Theorem 1: Let G be a finite non-abelian dihedral group of order $2n$. Let Ω be the set of all subsets of commuting elements of size two of the form of (a,b). If G acts on Ω by conjugation, then the generalized conjugacy class graph

$$\Gamma_G^{\Omega_c} = \begin{cases} K_5, & \text{if } n \text{ is even and } \dfrac{n}{2} \text{ is odd,} \\ K_4, & \text{if } n = 4 \text{ and } \dfrac{n}{2} \text{ is even,} \\ K_6, & \text{if } n \text{ is even, } n \neq 4 \text{ and } \dfrac{n}{2} \text{ is even,} \\ \text{an empty graph,} & \text{otherwise.} \end{cases}$$

Proof: In accordance with Proposition 3.2 in [5], there are three cases. The first case is when n is even and $\dfrac{n}{2}$ is odd. Based on the same proposition, the number of conjugacy classes is six, where five of them are of order $\dfrac{n}{2}$ and one is of order one. By Definition 1, the number of vertices is five. Since two vertices are adjacent if their cardinalities are not coprime, thus we have complete graph of K_5. However, in the second case in which n and $\dfrac{n}{2}$ are even, there are four conjugacy class of order $\dfrac{n}{2}$ and two conjugacy classes of order $\dfrac{n}{4}$. Since $\gcd\left(\dfrac{n}{2},\dfrac{n}{4}\right) \neq 1$ unless $n = 4$, hence $\Gamma_G^{\Omega_c}$ consists of complete component of K_4 if $n = 4$ and complete graph of K_6 if n and $\dfrac{n}{2}$ are even and $n \neq 4$. The third case is if n is odd, then there is only one conjugacy class. Hence the graph is an empty graph. The proof then follows.

According to the above theorem, the following proposition can be concluded.

Proposition 3: Let G be a finite non-abelian dihedral group of order $2n$. Let Ω be the set of all subsets of commuting elements of size two of the form of (a,b). If G acts on Ω by conjugation and the generalized conjugacy class graph

$$\Gamma_G^{\Omega_c} = \begin{cases} K_5, & \text{if } n \text{ is even and } \dfrac{n}{2} \text{ is odd,} \\ K_4, & \text{if } n = 4 \text{ and } \dfrac{n}{2} \text{ is even,} \\ K_6, & \text{if } n \text{ is even, } n \neq 4 \text{ and } \dfrac{n}{2} \text{ is even,} \\ \text{an empty graph,} & \text{otherwise.} \end{cases}$$

, then $\chi\left(\Gamma_G^{\Omega_c}\right) = \omega\left(\Gamma_G^{\Omega_c}\right) = 5$ if n is even and $\dfrac{n}{2}$ is odd. Next, $\chi\left(\Gamma_G^{\Omega_c}\right) = \omega\left(\Gamma_G^{\Omega_c}\right) = 4$ if $n = 4$ and $\chi\left(\Gamma_G^{\Omega_c}\right) = \omega\left(\Gamma_G^{\Omega_c}\right) = 6$ if n is even, $n \neq 4$ and $\dfrac{n}{2}$ is odd.

Proof: In the first case, the chromatic number and the clique number are equal to five since the generalized conjugacy class consists of a complete graph of five vertices. However, in the second case when $n = 4$ $\chi\left(\Gamma_G^{\Omega_c}\right) = \omega\left(\Gamma_G^{\Omega_c}\right) = 4$, since $\Gamma_G^{\Omega_c}$ contains a complete graph of K_4. The chromatic number and clique number are equal to six when n and $\dfrac{n}{2}$ are even. Otherwise, there is no chromatic number and clique number since generalized conjugacy class graph is empty.

In the following, we find the generalized conjugacy class graph of quaternion groups and its generalized group.

Theorem 2: Let G be a finite quaternion group of order 2^n. Let G be S a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation, then $\Gamma_G^{\Omega_c}$ is a null graph.

Proof: According to Theorem 3.2 in [5], $\Gamma_G^{\Omega_c}$ is a null graph since we have only one conjugacy class, namely Ω.

Theorem 3: Let G be a finite non-abelian semi-dihedral group, $G \cong \langle a,b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$. Let G be S a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $\Gamma_G^{\Omega_c} = K_3$.

Proof: Based on Theorem 3.1 in [11], there are four conjugacy classes where two of them are of order 2^{n-1} and one of order 2^{n-2}. Thus the number of vertices in $\Gamma_G^{\Omega_c}$ is three. Since two vertices are adjacent if their cardinalities are set-wise relatively prime, hence $\gcd\left(2^{n-1}, 2^{n-2}\right) \neq 1$. Therefore it consists of a complete graph of K_3, as claimed.

Based on the Theorem 3, the following corollary can be concluded.

Corollary 1: Let G be a finite non-abelian semi-dihedral group, $G \cong \langle a,b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $\Gamma_G^{\Omega_c} = K_3$. then $\chi\left(\Gamma_G^{\Omega_c}\right) = \omega\left(\Gamma_G^{\Omega_c}\right) = 3$.

Proof: The proof is clear since $\Gamma_G^{\Omega_c}$ consists of one complete graph.

Corollary 2: Let G be a finite non-abelian semi-dihedral group, $G \cong \langle a,b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$. Let G be S a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $\Gamma_G^{\Omega_c} = K_3$, then $\Gamma_G^{\Omega_c}$ is connected and Hamiltonian.

Proof: The generalized conjugacy class $\Gamma_G^{\Omega_c}$ is connected since it is a complete graph of three vertices and based on Lemma 1 it is a Hamiltonian graph.

CONCLUSION

In this paper, the graph related to conjugacy class has been generalized by defining a graph called generalized conjugacy class graph whose vertices are non-central conjugacy classes under group action on a set. Furthermore, the generalized conjugacy class graph was found for some finite non-abelian groups including the dihedral groups, quaternion groups and semi-dihedral groups where some graph properties were obtained.

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