A GENERALIZATION ON THE NTH COMMUTATIVITY DEGREE OF ALTERNATING GROUPS OF DEGREE 4 AND 5

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\textbf{Abstract}

The theory of commutativity degree is important in determining the abelianness of a group. The commutativity degree of a finite group $G$ is the probability that a pair of elements chosen randomly from a group $G$, commute. The concept of commutativity degree can be generalized to the $n$th commutativity degree of a group which is defined as the probability of commuting the $n$th power of a randomly chosen element with another random element from the same group. In this research, the $n$th commutativity degree of alternating groups of degree 4 and 5 are presented.

Keywords: Abelianness; commutativity degree; alternating group

\textbf{1.0 INTRODUCTION}

All groups mentioned in this paper are considered finite. The commutativity degree of a group $G$ is the probability that a selected chosen pair of elements of $G$ commute. It is denoted by $P(G)$. The definition of the commutativity degree is given as follows.

\textbf{Definition 1.1} [1] The commutativity degree of a group $G$, denoted as $P(G)$, can be written as

$$P(G) = \frac{|\{(x,y) \in G \times G | xy = yx\}|}{|G|^2}.$$
symmetric groups, $S_m$. Later, Gustafson [4] and Machale [1] showed that the commutativity degree of all nonabelian groups is less than or equal to $\frac{3}{8}$.

In 2006, Mohd Ali and Sarmin [5] extended the concept of commutativity degree of a group $G$ to the $n^{th}$ commutativity degree of $G$, denoted as $P_n(G)$, which is the probability that the $n^{th}$ power of a selected element commute with another element of $G$.

The formal definition of $n^{th}$ commutativity degree is given in the following.

**Definition 1.2** [5] The $n^{th}$ commutativity degree of a group $G$, denoted as $P_n(G)$, is defined as

$$P_n(G) = \left| \{(x,y) \in G \times G | x^ny = y^nx \} \right| / |G|^2$$

Note that for $n = 1$, $P_1(G) = P(G)$. In finding $P_n(G)$, the power of each element in $G$ is gradually raised until the power $n$ is achieved.

There are two approaches on finding the probability that a pair of elements commute. First by using the Cayley Table (or symmetrical 0-1 Table) and second by using the number of conjugacy classes. MacHale [1] used the 0-1 Table to find the probability that two elements commute in a group. In this research, the 0-1 Table is used to determine the $n^{th}$ commutativity degree of a group $G$.

In this research the $n^{th}$ commutativity degree of alternating groups of degree 4 of order 12 and alternating groups of degree 5 of order 60 are found.

### 2.0 PRELIMINARIES

In this section, we provide some preliminaries and basic definitions that are needed in this research.

**Definition 2.1** [6] Symmetric Group of Degree $m$

Let $A$ be the finite set $\{1,2,\ldots,m\}$. The group of all permutations of $A$ is the symmetric group on $m$ letters, and is denoted by $S_m$. The order of $S_m$ is $m!$.

**Definition 2.2** [7] Alternating Group of Degree $m$

The set of all even permutation in $S_m$ forms a subgroup of $S_m$ for $m \geq 2$. This subgroup is called the alternating group of degree $m$, and denoted by $A_m$. The order of $A_m$ is $\frac{m!}{2}$.

**Definition 2.3** [1] The 0-1 Table for a Group $G$

If $xy = yx$ for all $x, y$ in $G$, each of the boxes corresponding to $xy$ and $yx$ will be assigned the number 1. In other side, if $xy \neq yx$, the number 0 will be placed in each of these boxes.

### 3.0 RESULTS AND DISCUSSION

In this section, the results of $P_n(A_m)$, which is the $n^{th}$ commutativity degree of alternating groups of degree $m$ where $m = 4$ and 5 are determined using the 0-1 Table.

Clearly, $A_4$ is the alternating group of degree 4. The elements of $A_4$ are $\{1\}$, $\{1,23\}$, $\{1,24\}$, $\{1,34\}$, $\{1,32\}$, $\{1,42\}$, $\{1,43\}$, $\{2,34\}$, $\{2,34\}$, $\{1,23\}$ and $\{1,24\}$.

To compute the multiplication table for $A_4$, we let $P_1(A_4)$ = 0.97.

The Cayley table of $A_4$ is given in the following:

<table>
<thead>
<tr>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
<th>$\beta_6$</th>
<th>$\beta_7$</th>
<th>$\beta_8$</th>
<th>$\beta_9$</th>
<th>$\beta_{10}$</th>
<th>$\beta_{11}$</th>
<th>$\beta_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
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<td>$\beta_3$</td>
<td>$\beta_4$</td>
<td>$\beta_5$</td>
<td>$\beta_6$</td>
<td>$\beta_7$</td>
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<td>$\beta_9$</td>
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<td>$\beta_4$</td>
<td>$\beta_5$</td>
<td>$\beta_6$</td>
<td>$\beta_7$</td>
<td>$\beta_8$</td>
<td>$\beta_9$</td>
<td>$\beta_{10}$</td>
<td>$\beta_{11}$</td>
<td>$\beta_{12}$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>$\beta_4$</td>
<td>$\beta_5$</td>
<td>$\beta_6$</td>
<td>$\beta_7$</td>
<td>$\beta_8$</td>
<td>$\beta_9$</td>
<td>$\beta_{10}$</td>
<td>$\beta_{11}$</td>
<td>$\beta_{12}$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>$\beta_5$</td>
<td>$\beta_6$</td>
<td>$\beta_7$</td>
<td>$\beta_8$</td>
<td>$\beta_9$</td>
<td>$\beta_{10}$</td>
<td>$\beta_{11}$</td>
<td>$\beta_{12}$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
<td>$\beta_3$</td>
</tr>
</tbody>
</table>

From Table 3.1, we can produce the 0-1 Table for $A_4$ as shown in the following.
From Table 3.2, 48 pairs of elements commute with each other. Therefore, \( P(A_n) = \frac{48}{144} = \frac{1}{3} \).

In Table 3.3 and Table 3.4, the powers of each element in \( A_n \) are computed up to a certain value (until it can be generalized) and the value of \( P_n(A_n) \) is computed for \( n = 1, 2, 3, \ldots, 12 \).

### Table 3

\( P_n(A_n) \) for \( n = 2, 3, 4, 5 \) and 6

<table>
<thead>
<tr>
<th>( x \in A_4 )</th>
<th>( x^2 )</th>
<th>( x^3 )</th>
<th>( x^4 )</th>
<th>( x^5 )</th>
<th>( x^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>( \beta_1^2 = \beta_1 )</td>
<td>( \beta_1^3 = \beta_1 )</td>
<td>( \beta_1^4 = \beta_1 )</td>
<td>( \beta_1^5 = \beta_1 )</td>
<td>( \beta_1^6 = \beta_1 )</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>( \beta_2^2 = \beta_2 )</td>
<td>( \beta_2^3 = \beta_2 )</td>
<td>( \beta_2^4 = \beta_2 )</td>
<td>( \beta_2^5 = \beta_2 )</td>
<td>( \beta_2^6 = \beta_2 )</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>( \beta_3^2 = \beta_3 )</td>
<td>( \beta_3^3 = \beta_3 )</td>
<td>( \beta_3^4 = \beta_3 )</td>
<td>( \beta_3^5 = \beta_3 )</td>
<td>( \beta_3^6 = \beta_3 )</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>( \beta_4^2 = \beta_4 )</td>
<td>( \beta_4^3 = \beta_4 )</td>
<td>( \beta_4^4 = \beta_4 )</td>
<td>( \beta_4^5 = \beta_4 )</td>
<td>( \beta_4^6 = \beta_4 )</td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>( \beta_5^2 = \beta_5 )</td>
<td>( \beta_5^3 = \beta_5 )</td>
<td>( \beta_5^4 = \beta_5 )</td>
<td>( \beta_5^5 = \beta_5 )</td>
<td>( \beta_5^6 = \beta_5 )</td>
</tr>
<tr>
<td>( \beta_6 )</td>
<td>( \beta_6^2 = \beta_6 )</td>
<td>( \beta_6^3 = \beta_6 )</td>
<td>( \beta_6^4 = \beta_6 )</td>
<td>( \beta_6^5 = \beta_6 )</td>
<td>( \beta_6^6 = \beta_6 )</td>
</tr>
<tr>
<td>( \beta_{10} )</td>
<td>( \beta_{10}^2 = \beta_{10} )</td>
<td>( \beta_{10}^3 = \beta_{10} )</td>
<td>( \beta_{10}^4 = \beta_{10} )</td>
<td>( \beta_{10}^5 = \beta_{10} )</td>
<td>( \beta_{10}^6 = \beta_{10} )</td>
</tr>
<tr>
<td>( \beta_{11} )</td>
<td>( \beta_{11}^2 = \beta_{11} )</td>
<td>( \beta_{11}^3 = \beta_{11} )</td>
<td>( \beta_{11}^4 = \beta_{11} )</td>
<td>( \beta_{11}^5 = \beta_{11} )</td>
<td>( \beta_{11}^6 = \beta_{11} )</td>
</tr>
<tr>
<td>( \beta_{12} )</td>
<td>( \beta_{12}^2 = \beta_{12} )</td>
<td>( \beta_{12}^3 = \beta_{12} )</td>
<td>( \beta_{12}^4 = \beta_{12} )</td>
<td>( \beta_{12}^5 = \beta_{12} )</td>
<td>( \beta_{12}^6 = \beta_{12} )</td>
</tr>
</tbody>
</table>

### Table 4

\( P_n(A_n) \) for \( n = 7, 8, 9, 10, 11 \) and 12

<table>
<thead>
<tr>
<th>( x^7 )</th>
<th>( x^8 )</th>
<th>( x^9 )</th>
<th>( x^{10} )</th>
<th>( x^{11} )</th>
<th>( x^{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>( \beta_1^{12} = \beta_1 )</td>
<td>( \beta_1^{11} = \beta_1 )</td>
<td>( \beta_1^{10} = \beta_1 )</td>
<td>( \beta_1^9 = \beta_1 )</td>
<td>( \beta_1^8 = \beta_1 )</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>( \beta_2^{12} = \beta_2 )</td>
<td>( \beta_2^{11} = \beta_2 )</td>
<td>( \beta_2^{10} = \beta_2 )</td>
<td>( \beta_2^9 = \beta_2 )</td>
<td>( \beta_2^8 = \beta_2 )</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>( \beta_3^{12} = \beta_3 )</td>
<td>( \beta_3^{11} = \beta_3 )</td>
<td>( \beta_3^{10} = \beta_3 )</td>
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<td>( \beta_3^8 = \beta_3 )</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>( \beta_4^{12} = \beta_4 )</td>
<td>( \beta_4^{11} = \beta_4 )</td>
<td>( \beta_4^{10} = \beta_4 )</td>
<td>( \beta_4^9 = \beta_4 )</td>
<td>( \beta_4^8 = \beta_4 )</td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>( \beta_5^{12} = \beta_5 )</td>
<td>( \beta_5^{11} = \beta_5 )</td>
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<td>( \beta_5^8 = \beta_5 )</td>
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<td>( \beta_6^{12} = \beta_6 )</td>
<td>( \beta_6^{11} = \beta_6 )</td>
<td>( \beta_6^{10} = \beta_6 )</td>
<td>( \beta_6^9 = \beta_6 )</td>
<td>( \beta_6^8 = \beta_6 )</td>
</tr>
</tbody>
</table>

From Table 3.3 and Table 3.4, we can generalize the \( n^{10} \) commutativity degree of alternating group of degree 4, \( P_n(A_4) \) as in the following theorem.

**Theorem 3.1** Let \( A_4 \) be an alternating group of degree 4. Then for \( n, k \in \mathbb{Z}^+ \) where \( k = 0, 1, 2, \ldots, P_n(A_4) \) is given as follows:

\[
P_n(A_4) = \begin{cases} 
1 & n = 1 + 6k, n = 5 + 6k \\
\frac{1}{2} & n = 2 + 6k, n = 4 + 6k \\
\frac{5}{6} & n = 3 + 6k \\
1 & n = 6k 
\end{cases}
\]

**Proof** For all elements \( x \) in \( A_4 \), the order of \( x \) is 1, 2 or 3. Furthermore, for any \( x \in A_4 \), \( x^6 = e \) and \( x^n = e \) for \( n = 6k \) where \( k \in \mathbb{Z}^+ \).

The number of \( (x, y) \) where \( x \cdot y = y \cdot x \) also equal to the number of \( (x, y) \) when \( x^5 \cdot y = y \cdot x^5 \), \( x^7 \cdot y = y \cdot x^7 \) and \( x^{11} \cdot y = y \cdot x^{11} \).

Now we need to prove that \( x^5 \cdot y = y \cdot x^5 \) and \( x^7 \cdot y = y \cdot x^7 \) and \( x^{11} \cdot y = y \cdot x^{11} \) can be reduced to \( x \cdot y = y \cdot x \).

Suppose \( x^4 = e \). This implies \( x^{-1} = x^3 \). Therefore \( x^3 \cdot y = y \cdot x^3 \) is the same as \( x^{-1} \cdot y = y \cdot x^{-1} \). By cancellation we have \( x \cdot y = y \cdot x \).
Next $x^2 \cdot y = y \cdot x^2$ can be written as $x \cdot x^2 \cdot y = y \cdot x \cdot x^4$ 
$x \cdot e \cdot y = y \cdot x \cdot e 
 x \cdot y = y \cdot x$.

By the same calculations and argument, it can be shown that $x^{11} \cdot y = y \cdot x^{11}$ can be reduced to $x \cdot y = y \cdot x$.

Next $x^4 \cdot y = y \cdot x^4$ and $x^{10} \cdot y = y \cdot x^{10}$ are equal to $x^2 \cdot y = y \cdot x^2$, and $x^2 \cdot y = y \cdot x^2$ is equal to $x^3 \cdot y = y \cdot x^3$.

Suppose $x^4 = e$. This implies $(x^2)^{-1} = x^4$. Therefore $x^4 \cdot y = y \cdot x^4$ is the same as $(x^2)^{-1} \cdot y = y \cdot (x^2)^{-1}$. By cancellation we have $x^2 \cdot y = y \cdot x^2$.

Next $x^8 \cdot y = y \cdot x^8$ can be written as 
$x^8 \cdot x^2 \cdot x^4 \cdot y = y \cdot x^2 \cdot x^2 \cdot x^4$
$x^2 \cdot e \cdot y = y \cdot x^2 \cdot e$
$x^2 \cdot y = y \cdot x^2$.

By the same calculations and argument, it can be shown that $x^{10} \cdot y = y \cdot x^{10}$ can be reduced to $x^2 \cdot y = y \cdot x^2$.

Next $x^4 \cdot y = y \cdot x^4$ can be written as 
$x^4 \cdot x^2 \cdot x^4 \cdot y = y \cdot x^3 \cdot x^3 \cdot x^3$
$x^3 \cdot e \cdot y = y \cdot x^3 \cdot e$
$x^3 \cdot y = y \cdot x^3$.

Clearly $x^4$ is an identity in $A_4$, then $x^{12} \cdot y = y \cdot x^{12}$ can also be reduced to $x^4 \cdot y = y \cdot x^4$.

By some calculations,

$x^{1+4} \cdot y = y \cdot x^{1+4}$ is equal to $x \cdot y = y \cdot x$.

Suppose $x^{4} = e$, then,

$x^{1+4} \cdot y = y \cdot x^{1+4}$
$x \cdot x^{1+4} \cdot y = y \cdot x \cdot x^{1+4}$
$x \cdot e \cdot y = y \cdot x \cdot e$
$x \cdot y = y \cdot x$.

$x^{5+4} \cdot y = y \cdot x^{5+4}$ is equal to $x^5 \cdot y = y \cdot x^5$.

Suppose $x^{4} = e$, then,

$x^{5+4} \cdot y = y \cdot x^{5+4}$
$x^{5} \cdot x^{4} \cdot y = y \cdot x^{5} \cdot x^{4}$
$x^{5} \cdot e \cdot y = y \cdot x^{5} \cdot e$
$x^{5} \cdot y = y \cdot x^{5}$

$x^{2+4k} \cdot y = y \cdot x^{2+4k}$ is equal to $x^2 \cdot y = y \cdot x^2$.

Suppose $x^{4} = e$, then,

$x^{2+4k} \cdot y = y \cdot x^{2+4k}$
$x^{2} \cdot x^{4k} \cdot y = y \cdot x^{2} \cdot x^{4k}$
$x^{2} \cdot e \cdot y = y \cdot x^{2} \cdot e$
$x^{2} \cdot y = y \cdot x^{2}$

Next can be written as 

Suppose $x^{4} = e$,

Using similar method, we found the generalization of the $n^{th}$ commutativity degree of alternating group of degree $5$, $P_n(A_5)$ is given as follows.

**Theorem 3.2** Let $A_5$ be an alternating group of degree $5$. Then for $n$, $k \in \mathbb{Z}^+$ where $k = 0, 1, 2, ..., P_n(A_5)$ is given as follows:

$$P_n(A_5) = \left\{ \begin{array}{l}
1, \quad n = 1 + 30k, \quad n = 7 + 30k, \quad n = 11 + 30k, \quad n = 13 + 30k, \\
12, \quad n = 17 + 30k, \quad n = 19 + 30k, \quad n = 23 + 30k, \quad n = 29 + 30k, \\
1441, \quad n = 3 + 30k, \quad n = 9 + 30k, \quad n = 21 + 30k, \quad n = 27 + 30k, \\
3600, \quad n = 19 + 30k, \quad n = 4 + 30k, \quad n = 8 + 30k, \quad n = 14 + 30k, \\
60, \quad n = 6 + 30k, \quad n = 22 + 30k, \quad n = 25 + 30k, \quad n = 28 + 30k, \\
1619, \quad n = 5 + 30k, \quad n = 25 + 30k, \\
2281, \quad n = 6 + 30k, \quad n = 12 + 30k, \quad n = 18 + 30k, \quad n = 24 + 30k, \\
3600, \quad n = 12 + 30k, \quad n = 10 + 30k, \quad n = 20 + 30k, \\
3660, \quad n = 30 + 30k, \\
2399, \quad n = 23 + 30k, \\
30, \quad n = 15 + 30k, \\
1, \quad n = 30 + 30k.
\end{array} \right.$$
4.0 CONCLUSION

As a conclusion, the $n^{th}$ commutativity degree of alternating groups of degree 4 and alternating groups of degree 5 are determined. The 0-1 Table was used in finding $P_n(A_4)$ and $P_n(A_5)$.

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