AIP Conference Proceedings

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Citation: AIP Conference Proceedings **1605**, 601 (2014); doi: 10.1063/1.4887657
View online: http://dx.doi.org/10.1063/1.4887657
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1605?ver=pdfcov
Published by the AIP Publishing

The Productivity Degree of Two Subgroups of Dihedral Groups

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Abstract. The commutativity degree of a group is the probability that two randomly chosen elements of G commute. The concept of commutativity degree is then extended to the relative commutativity degree of a group, which is defined as the probability for an element of H and an element of G to commute to one another. In this research, the relative commutativity degree concept is extended to the probability of a product of two subgroups of a group G productivity degree of two subgroups of a group G which is defined as the ratio of the order of the intersection of HK and KH with the order of their union where H and K are the subgroups of a group G. This research focuses only on the dihedral groups.

Keywords: Productivity degree of two subgroups, Dihedral group
PACS: 02.20.-a

INTRODUCTION

The theory of commutativity degree in group theory is one of the oldest areas in group theory and plays a major role in determining the abelianness of the group. It has been attracted by many researchers and it is studied in various directions. Many papers by Miller, Lescot and Rusin as in [1,2,3], give explicit formulas of $P(G)$ for some particular finite groups G. The concept of commutativity degree can be generalized and modified in many directions. For instance, two subgroups H and K of G permute if $HK = KH$. Hence, by changing the role of elements to subgroups in a finite group, one can obtain a modification of the commutativity degree of a finite group. If G is a finite group, then the commutativity degree of G, denoted by $P(G)$, is the probability that two randomly chosen element of G commute. The first appearance of this concept was in 1944 by Miller [1]. Then, the idea to compute $P(G)$ for symmetric groups has been introduced by Erdos and Turan [4] at the end of the 60s.

For any finite group G, and if H is a subgroup then the relative commutativity degree of G, denoted by $P(H,G)$, is the probability for an element of H and an element of G to commute to one another. Similarly, if K is another subgroup of G then the probability for an element of H to commute to an element of K, is denoted by $P(H,K)$. This probability is called the relative commutativity degree of two subgroups of a finite group. This concept was first introduce by Erfanian *et al*. in [5]. Meanwhile, in 2011, Erfanian *et al*. [6] defined the relative n-th commutativity degree as the probability that the n-th power of a random element of H commutes with a random element of G. This relative commutativity degree can be extended to the productivity degree of two subgroups of G where it is defined as the ratio of the order of the intersection of HK and KH with the order of their union where H and K are the subgroups of a group G

$$\mathrm{Pro}_G(H,K) = \frac{|HK \cap KH|}{|HK \cup KH|}.$$

In this paper, the productivity degree of two subgroups of dihedral groups which is denoted as $\mathrm{Pro}_D(H,K)$, is computed. It is can easily be seen that $\mathrm{Pro}_G(H,K) = 1$ if H and K are trivial or center of a group. It will also give the same result if $H = K$.

Proceedings of the 21st National Symposium on Mathematical Sciences (SKSM21)
AIP Conf. Proc. 1605, 601-604 (2014); doi: 10.1063/1.4887657

PRELIMINARIES

In this section, some definitions and theorems used in this research are listed below.

Definition 1 [7] Dihedral Groups of Degree *n*

For each $n \in \mathbb{Z}$, and $n \geq 2$, D_{2n} is denoted as the set of symmetries of a regular *n*-gon. Furthermore, the order of D_{2n} is $2n$ or equivalently $|D_{2n}| = 2n$. The Dihedral groups, D_{2n} can be represented in a form of generators and relations given in the following representation:

$$D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, yx = x^{-1}y \rangle.$$

Next, the Cayley Table for the dihedral group of order 6, D_6 and the dihedral group of order 8, D_8 are given as in the following tables and will be used in the next section.

TABLE (1). Cayley Table for D_6

*	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

TABLE (2). Cayley Table for D_8

*	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^3b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	e	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	e	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	e	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	e

RESULTS

In this section, the generalization of $\text{Pro}_{D_{2n}}(H,K)$ are presented through the following propositions and theorems. Furthermore, some examples are given to illustrate our results.

Proposition 1

Let G be a Dihedral group of order $2n$ and n is odd. Suppose H and K are subgroups of D_{2n} . If $|H| = 2$ and $|K| = 2$, then $\text{Pro}_{D_{2n}}(H,K) = \frac{3}{5}$.

Proof

Suppose H and K are subgroups of D_{2n} for n is odd and e, x be two elements in H and e, y elements in K *i.e* $H=\{e,x\}$ and $K=\{e,y\}$. Then $HK=\{e,y,x,xy\}$ and $KH=\{e,x,y,yx\}$. Therefore, $HK\bigcap KH=\{e,y,x\}$ implies $|HK\bigcap KH|=3$. Meanwhile, $HK\bigcup KH=\{e,y,x,xy,yx\}$ implies $|HK\bigcup KH|=5$ since $xy\neq yx$. Thus, by definition, $\text{Pro}_{D_{2n}}(H,K)=\dfrac{3}{5}$.

Example 1

Given G is dihedral group of order 6, D_6. Let $H=\{e,b\}$ and $K=\{e,ab\}$ since $|H|=2$ and $|K|=2$. By refering to Table 1, $HK=\{e,ab,b,a^2\}$ and $KH=\{e,b,ba,a\}$. Since $b*ab\neq ab*b$, then $|HK\bigcap KH|=3$. Clearly $|HK\bigcup KH|=5$ Thus, $\text{Pro}_{D_{2n}}(H,K)=\dfrac{3}{5}$.

Theorem 1

If H is a normal subgroup of D_{2n} and $K\leq D_{2n}$ then $\text{Pro}_{D_{2n}}(H,K)=1$.

Proof

Let H be normal subgroup and K be another subgroups of Dihedral group. Suppose $h\in H$, $k\in K$ and $H\lhd D_{2n}$. Thus $gh=hg$ for all $g\in D_{2n}$. Since $k\in K$ implies $k\in D_{2n}$. Hence $hk=kh$.

Example 2

Given G is dihedral group of order 8, D_8. Let $H=\{e,a,a^2,a^3\}$ which is a normal subgroup of G and $K=\{e,a^2b\}$.By refering to Table 2, $HK=\{e,a^2b,a,a^3b,a^2,b,a^3,ab\}$ and $KH=\{e,a,a^2,a^3,a^2b,ab,b,a^3b\}$. By using definition, $|HK\bigcap KH|=8$ and $|HK\bigcup KH|=8$ implies $\dfrac{|HK\bigcup KH|}{|HK\bigcap KH|}=1$. Thus, $\text{Pro}_{D_{2n}}(H,K)=1$.

Theorem 2

Let H and K be any cyclic subgroups of Dihedral group of order $2p$, where p is prime. If $|H|=1$ or p or $|K|=1$ or p, then $\text{Pro}_{D_{2n}}(H,K)=1$. Conversely, if $|H|=2$ and $|K|=2$ then $\text{Pro}_{D_{2n}}(H,K)=\dfrac{3}{5}$.

Proof

Suppose $|H|=1$ or p. There is only a subgroup of Dihedral group of order $2p$ whose element is one i.e identity and clearly p belongs to normal subgroup. By Proposition 1, if H is identity and $K\leq D_{2p}$ then $\text{Pro}_{D_{2n}}(H,K)=1$. By Theorem 1, if H is a normal subgroup of D_{2p} and $K\leq D_{2p}$, then $P_{D_{2p},S}(H,K)=1$.. Conversely, by Proposition 3, if $H\leq D_{2p}$ and $K\leq D_{2p}$ where $|H|=2$ and $|K|=2$ then $\text{Pro}_{D_{2n}}(H,K)=\dfrac{3}{5}$.

Example 3

Given G is dihedral group of order 6, D_6. Let $H=\{e,a,a^2\}$ since $|H|=p$ and let K be another subgroup i.e $K=\{e,ab\}$. Then Table 1 gives $HK=\{e,ab,a,a^2b,a^2,b\}$ meanwhile $KH=\{e,a,a^2,ab,b,a^2b\}$. By using definition , $|HK\cap KH|=6$ and $|HK\cup KH|=6$ implies $\frac{|HK\cup KH|}{|HK\cap KH|}=1$. Thus $\text{Pro}_{D_{2n}}(H,K)=1$. Similar result is holds if $H=\{e\}$ since the identity commute with every elements. Conversely, suppose $H=\{e,b\}$ and $K=\{e,ab\}$ since $|H|=2$ and $|K|=2$. From Table 1, $HK=\{e,ab,b,a^2\}$ and $KH=\{e,b,ab,a\}$. Since the element b and ab do not commute, thus $|HK\cap KH|=3$ and $|HK\cup KH|=5$ Therefore, $\text{Pro}_{D_{2n}}(H,K)=\frac{3}{5}$.

CONCLUSION

In this paper, the probability of a product of two subgroups of dihedral groups has been obtained. From the results obtained, it is shown that $\text{Pro}_{D_{2n}}(H,K)$ is equal to 1 if H is a normal subgroup and if $|H|=1$ or p or $|K|=1$ or p for all order of dihedral groups where p is prime. Meanwhile, $\text{Pro}_{D_{2n}}(H,K)$ is equal to 3/5 if $|H|=2$ and $|K|=2$ for n is odd.

ACKNOWLEDGMENTS

The authors would like to acknowledge Ministry of Education (MOE) Malaysia, Universiti Teknologi Malaysia, Johor Bahru, Malaysia for the financial funding of this research through the Research University Grant (RUG) with vote No. 07J70. The first author is also indebted to MOE for her MyPhD Scholarship.

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