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On the Probability that a Metacyclic 2-Group Element Fixes a Set and its Generalized Conjugacy Class Graph

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Abstract. Let G be a metacyclic 2-group. The probability that two random elements commute in G is the quotient of the number of commuting elements by the square of the order of G. This concept has been generalized and extended by several authors. One of these extensions is the probability that an element of a group fixes a set, where the set consists of all subsets of commuting elements of G of size two that are in the form (a,b), where a and b commute and $\text{lcm}(|a|,|b|)=2$. In this paper, the probability that a group element fixes a set is found for metacyclic 2-groups of negative type of nilpotency class at least two. The results obtained on the size of the orbits are then applied to graph theory, more precisely to generalized conjugacy class graph.

Keywords: Commutativity degree, conjugacy class graph, graph theory.
PACS: 02.10Ox; 02.20-a.

INTRODUCTION

Throughout this paper, Γ denotes a simple undirected graph and G denotes a finite non-abelian group. In the following, a brief description about commutativity degree is given.

The probability that a pair of elements x and y, selected randomly from a group G, commute is called the commutativity degree of G, and was first introduced in 1944 [1]. This concept since then has been generalized and extended by several authors. One of these generalizations states the probability that a group element fixes a set, which was extended by Omer *et al.* [2] in 2013.

Next, we discuss some fundamental concepts in graph theory.

The first appearance of graph theory was in 1736 when Leonard Euler considered the Konigsberg bridge problem. Euler solved this problem by drawing a graph with points and lines. Years later, the usefulness of graph theory has been proven in a large number of diverse fields.

The following are some basic concepts in graph theory that are needed in this study. These concepts can be found in the references ([3], [4]).

A graph Γ is a mathematical structure consisting of two sets, namely vertices and edges, which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The graph is called directed if its edges are identified with ordered pair of vertices; otherwise, it is called undirected. Two vertices are adjacent if they are linked by an edge. A complete graph on n vertices, denoted by K_n, is a graph where each ordered pair of distinct vertices is adjacent. The graph Γ is said to be null if there are no vertices in Γ.

In 1990, a new graph which is related to conjugacy class was introduced by Bertram *et al.* [6]. The vertices of this graph are non central conjugacy classes i.e. $|V(\Gamma)| = K(G) - |Z(G)|$ where $K(G)$ is the number of conjugacy classes of the group G and $Z(G)$ is the center of G. A pair of vertices of this graph is connected by an edge if their cardinalities are not coprime.

In 2013, Omer *et al.* [7] extended the work on the conjugacy class graph by introducing a generalized conjugacy class graph, whose vertices are non-central orbits under group action on a set.

Next, we discuss the classification of metacyclic p-groups as proposed by Beuerle [8].

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In 2005, Beuerle classified the metacyclic p-groups to metacyclic p-groups of nilpotency of class two and metacyclic p-groups of nilpotency class at least three. Based on Beuerle's classification, the metacyclic p-groups of nilpotency class two are partitioned into two families of non-isomorphic p-groups, stated as follows:

(1) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = 1, [a,b] = a^{2^{\alpha-\lambda}} \rangle$, where $\alpha, \beta, \lambda \in \mathbb{N}, \alpha \geq 2\lambda, \beta \geq \lambda \geq 1$.

(2) $G \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$, a quaternion group of order 8, Q_8.

In addition, the metacyclic 2-groups of negative type of class at least three are partitioned into eight families [8]. The following are some of the negative types which had been considered in the scope of this research:

(3) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = a^{2^{\alpha-1}}, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3$.

(4) $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3$,

(5) $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3$,

(6) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3, \beta > 1$,

(7) $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3, \beta > 1$,

(8) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta > \lambda > 1$,

(9) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1$.

(10) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3, \beta > 1$.

This paper is divided into three sections. The first section focuses on some background topics in graph theory and algebra, while the second section provides some earlier and recent publications that are related to the probability that a group element fixes a set and conjugacy class graph. In the third section, we present our results on the probability that a metacyclic 2-group element fixes a set and determine the conjugacy class graphs of the metacyclic 2-groups of negative type of class at least three.

PRELIMINARIES

This section presents some works related to the probability that an element of a group fixes a set and graph theory. We commence with brief information about the probability that a group element fixes a set, followed by some related works on graph theory, more precisely on graphs related to conjugacy classes.

In 2013, Omer *et al.* [2] introduced the probability that a group element fixes a set. The following theorem is used in this paper.

Theorem 1 [2] **:** Let G be a finite group. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $\mathrm{lcm}(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω. Then the probability that an element of a group fixes a set is given by: $P_G(\Omega) = \dfrac{K(\Omega)}{|\Omega|}$, where $K(\Omega)$ is the number of orbits of Ω in G.

The work in [2] has then been extended by finding the probability for some finite non-abelian groups, including dihedral groups, quaternion groups, symmetric groups and alternating groups [2], [9] and [10]. In this study, the probability that a metacyclic 2-group element fixes a set is computed. The results obtained are then connected to graph theory by using the orbits obtained under group acting on a set to conjugacy class graphs

Some related works on conjugacy class graph are also discussed in this paper. Bianchi *et al*. [11] studied the regularity of the graph related to conjugacy classes and provided some results. In 2005, Moreto *et al*. [12] classified the groups in which conjugacy class sizes are not coprime for any five distinct classes. You *et al.* [13] also classified the groups in which conjugacy classes are not set-wise relatively prime for any four distinct classes. Moradipour *et al.* [14] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups.

Recently, Omer *et al.* [6] generalized the conjugacy class graph by defining the generalized conjugacy class graph whose vertices are non-central orbits under group action on a set. The following is the definition of the generalized conjugacy class graph

Definition 1 [6]**:** Let G be a finite group and Ω a subset of G. Let A be the set of commuting element in Ω , i.e $\{\omega \in \Omega : \omega g = g\omega, g \in G\}$. Then, the generalized conjugacy class graph $\Gamma_G^{\Omega_c}$ is defined as a graph whose vertices are non-central orbits under group action on a set, that is $\left|V\left(\Gamma_G^{\Omega_c}\right)\right| = K(\Omega) - |A|$. Two vertices ω_1 and ω_2 in $\Gamma_G^{\Omega_c}$ are adjacent if their cardinalities are not coprime, i.e $\gcd(\omega_1, \omega_2) \neq 1$.

In addition, the generalized conjugacy class graph is determined for the symmetric groups and alternating groups [8].

RESULTS AND DISCUSSION

This section consists of two parts. The first part focuses on the probability that an element of a metacyclic 2-group fixes a set, whereas the second part relates the obtained results to the generalized conjugacy class graph. Throughout this section, let S be a set of elements of G of size two in the form of (a,b), where a and b commute and $\operatorname{lcm}(|a|,|b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two.

The Probability that a Metacyclic 2-Group Element Fixes a Set

In this section, the probability that a group element fixes a set is found for metacyclic 2-groups of negative type of nilpotency class at least three, starting with type (3).

Theorem 2: Let G be a group of type (3), $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = a^{2^{\alpha-1}}, [b,a] = a^{-2}\rangle$, where $\alpha \geq 3$. If G acts on Ω by conjugation, then $P_G(\Omega) = \frac{4}{11}$.

Proof: If G acts on Ω by conjugation, then there exists $\psi : \Omega \times G \to \Omega$ such that $\psi_g(\omega) = g\omega g^{-1}, \omega \in \Omega$ and $g \in G$. Thus, the elements of Ω are stated as follows: There is only one element in the form of $\left(1, a^{2^{\alpha-1}}\right)$, four elements are in the form of $\left(1, a^{2^{\alpha-3i}}b\right), 0 \leq i \leq 2^{\alpha}$, where i is odd, four elements are in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-3i}}b\right), 0 \leq i \leq 2^{\alpha}$ where i is odd and two elements are in the form of $\left(a^{2^{\alpha-3i}}b, a^{2^{\alpha-3i}+2^{\alpha-1}}b\right), 0 \leq i \leq 2^{\alpha}$ where i is odd. From this, it follows that $|\Omega| = 11$. If G acts on Ω by conjugation, then $\operatorname{cl}(\omega) = g\omega g^{-1}, \omega \in \Omega, g \in G$. The orbits under the action of G on Ω can be described as follows: One orbit of $\left\{\left(1, a^{2^{\alpha-3i}}b\right), 0 \leq i \leq 2^{\alpha}\right\}$ of size four where i is odd, one orbit is in the form $\left\{\left(a^{2^{\alpha-1}}, a^{2^{\alpha-3i}}b\right), 0 \leq i \leq 2^{\alpha}\right\}$ of size four where i is odd, one orbit is in the form of $\left\{\left(a^{2^{\alpha-3i}}b, a^{2^{\alpha-3i}+2^{\alpha-1}}b\right), 0 \leq i \leq 2^{\alpha}\right\}$ of size two and one orbit is in the form $\left\{\left(1, a^{2^{\alpha-1}}\right)\right\}$. Therefore, there are four orbits. Using Theorem 1, the probability that an element of a group fixes a set in this case is equal to $\frac{4}{11}$. The proof is thus completed.

Theorem 3: Let G be a group of type (4), $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{-2}\rangle$, where $\alpha \geq 3$. If G acts on Ω by conjugation, then $P_G(\Omega) = \frac{4}{11}$.

Proof: The proof is similar to the proof of Theorem 2.

Theorem 4: Let G be a group of type (5), $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3$. If G acts on Ω by conjugation, then

$$P_G(\Omega) = \begin{cases} \frac{7}{21}, & \text{if } \alpha = 3, \\ \frac{2}{3}, & \text{otherwise.} \end{cases}$$

Proof: In the case that α is equal to 3, the elements of Ω are stated as follows: One element is in the form of $\left(1, a^{2^{\alpha-1}}\right)$, there are 2^{α} elements in the form of $\left(1, a^i b\right), 0 \leq i \leq 2^{\alpha}$, there are 2^{α} elements in the form of $\left(a^{2^{\alpha-1}}, a^i b\right), 0 \leq i \leq 2^{\alpha}$, and $2^{\alpha-1}$ elements in which $\omega \in \Omega$ are in the form of $\left(a^i, a^{i+2^{\alpha-1}} b\right), 0 \leq i \leq 2^{\alpha}$, from which it follows $|\Omega| = 2^{\alpha+1} + 2^{\alpha-1} + 1$. Since the action here is by conjugation, thus $\text{cl}(\omega) = \left\{\omega \in \Omega : g\omega g^{-1}, g \in G\right\}$. Therefore, the orbits under group action on Ω are described as follows: There is one orbit in the form of $\left\{\left(1, a^i b\right), 0 \leq i \leq 2^{\alpha}\right\}$ of size $2^{\alpha-1}$, where i is even and one orbit in the form $\left\{\left(1, a^i b\right), 0 \leq i \leq 2^{\alpha}\right\}$ where i is odd. In addition, there is one orbit in the form of $\left\{\left(a^{2^{\alpha-1}}, a^i b\right), 0 \leq i \leq 2^{\alpha}\right\}$ having size $2^{\alpha-1}$ where i is even, one orbit is in the form of $\left\{\left(a^{2^{\alpha-1}}, a^i b\right), 0 \leq i \leq 2^{\alpha}\right\}$ where i is odd, one orbit is in the form of $\left\{\left(1, a^{2^{\alpha-1}}\right)\right\}$ and two orbits are in the form of $\left\{\left(a^i, a^{i+2^{\alpha-1}} b\right), 0 \leq i \leq 2^{\alpha}\right\}$ of size two for both cases of i (even and odd). From this, it follows that, the number of orbits is seven. Using Theorem 1, $P_G(\Omega) = \frac{7}{21}$. However, when $\alpha > 3$, the elements of Ω are described as follows: One element in the form of $\left(1, a^{2^{\alpha-1}}\right)$, two elements are in the form of $\left(1, a^i b\right), 0 \leq i \leq 2^{\alpha}$, two elements are in the form of $\left(a^{2^{\alpha-1}}, a^i b\right), 0 \leq i \leq 2^{\alpha}$, and one element is in the form of $\left(b, a^{2^{\alpha-1}} b\right)$, in which it follows that $|\Omega| = 6$. Since the action here is by conjugation, thus there are four orbits. Using Theorem 1, the probability that a group element fixes a set, $P_G(\Omega) = \frac{2}{3}$, thus the proof follows.

Next, the probability that a metacyclic 2-group element fixes a set is computed for groups of types (6), (7) and (8), respectively.

Theorem 5: Let G be a group of type (6), $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3, \beta > 1$. If G acts on Ω by conjugation, then

$$P_G(\Omega) = \begin{cases} 1, & \text{if } \alpha = 3, \\ \frac{2}{3}, & \text{otherwise.} \end{cases}$$

Proof: If $\alpha = 3$ and G acts on Ω by conjugation, then there exists $\phi : \Omega \times G \to \Omega$ such that $\phi_g(\omega) = g\omega g^{-1}, \omega \in \Omega$ and $g \in G$. The elements of Ω are described as follows: There are two elements in the form of $\left(1, a^{2^{\alpha-1} i} b^{2^{\beta-1}}\right), 0 \leq i \leq 2^{\alpha}$, two elements are in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1} i} b^{2^{\beta-1}}\right), 0 \leq i \leq 2^{\alpha}$, one element is in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$, and one element is in the form of $\left(1, a^{2^{\alpha-1}}\right)$. Thus $|\Omega| = 6$. If G acts on Ω by conjugation, then

$\mathrm{cl}(\omega)=\left\{\omega\in\Omega : g\omega g^{-1}, g\in G\right\}$. Since there are six different types of elements in Ω, therefore the number of orbits is the same as $|\Omega|$. Using Theorem 1, thus $P_G(\Omega)=1$. However, in the case that $\alpha>3$, then the elements of Ω are six and have the following forms. Two elements are in the form of $\left(1,a^{2^{\alpha-1}i}b^{2^{\beta-1}}\right), 0\le i\le 2^{\alpha}$, two elements are in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-1}i}b^{2^{\beta-1}}\right), 0\le i\le 2^{\alpha}$, one element is in the form of $\left(b^{2^{\beta-1}},a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$ and one element is in the form of $\left(1,a^{2^{\alpha-1}}\right)$. Thus, when G acts on Ω by conjugation, $P_G(\Omega)$ is similar to the second part of Theorem 4. The proof then follows.

Theorem 6: Let G be a group of type (7), $G\cong\langle a,b : a^{2^{\alpha}}=b^{2^{\beta}}=1,[b,a]=a^{2^{\alpha-1}-2}\rangle$, where $\alpha\ge 3,\beta>1$. If G acts on Ω by conjugation, then

$$P_G(\Omega)=\begin{cases}1, & \text{if } \alpha=3,\\ \dfrac{2}{3}, & \text{otherwise.}\end{cases}$$

Proof: The proof is similar to the proof of Theorem 5.

Theorem 7: Let G be a group of type (8), $G\cong\langle a,b : a^{2^{\alpha}}=1,\ b^{2^{\beta}}=a^{2^{\alpha-1}},[b,a]=a^{2^{\alpha-\gamma}-2}\rangle$, where $\alpha-\gamma>1$, and $\beta>\gamma>1$. If G acts on Ω by conjugation, then $P_G(\Omega)=\dfrac{2}{3}$.

Proof: There are six elements of Ω, divided as follows: Two elements are in the form of $\left(1,a^{2^{\alpha-2}i}b^{2^{\beta-1}}\right), 0\le i\le 2^{\alpha}$, where i is odd, two elements are in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-2}i+2^{\alpha-1}}b^{2^{\beta-1}}\right), 0\le i\le 2^{\alpha}$, where i is odd, one element is in the form of $\left(1,a^{2^{\alpha-1}}\right)$ and one element is in the form of $\left(b^{2^{\beta-1}},a^{2^{\alpha-2}+2^{\alpha-1}}b^{2^{\beta-1}}\right)$. By conjugate action, there are four orbits. The proof then follows.

Theorem 8: Let G be a group of type (9), $G\cong\langle a,b : a^{2^{\alpha}}=1,\ b^{2^{\beta}}=1,[b,a]=a^{2^{\alpha-\gamma}-2}\rangle$, where $\alpha-\gamma>1$. If G acts on Ω by conjugation, then

$$P_G(\Omega)=\begin{cases}1, & \text{if } \alpha=3,\\ \dfrac{2}{3}, & \text{otherwise.}\end{cases}$$

Proof: The proof is similar to the proof of Theorem 5.

Next, the probability that a metacyclic 2-group element fixes a set is obtained for the group of type (9) and (10).

Theorem 9: Let G be a group of type (10), $G\cong\langle a,b : a^{2^{\alpha}}=1,\ b^{2^{\beta}}=a^{2^{\alpha-1}},[b,a]=a^{-2}\rangle$, where $\alpha\ge 3,\beta>1$. If G acts on Ω by conjugation, then $P_G(\Omega)=\dfrac{2}{3}$.

Proof: The elements of Ω are described as follows: One element is in the form of $\left(1,a^{2^{\alpha-1}}\right)$, two elements are in the form of $\left(1,a^{2^{\alpha-2}i}b^{2^{\beta-1}}\right), 0\le i\le 2^{\alpha}$ where i is odd, two elements are in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-2}i+2^{\alpha-1}}b^{2^{\beta-1}}\right), 0\le i\le 2^{\alpha}$, where i is odd, and one element is in the form of $\left(a^{2^{\alpha-1}}b^{2^{\beta-1}},a^{2^{\alpha-2}+2^{\alpha-1}}b^{2^{\beta-1}}\right)$, from which it follows $|\Omega|=6$. Since G acts on Ω by conjugation, thus there are four orbits. Using Theorem 1, $P_G(\Omega)=\dfrac{4}{6}=\dfrac{2}{3}$. The proof is then completed.

Next, the results are obtained; more specifically the sizes of orbits, are applied to the generalized conjugacy class graph.

Generalized Conjugacy Class Graph

This section discusses the results on the sizes of the orbits which are related to generalized conjugacy class graph (refer to Definition 1). The following theorem is key to getting immediate results from the probability that a group element fixes a set to generalized conjugacy class graph.

Theorem 10: Let G be a finite non-Abelian group and Ω the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $P_G(\Omega)=1$, then the generalized conjugacy class graph, $\Gamma_G^{\Omega_c}$ is null.

Proof: If $P_G(\Omega)=1$, then based on Theorem 1, $K(\Omega)=|\Omega|$. Thus, for all $\omega\in\Omega$, $\mathrm{cl}(\omega)=\omega$. By Definition 1, the number of vertices is $\left|V\left(\Gamma_G^{\Omega_c}\right)\right|=0$. Thus the graph is null i.e $\Gamma_G^{\Omega_c}=K_0$.

Now, the generalized conjugacy class graph of metacyclic 2-groups is found, starting with the group of type (3).

Theorem 11: Let G be a group of type (3), $G\cong\langle a,b : a^{2^\alpha}=1, b^2=a^{2^{\alpha-1}}, [b,a]=a^{-2}\rangle$, where $\alpha\geq 3$. If G acts on Ω by conjugation, then $\Gamma_G^{\Omega_c}=K_3$.

Proof: According to Theorem 2, there are four orbits. Among these orbits, two of them have size four, one orbit has size two and one orbit has size one. By Definition 1, $\left|V\left(\Gamma_G^{\Omega_c}\right)\right|=3$. By the vertices adjacency of $\Gamma_G^{\Omega_c}$, two of the vertices ω_1 and ω_2 are adjacent if their cardinalities are not coprime (i.e $\gcd(|\omega_1|,|\omega_2|)\neq 1$) hence $\Gamma_G^{\Omega_c}$ consists of a complete graph of K_3. Figure 1 shows the complete graph of three vertices.

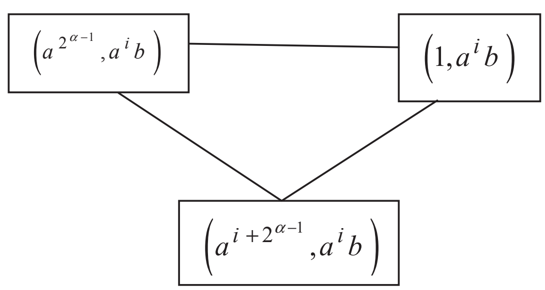

FIGURE 1: Complete graph of K_3

Theorem 12: Let G be a group of type (4), $G\cong\langle a,b : a^{2^\alpha}=b^2=1,\ [b,a]=a^{-2}\rangle$, where $\alpha\geq 3$. If G acts on Ω by conjugation, then $\Gamma_G^{\Omega_c}=K_3$.

Proof: The proof is similar to the proof in Theorem 11.

Theorem 13: Let G be a group of type (5), $G\cong\langle a,b : a^{2^\alpha}=b^2=1, [b,a]=a^{2^{\alpha-1}-2}\rangle$, where $\alpha\geq 3$. If G acts on Ω by conjugation, then

$$\Gamma_G^{\Omega_c}=\begin{cases} K_6, & \text{if } \alpha=3, \\ K_3, & \text{otherwise.}\end{cases}$$

Proof: Based on Theorem 4 when $\alpha=3$, there are seven orbits, four of them have size four and two have size two. Since two vertices are adjacent if their cardinalities are not coprime i.e gcd(4, 2) $\neq 1$, hence $\Gamma_G^{\Omega_c}$ consists of one complete component of K_6. If $\alpha>3$, the proof is similar to the proof of Theorem 11.

Theorem 14: Let G be a group of type (6), $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3, \beta > 1$. If G acts on Ω by conjugation, then

$$\Gamma_G^{\Omega_c} = \begin{cases} K_0, & \text{if } \alpha = 3, \\ K_2, & \text{otherwise.} \end{cases}$$

Proof: Based on Theorem 5, when $\alpha = 3$, the probability that a group element fixes a set, $P_G(\Omega) = 1$. Using Theorem 10, the generalized conjugacy class graph in this case is null. If $\alpha > 3$, the number of vertices in $\Gamma_G^{\Omega_c}$ is three. Since two vertices are connected if they are not coprime, thus $\Gamma_G^{\Omega_c}$ consists of one complete component of K_2 and one isolated vertex. The proof then follows.

Theorem 15: Let G be a group of type (7), $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3, \beta > 1$. If G acts on Ω by conjugation, then

$$\Gamma_G^{\Omega_c} = \begin{cases} K_0, & \text{if } \alpha = 3, \\ K_2, & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to the proof in Theorem 14.

Theorem 16: Let G be a group of type (8), $G \cong \langle a,b : a^{2^{\alpha}} = 1,\ b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1$, and $\beta > \gamma > 1$. If G acts on Ω by conjugation, then $\Gamma_G^{\Omega_c} = K_2$.

Proof: According to Theorem 7 and Definition 1, the number of vertices in $\Gamma_G^{\Omega_c}$ is three. Based on vertices adjacency, there is only one complete component of K_2 and one isolated vertex. The proof is thus completed.

Theorem 17: Let G be a group of type (9), $G \cong \langle a,b : a^{2^{\alpha}} = 1,\ b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1$, and $\beta \geq \gamma > 1$. If G acts on Ω by conjugation, then

$$\Gamma_G^{\Omega_c} = \begin{cases} K_0, & \text{if } \alpha = 3, \\ K_2, & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to the proof of Theorem 14.

Theorem 18: Let G be a group of type (10), $G \cong \langle a,b : a^{2^{\alpha}} = 1,\ b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3, \beta > 1$. If G acts on Ω by conjugation, then $\Gamma_G^{\Omega_c} = K_2$.

Proof: According to Theorem 9, there are two orbits having size two. According to Definition 1, $\Gamma_G^{\Omega_c}$ consists of a complete component of K_2 and one isolated vertex, namely $\left(a^{2^{\alpha-1}} b^{2^{\beta-1}}, a^{2^{\alpha-2}+2^{\alpha-1}} b^{2^{\beta-1}}\right)$. The proof is then complete.

CONCLUSION

In this paper, the probability that a group element fixes a set is found for metacyclic 2-groups of negative type of nilpotency class at least three. The results obtained on the sizes of the orbits are then applied to the generalized conjugacy class graph.

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