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Multiplicative Degree of Some Dihedral Groups

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Abstract. Let *G* be a group and *H* any subgroup of *G*. The commutativity degree of a finite group *G* is defined as the probability that a pair of elements *x* and *y*, chosen randomly from a group *G*, commute. The concept of commutativity degree has been extended to the relative commutativity degree of a subgroup *H*, which is defined as the probability that a random element of a subgroup, *H* commutes with another random element of a group *G*. This research extends the concept of relative commutativity degree to the multiplicative degree of a group *G*, which is defined as the probability that the product of a pair of elements *x*, *y* chosen randomly from a group *G*, is in *H*. This research focuses on some dihedral groups.

INTRODUCTION

In this paper, *G* is considered as a finite group. The commutativity degree of a group *G* is the probability that a selected chosen pair of elements of a group *G* commute, denoted by *P*(*G*), and it was firstly introduced by Miller [1] in 1944. The commutativity degree has been investigated by several authors [2-6] and some formulas of *P*(*G*) have been found for some finite groups *G*.

Sherman [7] used this concept of probability and proved that the probability cannot be arbitrarily close to 1 if *G* is a finite nonabelian group. Gustafson [3] and Machale [4] showed that the commutativity degree of all finite groups is less than or equal to $\frac{5}{8}$.

The concept of commutativity degree has been extended to the relative commutativity degree of a subgroup *H*, which is the probability for an element of a subgroup *H* and an element of a group *G* to commute with one another. This concept has been generalized by Erfanian *et al*. [8] in 2007, where the definition of the relative commutativity degree, denoted as $P(H,G)$ was introduced. In 2012, Abdul Hamid *et al*. [9] presented some results on *P*(*H*,*G*), where *G* is the Dihedral groups up to order 26. In the case that $H = G,$ we have $P(H,G) = P(G)$ and if *G* is abelian, then $P(H,G) = 1.$

Barzgar *et al*. [10] in 2013, studied the set of all relative commutativity degree of a subgroup *G* and computed the number of relative commutativity degree for some classes of finite groups including dihedral groups, generalized quaternion groups and quasi-dihedral groups.

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Inspired by this concept, we introduced a new extended relative commutativity degree, called the multiplicative degree of a group G. This multiplicative degree is defined as the probability that the product of a pair of elements x and y chosen randomly from a group G, is in H.

PRELIMINARIES

In this section, some preliminaries and basic definitions that are required in this research are provided as follows.

Definition 1 [11] **Dihedral Groups of Degree *n***
For $n \geq 3$, dihedral groups, D_n is denoted as the set of symmetries of a regular n-gon. Furthermore, the order of D_n is $2n$, or equivalently $|D_n| = 2n$. The Dihedral groups, D_n can be represented in a form of generators and relations given as in the following;

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ ba = a^{-1}b \rangle.$$

Definition 2 [1] **The Commutativity Degree of a Group**
The commutativity degree of a group G, $P(G)$, is defined as

$$P(G) = \frac{\left|\left\{(x, y) \in G \times G \,\middle|\, xy = yx\right\}\right|}{|G|^2}.$$

Definition 3 [8] **The Relative Commutativity Degree of a Subgroup of a Group**
The relative commutativity degree of a subgroup H of a group G, $P(H,G)$, is defined as

$$P(H, G) = \frac{\left|\left\{(h, g) \in H \times G \,\middle|\, [h, g] = 1\right\}\right|}{|H||G|}.$$

Definition 4 The Multiplicative Degree of a Group
Let G be a finite nonabelian group and H any subgroup of G. For any $x, y \in G,$ then the multiplicative degree of a group G, denoted as $P_{xy}(G)$, is defined as

$$P_{xy}(G) = \frac{\left|\left\{(x, y) \in G \times G : xy \in H\right\}\right|}{|G|^2}.$$

In the next section, new results of the multiplicative degree of some dihedral groups are presented.

RESULTS AND DISCUSSION

In this section, the result of $P_{xy}(D_n),$ that is the multiplicative degree of some dihedral groups is presented. There are three propositions in this paper. In the first proposition, the multiplicative degree of a dihedral group when both of the elements x and y are in the subgroup H of D_n, is given.

Proposition 1
Let D_n be a dihedral group of order $2n$, where $n \geq 3$ and H be any subgroup of D_n. Suppose $x, y \in H$ then $P_{xy}(D_n) = \left(\frac{|H|}{|D_n|}\right)^2$.

Proof

Let $H \le D_n$, then for every $x, y \in H$ we have $xy \in H$ and $e_{D_n} \in H$. By the Definition 4, $P_{xy}(D_n) = \dfrac{|H|^2}{|D_n|^2}$. Therefore,

$$P_{xy}(D_n) = \left(\frac{|H|}{|D_n|}\right)^2. \square$$

In the following two propositions, the multiplicative degree of two cyclic subgroups of order n and $\frac{n}{2}$ of D_n respectively, are given. These propositions give the multiplicative degree of a dihedral group when both of the elements x and y are in D_n but not in the subgroup H.

Proposition 2

Let D_n be a dihedral group of order $2n$, where $n \ge 3$. Suppose H is a cyclic subgroup of D_n of order n. Let $A = \{x, y \in D_n \setminus H \text{ but } xy \in H\}$. If $x, y \in A$ then $P_{xy}(D_n) = \frac{1}{4}$.

Proof

Suppose H is a cyclic subgroup of D_n of order n. Now, let $A = \{x, y \in D_n \setminus H \text{ but } xy \in H\}$. Here A is not an empty set since for $x, y \in D_n \setminus H$ and $y = x^{-1}$, we have $e_{D_n} \in H$. Take $s \in A$. Then the possible number of s is n since $|H| = n$, $|D_n| = 2n$ implies $|s| = |D_n| - |H| = n$.

Let $x = b$ and $y = a^n b$, by the relation in the presentation of D_n (see Definition 1), we have

$$\begin{aligned} xy &= ba^n b \\ &= baaaa...ab \\ &= a^{-1}ba^{n-1}b \\ &= a^{-1}a^{-1}ba^{n-2}b \\ &= a^{-2}ba^{n-2}b \\ &= a^{-2}a^{-1}ba^{n-3}b \\ &= a^{-3}ba^{n-3}b. \end{aligned}$$

By continuing in the same way we have

$$\begin{aligned} xy &= a^{-n}ba^{n-n}b \\ &= a^{-n}bb \\ &= a^{-n}b^2 \\ &= a^{-n}e \\ &= a^{-n}. \end{aligned}$$

By the same calculations, it can be shown that :

$$\begin{aligned} &\text{If } x = ab,\ y = a^n b \text{ then } xy = a^{1-n} \\ &\text{If } x = a^2 b,\ y = a^n b \text{ then } xy = a^{2-n} \\ &\text{If } x = a^3 b,\ y = a^n b \text{ then } xy = a^{3-n} \\ &\quad . \\ &\quad . \\ &\quad . \\ &\text{If } x = a^n b,\ y = a^n b \text{ then } xy = a^{n-n} = 1. \end{aligned}$$

Therefore the possible number of pairs of $x, y \in A$ is equal to $|A| = n^2$. Thus by Definition 4, $P_{xy}(D_n) = \frac{n^2}{(2n)^2} = \frac{n^2}{4n^2} = \frac{1}{4}$. □

The following is an example that explains how the multiplicative degree of a cyclic subgroup of order n can be computed.

Example 1

Let D_{10} be a dihedral group of order 20. Then we can write $D_{10} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b\}$. Let H be a subgroup of D_{10}, $H = \langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9\}$. We have $A = \{x, y \in D_{10} \setminus H \text{ but } xy \in H\}$. Take $s \in A$, then the possible number of s is 10 since $|s| = |D_{10}| - |H| = 20 - 10 = 10$. Therefore $|A| = 10^2$. Thus by Definition 4, $P_{xy}(D_{10}) = \frac{|A|}{|D_{10}|^2} = \frac{10^2}{(20)^2} = \frac{1}{4}$.

Proposition 3

Let D_n be a dihedral group of order $2n$, where $n \geq 5$ and n is even. Let H be a cyclic subgroup of D_n of order $\frac{n}{2}$ and $A = \{x, y \in D_n \setminus H \text{ but } xy \in H\}$. If $x, y \in A$ then $P_{xy}(D_n) = \frac{3}{16}$.

Proof

Suppose D_n is a dihedral group of order $2n$, where $n \geq 5$ and n is even. Suppose H is a cyclic subgroup of D_n of order $\frac{n}{2}$ which is $H = \langle a^2 \rangle$ and let $A = \{x, y \in D_n \setminus H \text{ but } xy \in H\}$. The group D_n has $2n$ elements listed in the following:

$$D_n = \{1, a, a^2, ..., a^{n-1}, b, ab, ..., a^{n-1}b\}.$$

The group D_n consists of n rotations and n reflections which are the elements in the sets $\{a, a^2, a^3, ..., a^{n-1}\}$ and $\{b, ab, a^2b, ..., a^{n-1}b\}$, respectively.

For both x and y are rotations and $m = 0, 1, 2, ..., n-1$, we have the following :

If $x = a$ and $y = a^{2m+1}$ then,

$$\begin{aligned} xy &= a \cdot a^{2m+1} \\ &= a^{2(m+1)} \in H. \end{aligned}$$

If $x = a^3$ and $y = a^{2m+1}$ then,

$$\begin{aligned} xy &= a^3 \cdot a^{2m+1} \\ &= a^{2m+4} \\ &= a^{2(m+2)} \in H. \end{aligned}$$

If $x = a^5$ and $y = a^{2m+1}$ then,

$$\begin{aligned} xy &= a^5 \cdot a^{2m+1} \\ &= a^{2m+6} \\ &= a^{2(m+3)} \in H. \end{aligned}$$

By continuing in the same way we have the following :

If $x = a^{n-1}$ and $y = a^{2m+1}$ then,

$$\begin{aligned} xy &= a^{n-1} \cdot a^{2m+1} \\ &= a^{n-1+(2m+1)} \\ &= a^{n+2m} \in H. \end{aligned}$$

Here, the possible number of pair of $x, y \in A$ is $|H|^2$, namely $\left(\frac{n}{2}\right)^2$.

For both x and y are reflections and $i = 0,1,2,...,n-2$. We consider the relation in the presentation of D_n (see Definition 1).

If $x = b$ and $y = a^{2i}b$ then,

$$\begin{aligned} xy &= b \cdot a^{2i}b \\ &= ba^2a^2a^2...a^2b \\ &= a^{-1}baa^{2i-1}b \\ &= a^{-1}a^{-1}ba^{2i-1}b \\ &= a^{-2}a^{-1}baa^{2i-2}b \\ &= a^{-3}a^{-1}ba^{2i-2}b \\ &= a^{-4}ba^{2i-2}b \\ &\;\;\vdots \\ &= a^{-(i+i)}ba^{2(i-i)}b \\ &= a^{-2i}bb \\ &= a^{-2i}b^2 \\ &= \left(a^{2i}\right)^{-1} \in H. \end{aligned}$$

If $x = a^2b$ and $y = a^{2i}b$ then,

$$
\begin{aligned}
xy &= a^2b \cdot a^{2i}b \\
&= a^2ba^2a^2a^2...a^2b \\
&= a^2a^{-1}baa^{2i-1}b \\
&= aa^{-1}ba^{2i-1}b \\
&= a^{-1}baa^{2i-2}b \\
&= a^{-1}a^{-1}ba^{2i-2}b \\
&= a^{-2}ba^{2i-2}b \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a^{-(i+i)}ba^{2(i-i)}b \\
&= a^{-2i}bb \\
&= a^{-2i}b^2 \\
&= \left(a^{2i}\right)^{-1} \in H.
\end{aligned}
$$

By the same calculations, the following can be shown :

If $x = a^{n-2}b,\ y = a^{2i}b$ then

$$
\begin{aligned}
xy &= a^{n-2}b \cdot a^{2i}b \\
&= a^{n-2}ba^2a^2a^2...a^2b \\
&= a^{n-2}a^{-1}baa^{2i-1}b \\
&= a^{n-3}a^{-1}ba^{2i-1}b \\
&= a^{n-4}a^{-1}baa^{2i-2}b \\
&= a^{n-5}a^{-1}ba^{2i-2}b \\
&= a^{n-6}ba^{2i-2}b \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a^{n-n}ba^{2(i-i)}b \\
&= b^2 \\
&= 1 \in H.
\end{aligned}
$$

For both x and y are reflections and $j = 0,1,2,...,n-1$, we consider the relation in the presentation of D_n (see Definition 1).

If $x = ab$ and $y = a^{2j+1}b$ then,

$$
\begin{aligned}
xy &= ab \cdot a^{2j+1}b \\
&= aba^{2j}ab \\
&= aba^2a^2a^2 \ldots a^2ab \\
&= aa^{-1}baa^{2j-1}ab \\
&= a^{-1}ba^{2j-1}ab \\
&= a^{-1}a^{-1}baa^{2j-2}ab \\
&= a^{-2}a^{-1}ba^{2j-2}ab \\
&= a^{-3}ba^{2j-2}ab \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a^{-(j+j)}ba^{2(j-j)}b \\
&= a^{-2j}bb \\
&= a^{-2j}b^2 \\
&= \left(a^{2j}\right)^{-1} \in H.
\end{aligned}
$$

If $x = a^3b$ and $y = a^{2j+1}b$ then,

$$
\begin{aligned}
xy &= a^3b \cdot a^{2j+1}b \\
&= a^3ba^{2j}ab \\
&= a^3ba^2a^2a^2 \ldots a^2ab \\
&= a^3a^{-1}baa^{2j-1}ab \\
&= a^2a^{-1}ba^{2j-1}ab \\
&= aa^{-1}baa^{2j-2}ab \\
&= a^{-1}ba^{2j-2}ab \\
&= a^{-1}a^{-1}baa^{2j-3}ab \\
&= a^{-2}a^{-1}ba^{2j-3}ab \\
&= a^{-3}ba^{2j-3}ab \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a^{-(j+j)}ba^{2(j-j)}b \\
&= a^{-2j}bb \\
&= a^{-2j}b^2 \\
&= \left(a^{2j}\right)^{-1} \in H.
\end{aligned}
$$

Similarly, we have the following :

If $x = a^{n-1}b,\ y = a^{2j+1}b$ then

$$
\begin{aligned}
xy &= a^{n-1}b \cdot a^{2j+1}b \\
&= a^{n-1}ba^2a^2a^2 \ldots a^2ab \\
&= a^{n-1}a^{-1}baa^{2j-1}ab \\
&= a^{n-2}a^{-1}ba^{2j-1}ab \\
&= a^{n-3}a^{-1}baa^{2j-2}ab \\
&= a^{n-4}a^{-1}ba^{2j-2}ab \\
&= a^{n-5}ba^{2j-2}ab \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a^{n-n}ba^{2(i-i)}b \\
&= b^2 \\
&= 1 \in H.
\end{aligned}
$$

Here, the possible number of pair of $x, y \in A$ is $|H| \cdot n$ implies $\left(\frac{n}{2}\right) \cdot n.$

Therefore the possible number of pair of $x, y \in A$ is $|A| = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right)n = \frac{3n^2}{4}$. Thus, by Definition 4,

$$P_{xy}(D_n) = \frac{\left(\frac{3n^2}{4}\right)}{(2n)^2} = \frac{3}{16}. \ \square$$

Remark: This calculation involves multiplying the element of x and y where x and y either both rotations or both reflections. D_4 is not included in this calculation because for x, y in A, x and y can be both rotations, both reflections, one of them rotation and another one reflection and vice versa.

The following is an example that explains how the multiplicative degree of a cyclic subgroup of order $\frac{n}{2}$ can be computed.

Example 2:
Let D_8 be a dihedral group of order 16. Then we can write $D_8 = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$.
Let H be a subgroup of D_8, $H = \langle a^2 \rangle = \{1, a^2, a^4, a^6\}$. We have $A = \{x, y \in D_8 \setminus H \text{ but } xy \in H\}$. Therefore $|A| = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right)n = (4)^2 + (4 \times 8) = 48.$ Thus by Definition 4, $P_{xy}(D_8) = \frac{|A|}{|D_8|^2} = \frac{48}{(16)^2} = \frac{3}{16}.$

CONCLUSION

In this research, the multiplicative degree of some of dihedral groups is found. For every $x, y \in H$ we have $xy \in H$ then the multiplicative degree of a dihedral group is equal to $\left(\frac{|H|}{|D_n|}\right)^2$. Meanwhile, for every $x, y \in D_n \setminus H$ but $xy \in H$ and H is a cyclic subgroup of D_n of order n and $\frac{n}{2}$ then the multiplicative degree of a dihedral group is equal to $\frac{1}{4}$ and $\frac{3}{16}$ respectively.

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