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The Probability That an Element of a Metacylic 3-Group Fixes a Set of Size Three

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Abstract. Let G be a metacylic 3-group of negative type of nilpotency class at least three. In this paper, Ω is a set of all subsets of all commuting elements of G of size three in the form of (a,b), where a and b commute. The probability that an element of a group G fixes a set Ω is one of extensions of the commutativity degree that can be obtained under group action on set. This probability is the ratio of the number of orbits to the order of Ω. In this paper, the probability that an element of a group G fixes a set Ω is computed by using conjugate action.

Keywords: metacyclic 3-group, commutativity degree, conjugate action

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INTRODUCTION

In the first section, some backgrounds related to commutativity degree and probability are provided. Miller [1] was the first researcher who introduced the concept of commutativity degree, which is a very important topic in finding the abelianness of a group. The definition of commutativity degree given by Miller is shown below.

Definition 1 [1] Let G be a finite group. The commutativity degree is the probability that two random elements (x,y) in G commute, as defined as follows:

$$P(G)=\frac{\left|\{(x,y)\in G\times G \mid xy=yx\}\right|}{\left|G\right|^2}.$$

In this paper, the probability that a group element fixes a set for metacyclic 3-group of negative type will be found by using conjugate action. This can be done by computing the number of orbits under the same group action on set.

Some basic concepts related to commutativity degree, probability and group action on set that are used in this paper are given in the following definitions.

Definition 2 [2] A group G is called metacyclic if it has a cyclic normal subgroup H such that the quotient group $\frac{G}{H}$ is also cyclic.

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Definition 3 [3] Let G be a finite group and S be a set. Then G acts on itself if there is a function $G \times G \to G$, such that

i. $gh(x) = g(hx), \forall g, h, x \in G.$

ii. $1_{G^X} = x, \forall x \in X.$

Definition 4 [4] Let G acts on a set Ω and $x \in \Omega$. The orbit of x denoted by $O(x)$ is the subset of Ω such that $O(x) = \{gx : g \in G\} \subseteq \Omega.$

If a group G act on itself by conjugation, the orbit $O(x)$ is $\{y \in G : y = axa^{-1} \text{ for some } a \in G\}.$ In the case that G acts on itself, the orbit is also called the conjugacy classes of x in $G.$ In this paper, the notation $K(\Omega)$ is used for the number of orbits under group action on a set.

In 2005, Beuerle [6] gives several classifications of finite metacyclic *p*-groups of class at least three. In this paper, we used the classification of metacyclic *p*-group of class at least three for odd prime, as given in the following theorem.

Theorem 5 [6] Let p be an odd prime and G a metacyclic p-group of nilpotency class at least three. Then G is isomorphic to exactly one group in the following list:

1. $G \cong \left\langle a, b \middle| a^{p^\alpha} = b^{p^\beta} = 1, [b, a] = a^{p^{\alpha-\delta}} \right\rangle$, where $\alpha, \beta, \delta \in \square, \delta - 1 \le \alpha < 2\delta, \delta \le \beta;$
2. $G \cong \left\langle a, b \middle| a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, [b, a] = a^{p^{\alpha-\delta}} \right\rangle$, where $\alpha, \beta, \delta, \varepsilon \in \square, \delta - 1 \le \alpha < 2\delta, \delta \le \beta, \alpha < \beta + \varepsilon.$

However, throughout this paper, we considered only on the first classification where *p* is an odd prime number, namely three.

This paper is divided into three sections. The first section browses on some backgrounds on commutativity degree and probability which also includes basic concepts of metacyclic group and its classification, and also group action on set. Meanwhile, the second section shows on previous researches and recent publications that have been done related to probability. In the third section, our main result on the probability that a group element fixes a set is presented.

PRELIMINARIES

The concept of commutativity degree and probability were later investigated by several researchers where they extended the finding of the probability of commuting elements in several groups.

Some problems on statistical group theory and commutativity degree in non-abelian groups were explored by Erdos and Turan [6] in 1968 where the concept of commutativity degree for symmetric group, S_n was introduced. Next, Gustafson [7] found that the probability of two elements commute in a group is equal to the number of conjugacy classes divided by the order of group, $\frac{k(G)}{|G|}$. After that, MacHale [8] showed that the probability of a random pair of elements of a group commute is at most $\frac{5}{8}$. Later in 1975, Sherman [9] defined the probability of an automorphism of a group fixes a random element from a set X, given as follows:

Definition 6 [9] Let G be a group. Let X be the non-empty set of G, where G is a group permutation of X. Then the probability of an automorphism of a group fixes a random element form X is defined as follows:

$$P_G(X) = \frac{\left|\left\{(g,x) \mid gx = x \forall g \in G, x \in X\right\}\right|}{|X||G|}.$$

Recently, Omer *et al.* [10] introduced an extension of probability, named as the probability that a group element fixes a set, which denoted as $P_G(\Omega)$. In the research, they found the probability for metacyclic 2-groups of positive type of nilpotency class at least three. After that, they showed the formula of finding the probability that a group element fixes a set of size two by using conjugation action [10], where some restrictions have been applied on the order of elements of Ω.

Definition 7 [10] Let G be a finite group and let X be a set of elements of size two in the form of (a,b) where a and b commute. Let S be the set of of all subsets of commuting elements of G of size 2 and G acts on S by conjugation. Then the probability that an element of a group fixes a set is given by:

$$P_G(S) = \frac{K}{|S|}$$

where K is the number of conjugacy classes of S in G.

Theorem 8 [10] Let G be a finite group. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute and $\operatorname{lcm}(|a|,|b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size 2 and G acts on Ω. Then the probability that an element of a group fixes a set is given by:

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$$

where $K(\Omega)$ is the number of orbits of Ω in G.

This probability was also found for metacyclic 2-groups of nilpotency class two and class at least three [11].

Next, El-Sanfaz *et al.* [12] extended the research done by Omer *et al.* [10] where they provided the necessary and sufficient conditions for all non-abelian groups in which $P_G(\Omega) = 1$ under conjugation action. Besides, they also computed the probability that a group element fixes a set for metacyclic 2-groups of negative type of nilpotency class two and at least three [13]. Later on, El-Sanfaz *et al.* [14] computed the probability that an element of a group fixes a set for semi-dihedral and quasi-dihedral groups. The results are then applied to graph theory, specifically generalized conjugacy class graph.

Theorem 9 [12] Let G be a finite non-abelian group. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute and $|a| = |b| = 2$. Let Ω be the set of of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then $P_G(\Omega) = 1$ if and only if all commuting elements a and b are in the center of G.

MAIN RESULT

This section provides our main result which is the computation of probability that an element of a metacyclic 3-group of negative type fixes a set. In this paper, we focuses only on the condition when $\alpha = 1, \beta = 1, \delta = 1$, as given in the following proposition.

Proposition 10 Let G be a group such that $G \cong \left\langle a,b \middle| a^{3^{\alpha}} = b^{3^{\beta}} = 1, [b,a] = a^{3^{\alpha-\delta}} \right\rangle$, where $\alpha, \beta, \delta \in \square, \delta - 1 \le \alpha < 2\delta, \delta \le \beta$ and $\alpha = \beta = \delta = 1$. Let S be a set of elements of G of size three in the form of (a,b) where a and b commute and $\mathrm{lcm}(|a|,|b|) = 3$. Let Ω be the set of all subsets of commuting elements of G of size three. If G acts on Ω by conjugation, then $P_G(\Omega) = \frac{5}{6}$.

Proof The elements of G of order three are $a^{3^{\alpha-1}i} 0 \le i \le 2, b$ and b^2. Therefore, the elements of Ω of size three are described as follows. There are two elements in the form of $(1, a^{3^{\alpha-1}i}), 0 \le i \le 3^{\alpha}$, one element in the form of $\left(a^{3^{\alpha-1}}, a^{2\times\left(3^{\alpha-1}\right)}\right)$, two elements in the form of $\left(1, b^{3^{\beta}}\right)$ and one element in the form of $\left(b^{3^{\beta}}, b^{3^{\beta-1}}\right)$. Therefore, $|\Omega| = 6$. Since the action here is by conjugation, thus there are five orbits divided as follows: There is one orbit is in the form of $\left\{(1,a),(1,a^2)\right\}$ of size two, two orbit of size three in the form of $\left\{(1,b),(1,ab),(1,a^2b)\right\}$ and $\left\{(1,b^2),(1,ab^2),(1,a^2b^2)\right\}$, one orbit in the form of $\left\{(a,a^2)\right\}$ of size one and lastly one orbit of $\left\{(b,b^2),(b,ab^2),(b,a^2b^2),(a^2b,b^2),(a^2b,a^2b^2),(ab,b^2),(ab,ab^2)\right\}$ of size seven. Using Theorem 2, $P_G(\Omega) = \frac{5}{6}$, as claimed.

CONCLUSION

In this paper, the probability that an element of a group fixes a set has been computed for metacyclic 3-groups of negative type of nilpotency class at least three. However, only the condition when $\alpha = 1, \beta = 1, \delta = 1$ has been computed where the probability turns out to be $\frac{5}{6}$ as given in the proposition.

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