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On the Capability of Nonabelian Groups of Order p^4

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Abstract. A group is capable if it is isomorphic to the central factor group. For finite groups of prime power order, the capability is closely related to their classification. In this research, by using the classification of groups of order p^4, we determine the capability of this group.

Keywords: Capability; Groups of order p^4;
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INTRODUCTION

Capability of groups was first introduced by Baer in [1]. A group G is capable if and only if it is isomorphic to the central factor group of another group. A group G is also said to be capable if and only if it is isomorphic to the inner automorphism group of a group H. Ellis in [2] proved that a group G is capable if and only if its exterior center, $Z^{\wedge}(G)$ is trivial, where $Z^{\wedge}(G)=\{g\in G \mid g\wedge x=1_{\wedge} \text{ for all } x\in G\}$ and $1_{\wedge}$ denotes the identity in the exterior square.

In 1979, Beyl *et al*. [3] proved that an extra special p-group is capable if and only if it is either a dihedral group of order 8 or of order p^3 and exponent $p > 2$ by characterizing the capable extra special p-groups in term of its epicenter. They found that a group is capable if and only if the epicenter, $Z^{*}(G)$ of the group is trivial, where

$$Z^{*}(G)=\bigcap\{\phi Z(E) \mid (E,\phi) \text{ is a central extension of } G\}.$$

Later, Niroomand and Parvizi in [4] classified all capable p-groups with commutator subgroup of order p and the abelianization of groups in term of its Schur multiplier, epicenter and exterior square. In 2012, the capability of groups of order p^2q has been computed by Samad *et al*. in [5]. Recently, Samad *et al*. in [6] determined the capability of groups of order p^3q, where p and q are distinct primes and $p < q$.

PRELIMINARIES

In this section, some preliminary results from previous researchers are stated. Those results are needed in determining the capability of all groups of order p^4 in the classification.

The classification of groups of order p^4, where p is an odd prime has been given by Burnside [7] is stated in the following theorem.

Theorem 2.1 [7] Let G be a group of order p^4, where p is an odd prime. Then exactly one of the following holds:

$$G\cong\mathbb{Z}_{p^4}. \quad (1)$$

$$G\cong\mathbb{Z}_{p^3}\times\mathbb{Z}_{p}. \quad (2)$$

$$G\cong\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2}. \quad (3)$$

$$G\cong\mathbb{Z}_{p^2}\times\mathbb{Z}_{p}\times\mathbb{Z}_{p}. \quad (4)$$

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$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \quad (5)$$

$$G \cong \left\langle x, y \middle| x^{p^3} = y^p = 1, x^y = x^{1+p^2} \right\rangle. \quad (6)$$

$$G \cong \left\langle x, y, z \middle| x^p = y^p = z^{p^2} = 1, [x, y] = [y, z] = 1, [x, y] = z^p \right\rangle. \quad (7)$$

$$G \cong \left\langle x, y \middle| x^{p^2} = y^{p^2} = 1, x^y = x^{1+p} \right\rangle. \quad (8)$$

$G \cong N \times \langle w \rangle$, *where*

$$N \cong \left\langle x, y, z \middle| x^p = y^p = z^p = 1, [x, y] = [y, z] = 1, [x, y] = z \right\rangle, \quad (9)$$

$$\langle w \rangle \cong \left\langle w \middle| w^p = 1 \right\rangle.$$

$$G \cong \left\langle x, y, z \middle| x^{p^2} = y^p = z^p = 1, [x, z] = [y, z] = 1, x^y = x^{1+p} \right\rangle. \quad (10)$$

$$G \cong \left\langle x, y, z \middle| x^{p^2} = y^p = z^p = 1, [x, y] = z, [x, z] = [y, z] = 1 \right\rangle. \quad (11)$$

$$G \cong \left\langle x, y, z \middle| x^{p^2} = y^p = z^p = 1, yx = x^{1+p} y, zx = x^{1+p} yz, zy = x^p yz \right\rangle. \quad (12)$$

$$G \cong \left\langle x, y, z \middle| x^{p^2} = y^p = z^p = 1, yx = x^{1+p} y, zx = x^{1+dp} yz, zy = x^{dp} yz, d \neq 0, 1 \pmod{p} \right\rangle. \quad (13)$$

$$G \cong \left\langle w, x, y, z \middle| w^p, x^p = y^p = z^p = 1, zy = wyz, xz = zx, wz = zw, xy = yx, wy = yw, wx = xw \right\rangle. \quad (14)$$

$$G \cong \left\langle w, x, y, z \middle| w^p, x^p = y^p = z^p = 1, zy = xyz, zx = wxz, wz = zw, xy = yx, wy = yw, wx = xw \right\rangle. \quad (15)$$

This classification consists of five abelian groups which are groups of type (1) to (5) and ten nonabelian groups; they are group of type (6) to (15). In this paper, we only consider the nonabelian groups of type (6) to (11).

The result concerning the commutator subgroup and center of groups of order p^4 is stated in the next theorem:

Theorem 2.2 [8] Given G is a nonabelian p-group of order p^4. Then one of the following holds:

i. $|Z(G)| = p^2$, $|G'| = p$, and $G' \subseteq Z(G)$.

ii. $|Z(G)| = p$, $|G'| = p^2$, and $Z(G) \subseteq G'$.

The following four theorems give preliminary results that will be used in order to prove the main results.

Theorem 2.3 [9] Let G be a group and $H, K \leq G$.

i. Then G/G' is abelian.

ii. Suppose $H \trianglelefteq G$. Then G/H is abelian if and only if $G' \subseteq H$.

iii. If $H, K \trianglelefteq G, K \leq H$, then $[H, K] \leq K \Leftrightarrow H/K \leq Z(G/K)$.

iv. If $G/Z(G)$ is cyclic, then G is abelian.

v. G is abelian if and only if $G' = 1$.

vi. If G is a nontrivial group and $\exp(G/G') \geq |G'|$, then $Z(G) \neq 1$, where $\exp(G/G')$ is the least common multiple of the orders of all elements of the group G/G'.

Theorem 2.4 [10] Let G be a group of order p^3, where p is an odd prime. Then exactly one of the following holds:

$$G \cong \mathbb{Z}_{p^3}. \quad (16)$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p. \quad (17)$$

$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{18}$$

$$G \cong \left\langle a,b \mid a^p = b^p = 1, [a,b]^a = [a,b] = [a,b]^b \right\rangle. \text{ In this case } \exp(G) = p. \tag{19}$$

$$G \cong \left\langle a,b \mid a^{p^2} = b^p = 1, a^b = a^{p+1} \right\rangle. \text{ In this case } \exp(G) = p^2. \tag{20}$$

The Schur multiplier of groups of order p^3, where p is an odd prime is stated in the following theorem.

Theorem 2.5 [11] Let G be a group of order p^3, where p is an odd prime. Then exactly one of the following holds:

$$M(G) \cong \begin{cases} 1 & ; G \text{ is of type (16) or (20)}, \\ \mathbb{Z}_p & ; G \text{ is of type (17)}, \\ \mathbb{Z}_p^2 & ; G \text{ is of type (19)}, \\ \mathbb{Z}_p^3 & ; G \text{ is of type (18)}. \end{cases}$$

Theorem 2.6 [3] Let Z be a central subgroup of a finite group G. Then the following conditions are equivalent:

i. $M(G) \cong M(G/Z)/(G' \cap Z)$.

ii. $Z \subseteq Z^*(G)$.

iii. The natural map $M(G) \to M(G/Z)$ is injective.

Next, some results on capability of groups which are used to compute the capability of groups of order p^4, where p is an odd prime are stated.

Theorem 2.7 [12] Let G be a finitely generated capable group. Then every central element z in G has order dividing $\exp\left(\left(G/\langle z\rangle\right)^{ab}\right)$.

Theorem 2.8 [13] Assume that G is a group with trivial Schur multiplier and finite $d\left(G/Z^\wedge(G)\right)$, where $d\left(G/Z^\wedge(G)\right)$ is the minimum number of generators of group $G/Z^\wedge(G)$. Then $C_G(x) = C_G^\wedge(x)$ for every element x of G. In particular, such group has $Z(G) = Z^\wedge(G)$.

Theorem 2.9 [3] G is capable if and only if the natural map $M(G) \to M\left(G/\langle x\rangle\right)$ has a nontrivial kernel for all $1 \neq x \in Z(G)$.

MAIN RESULTS

In this section, the capability of all nonabelian groups of order p^4 of type (6) to (11), where p is an odd prime in the classification is discussed. The capability of these groups is computed in the following theorem.

Theorem 3.1 Let G be a nonabelian group of order p^4 of type (6) to (11), where p is an odd prime. Then G is capable if G is a group of type (8) or type (9).

Proof: All nonabelian groups of order p^4 of type (6) until type (11) are satisfy the conditions in Theorem 2.2(i). For the group G of type (6), the order of commutator subgroup of G, $|G'| = p$, the order of abelianization of G, $|G^{ab}|$ is equal to p^3, where $G^{ab} = G/G'$. Since $\exp(G^{ab}) = p$ and $\exp(G^{ab}) = |G'|$, then by Theorem 2.3 (vi), it implies that $Z(G) \neq 1$. The computation of the Schur multiplier of G in [13] implies that, $M(G) = 1$. It is obvious that $d(G/Z^{\wedge}(G))$ is finite, then by Theorem 2.8, $Z(G) = Z^{\wedge}(G)$. Since $Z(G) \neq 1$ then $Z^{\wedge}(G) \neq 1$. Thus, G is not capable.

Next, we prove that the group of type (7) is not capable. For this group, there exist $z \in Z(G)$ and by choosing the order of z to be equal to p^2, it implies that the order of $G/\langle z\rangle$ is equal to p^2. For the group $G/\langle z\rangle$, we consider the following cases:

Case 1: $G/\langle z\rangle$ is abelian. Then $|G'| < |Z(G)|$, that is, $p < p^2$. Therefore, this case is impossible.

Case 2: $G/\langle z\rangle$ is nonabelian. Since the groups of order p^2 consists only abelian groups [10], then $G/\langle z\rangle$ is not isomorphic to a nonabelian group of order p^2. Therefore, this case is impossible.

Thus, G is not capable.

For the group G of type (8), $Z(G) = \{1, a, b, ab\} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and the Schur multiplier of G given by [14] is $M(G) = \mathbb{Z}_p$. The following cases are obtained:

Case 1: $|G/\langle b\rangle| = p^3, (G/\langle b\rangle)^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p$ and by Theorem 2.5, it implies that $M(G/\langle b\rangle) \cong \mathbb{Z}_p$.

Case 2: $|G/\langle a\rangle| = p^3, (G/\langle a\rangle)^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p$ and by Theorem 2.5, it implies that $M(G/\langle a\rangle) \cong \mathbb{Z}_p$.

Case 3: $|G/\langle ab\rangle| = p^3, (G/\langle ab\rangle)^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p$ and by Theorem 2.5, it implies that $M(G/\langle ab\rangle) \cong \mathbb{Z}_p$.

In these three cases, $M(G) \ncong M(G/Z(G))/(G' \cap Z(G))$ that is, the natural map $M(G) \to M(G/Z(G))$ is not injective. Thus the kernel of this map cannot be trivial. Therefore by Theorem 2.9, G is capable.

Next, we consider the group G of type (9). We choose an element $z \in Z(G)$ of order p. Thus, the order of $G/\langle z\rangle$ is equal to p^3. For the group $G/\langle z\rangle$, the following cases are possible:

Case 1: $G/\langle z\rangle$ is abelian. Then $|G'| < |Z(G)|$, that is, $p < p^2$. Therefore, this case is impossible.

Case 2: $G/\langle z\rangle$ is nonabelian, that is, $G/\langle z\rangle$ is isomorphic to a nonabelian group of order p^3 of type (19) (Theorem 2.4). Since the order of z divides $\exp\left((G/\langle z\rangle)^{ab}\right)$, where $\exp\left((G/\langle z\rangle)^{ab}\right) = p$.

Thus, by Theorem 2.7 G is capable.

For the group G of type (10), the computation is similar with the group of type (9). Let $z \in Z(G)$ and again by choosing z of order p, thus, the order of $G/\langle z\rangle$ is equal to p^3. The following cases are considered for the group $G/\langle z\rangle$:

Case 1: $G/\langle z\rangle$ is abelian. Then $|G'| < |Z(G)|$, that is, $p < p^2$. Therefore, this case is impossible.

Case 2: $G/\langle z\rangle$ is nonabelian, where $G/\langle z\rangle$ is isomorphic to a nonabelian group of order p^3 of type (10) (Theorem 2.4). Since $\exp\left((G/\langle z\rangle)^{ab}\right) = p^2$ then $|z| \nmid \exp\left((G/\langle z\rangle)^{ab}\right)$. Thus, by Theorem 2.7 it implies that G is not capable.

Lastly, for the group G of type (11), $Z(G) = \{1, a, b, ab\} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and by [14], the Schur multiplier, $M(G) = \mathbb{Z}_p \times \mathbb{Z}_p$. The following cases are obtained:

Case 1: $|G/\langle b\rangle| = p^3, (G/\langle b\rangle)^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p$ and Theorem 2.5 gives that $M(G/\langle b\rangle) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

Case 2: $|G/\langle a\rangle| = p^3, (G/\langle a\rangle)^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p$ and Theorem 2.5 gives that $M(G/\langle a\rangle) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

Case 3: $|G/\langle ab\rangle| = p^3, (G/\langle ab\rangle)^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p$ and Theorem 2.5 gives that $M(G/\langle ab\rangle) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

Since in three cases, the Schur multiplier of G is isomorphic to $M(G/Z)/(G' \cap Z(G))$, then by Theorem 2.6, the natural map $M(G) \to M(G/Z(G))$ is injective. Therefore by Theorem 2.9, G is not capable.

CONCLUSION

In this paper, the capability for all nonabelian groups of order p^4 of type (6) to (11), where p is an odd prime has been determined. We have proved that the nonabelian groups of order p^4 of type (6) to (11), where p is an odd prime is capable if G is a group of type (8) or type (9).

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