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Citation: AIP Conference Proceedings 1750, 050003 (2016); doi: 10.1063/1.4954591
View online: http://dx.doi.org/10.1063/1.4954591
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1750?ver=pdfcov
Published by the AIP Publishing

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Relative commutativity degree of some dihedral groups
Multiplicative Degree of Some Dihedral Groups

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\textbf{Abstract.} Let $G$ be a group and $H$ any subgroup of $G$. The commutativity degree of a finite group $G$ is defined as the probability that a pair of elements $x$ and $y$, chosen randomly from a group $G$, commute. The concept of commutativity degree has been extended to the relative commutativity degree of a subgroup $H$, which is defined as the probability that a random element of a subgroup, $H$ commutes with another random element of a group $G$. This research extends the concept of relative commutativity degree to the multiplicative degree of a group $G$, which is defined as the probability that the product of a pair of elements $x, y$ chosen randomly from a group $G$, is in $H$. This research focuses on some dihedral groups.

\section*{INTRODUCTION}

In this paper, $G$ is considered as a finite group. The commutativity degree of a group $G$ is the probability that a selected chosen pair of elements of a group $G$ commute, denoted by $P(G)$, and it was firstly introduced by Miller \cite{1} in 1944. The commutativity degree has been investigated by several authors \cite{2-6} and some formulas of $P(G)$ have been found for some finite groups $G$.

Sherman \cite{7} used this concept of probability and proved that the probability cannot be arbitrarily close to 1 if $G$ is a finite nonabelian group. Gustafson \cite{3} and Machale \cite{4} showed that the commutativity degree of all finite groups is less than or equal to $\frac{5}{8}$.

The concept of commutativity degree has been extended to the relative commutativity degree of a subgroup $H$, which is the probability for an element of a subgroup $H$ and an element of a group $G$ to commute with one another. This concept has been generalized by Erfanian \textit{et al.} \cite{8} in 2007, where the definition of the relative commutativity degree, denoted as $P(H,G)$ was introduced. In 2012, Abdul Hamid \textit{et al.} \cite{9} presented some results on $P(H,G)$, where $G$ is the Dihedral groups up to order 26. In the case that $H = G$, we have $P(H,G) = P(G)$ and if $G$ is abelian, then $P(H,G) = 1$.

Barzgar \textit{et al.} \cite{10} in 2013, studied the set of all relative commutativity degree of a subgroup $G$ and computed the number of relative commutativity degree for some classes of finite groups including dihedral groups, generalized quaternion groups and quasi-dihedral groups.
Inspired by this concept, we introduced a new extended relative commutativity degree, called the multiplicative degree of a group $G$. This multiplicative degree is defined as the probability that the product of a pair of elements $x$ and $y$ chosen randomly from a group $G$, is in $H$.

**PRELIMINARIES**

In this section, some preliminaries and basic definitions that are required in this research are provided as follows.

**Definition 1** [11] Dihedral Groups of Degree $n$
For $n \geq 3$, dihedral groups, $D_n$ is denoted as the set of symmetries of a regular $n$-gon. Furthermore, the order of $D_n$ is $2n$, or equivalently $|D_n| = 2n$. The Dihedral groups, $D_n$ can be represented in a form of generators and relations given as in the following:

$$D_n = \{a, b | a^n = b^2 = 1, \text{ } ba = a^{-1}b\}.$$  

**Definition 2** [1] The Commutativity Degree of a Group
The commutativity degree of a group $G$, $P(G)$, is defined as

$$P(G) = \frac{\left|\{(x, y) \in G \times G | xy = yx\}\right|}{|G|^2}.$$  

**Definition 3** [8] The Relative Commutativity Degree of a Subgroup of a Group
The relative commutativity degree of a subgroup $H$ of a group $G$, $P(H, G)$, is defined as

$$P(H, G) = \frac{\left|\{(h, g) \in H \times G | [h, g] = 1\}\right|}{|H||G|}.$$  

**Definition 4** The Multiplicative Degree of a Group
Let $G$ be a finite nonabelian group and $H$ any subgroup of $G$. For any $x, y \in G$, then the multiplicative degree of a group $G$, denoted as $P_{xy}(G)$, is defined as

$$P_{xy}(G) = \frac{\left|\{(x, y) \in G \times G : xy \in H\}\right|}{|G|^2}.$$  

In the next section, new results of the multiplicative degree of some dihedral groups are presented.

**RESULTS AND DISCUSSION**

In this section, the result of $P_{xy}(D_n)$, that is the multiplicative degree of some dihedral groups is presented. There are three propositions in this paper. In the first proposition, the multiplicative degree of a dihedral group when both of the elements $x$ and $y$ are in the subgroup $H$ of $D_n$, is given.

**Proposition 1**
Let $D_n$ be a dihedral group of order $2n$, where $n \geq 3$ and $H$ be any subgroup of $D_n$. Suppose $x, y \in H$ then $P_{xy}(D_n) = \left(\frac{|H|}{|D_n|}\right)^2$. 

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Proof

Let \( H \leq D_n \), then for every \( x, y \in H \) we have \( xy \in H \) and \( e_{\frac{\theta}{2\pi}} \in H \). By the Definition 4, \( P_{xy}(D_n) = \frac{|H|^2}{|D_n|} \). Therefore,

\[
P_{xy}(D_n) = \left( \frac{|H|}{|D_n|} \right)^2.
\]

In the following two propositions, the multiplicative degree of two cyclic subgroups of order \( n \) and \( \frac{n}{2} \) of \( D_n \) respectively, are given. These propositions give the multiplicative degree of a dihedral group when both of the elements \( x \) and \( y \) are in \( D_n \) but not in the subgroup \( H \).

**Proposition 2**

Let \( D_n \) be a dihedral group of order \( 2n \), where \( n \geq 3 \). Suppose \( H \) is a cyclic subgroup of \( D_n \) of order \( n \). Let \( A = \{ x, y \in D_n \setminus H \text{ but } xy \in H \} \). If \( x, y \in A \) then \( P_{xy}(D_n) = \frac{1}{4} \).

**Proof**

Suppose \( H \) is a cyclic subgroup of \( D_n \) of order \( n \). Now, let \( A = \{ x, y \in D_n \setminus H \text{ but } xy \in H \} \). Here \( A \) is not an empty set since for \( x, y \in D_n \setminus H \) and \( y = x^{-1} \), we have \( e_{\frac{\theta}{2\pi}} \in H \). Take \( s \in A \). Then the possible number of \( s \) is \( n \) since \( |H| = n \), \( |D_n| = 2n \) implies \( |s| = |D_n| - |H| = n \).

Let \( x = b \) and \( y = a^n b \), by the relation in the presentation of \( D_n \) (see Definition 1), we have

\[
xy = ba^n b
= baaaa...ab
= a^{-1}ba^{n+1}b
= a^{-1}a^{-1}ba^{n-2}b
= a^{-2}ba^{n-2}b
= a^{-2}a^{-1}ba^{n-3}b
= a^{-3}ba^{n-3}b.
\]

By continuing in the same way we have

\[
xy = a^{-m}ba^{n-m}b
= a^{-m}bb
= a^{-m}b^2
= a^{-m}e
= a^{-m}.
\]

By the same calculations, it can be shown that:

- If \( x = ab, y = a^n b \) then \( xy = a^{1-n} \)
- If \( x = a^7 b, y = a^n b \) then \( xy = a^{2-n} \)
- If \( x = a^3 b, y = a^n b \) then \( xy = a^{5-n} \)
  
  ... 
  
  ... 

- If \( x = a^3 b, y = a^n b \) then \( xy = a^{6-n} = 1 \).
Therefore the possible number of pairs of \( x, y \in A \) is equal to \( |A| = n^2 \). Thus by Definition 4,
\[
P_{xy}(D_n) = \frac{n^2}{(2n)^2} = \frac{n^2}{4n^2} = \frac{1}{4}. \quad \square
\]

The following is an example that explains how the multiplicative degree of a cyclic subgroup of order \( n \) can be computed.

**Example 1**

Let \( D_{10} \) be a dihedral group of order 20. Then we can write \( D_{10} = \{ e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, b, ab, a^2b, a^3b, a^4b, a^5b, a^7b, a^8b \} \). Let \( H \) be a subgroup of \( D_{10} \), \( H = \langle a \rangle = \{ e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 \} \). We have \( A = \{ x, y \in D_{10} \setminus H \text{ but } xy \in H \} \). Take \( s \in A \), then the possible number of \( s \) is 10 since \( |s| = |D_{10}| - |H| = 20 - 10 = 10 \). Therefore \( |A| = 10^2 \). Thus by Definition 4,
\[
P_{xy}(D_{10}) = \frac{|A|}{|D_{10}|^2} = \frac{10^2}{(20)^2} = \frac{1}{4}.
\]

**Proposition 3**

Let \( D_n \) be a dihedral group of order \( 2n \), where \( n \geq 5 \) and \( n \) is even. Let \( H \) be a cyclic subgroup of \( D_n \) of order \( \frac{n}{2} \) and \( A = \{ x, y \in D_n \setminus H \text{ but } xy \in H \} \). If \( x, y \in A \) then \( P_{xy}(D_n) = \frac{3}{16} \).

**Proof**

Suppose \( D_n \) is a dihedral group of order \( 2n \), where \( n \geq 5 \) and \( n \) is even. Suppose \( H \) is a cyclic subgroup of \( D_n \) of order \( \frac{n}{2} \) which is \( H = \langle a^2 \rangle \) and let \( A = \{ x, y \in D_n \setminus H \text{ but } xy \in H \} \). The group \( D_n \) has \( 2n \) elements listed in the following:
\[
D_n = \{ 1, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b \}.
\]

The group \( D_n \) consists of \( n \) rotations and \( n \) reflections which are the elements in the sets \( \{ a, a^2, \ldots, a^{n-1} \} \) and \( \{ b, ab, a^2b, \ldots, a^{n-1}b \} \), respectively.

For both \( x \) and \( y \) are rotations and \( m = 0, 1, 2, \ldots, n-1 \), we have the following:

If \( x = a \) and \( y = a^{2m+1} \) then,
\[
xy = a \cdot a^{2m+1} = a^{2m+2} \in H. \]

If \( x = a^3 \) and \( y = a^{2m+1} \) then,
\[
xy = a^3 \cdot a^{2m+1} = a^{2m+4} \in H.
\]

If \( x = a^5 \) and \( y = a^{2m+1} \) then,
\[
xy = a^5 \cdot a^{2m+1} = a^{2m+6} \in H.
\]

By continuing in the same way we have the following:
If \( x = a^{n-1} \) and \( y = a^{2m+1} \) then,

\[
xy = a^{n-1} \cdot a^{2m+1} \\
= a^{n-1+2m+1} \\
= a^{n+2m} \in H.
\]

Here, the possible number of pair of \( x, y \in A \) is \( |H|^2 \), namely \( \left( \frac{n}{2} \right)^2 \).

For both \( x \) and \( y \) are reflections and \( i = 0, 1, 2, \ldots, n-2 \). We consider the relation in the presentation of \( D_n \) (see Definition 1).

If \( x = b \) and \( y = a^{2i} \) then,

\[
xy = b \cdot a^{2i} \\
= ba^2a^2 \ldots a^2b \\
= a^{-1}baa^{2i-1}b \\
= a^{-1}a^{-1}ba^{2i-1}b \\
= a^{-2}a^{-1}ba^{2i-2}b \\
= a^{-3}a^{-1}ba^{2i-2}b \\
= a^{-4}ba^{2i-2}b \\
\hspace{2cm} \vdots \\
= a^{-(i+i)}ba^{2(i-i)}b \\
= a^{-2i}bb \\
= a^{-2i}b^2 \\
= (a^{2i})^{-1} \in H.
\]

If \( x = a^2b \) and \( y = a^{2i} \) then,
xy = a^2 b \cdot a^{2i} b \\
= a^2 \cdot a^{-2} b \cdot a^{2} ... a^{2i} b \\
= a^2 \cdot a^{-1} b \cdot a a^{2i-1} b \\
= a^{-1} b a a^{2i-1} b \\
= a^{-1} b a a^{2i-2} b \\
= a^{-1} a^{-1} b a a^{2i-2} b \\
= a^{-2} b a a^{2i-2} b \\
= a^{-2} b a^{2i-2} b \\
= \ldots \\
= a^{-i+i} b a a^{2(i-1)} b \\
= a^{-2i} b b \\
= a^{-2i} b^2 \\
= (a^{2i})^{-1} \in H.

By the same calculations, the following can be shown:
If \( x = a^{n-2} b \), \( y = a^{2i} b \) then

\[ xy = a^{n-2} b \cdot a^{2i} b \]
\[ = a^{n-2} b a^{2-2} a^{2i} a ... a^{2i} b \]
\[ = a^{n-2} a^{-1} b a a^{2i-1} b \]
\[ = a^{n-3} a^{-1} b a a^{2i-1} b \]
\[ = a^{n-4} a^{-1} b a a^{2i-2} b \]
\[ = a^{n-5} a^{-1} b a a^{2i-2} b \]
\[ = a^{n-6} b a a^{2i-2} b \]
\[ = \ldots \]
\[ = a^{-n} b a a^{2(i-1)} b \]
\[ = b^2 \]
\[ = 1 \in H. \]

For both \( x \) and \( y \) are reflections and \( j = 0,1,2,\ldots,n-1 \), we consider the relation in the presentation of \( D_n \) (see Definition 1).

If \( x = ab \) and \( y = a^{2j+1} b \) then,
\[
xy = ab \cdot a^{2j+1}b \\
= aba^{2j}ab \\
= aba^2a^2 \ldots a^2 ab \\
= aa^{-1}baa^{2j-1}ab \\
= a^{-1}ba^{2j-1}ab \\
= a^{-1}a^{-1}baa^{2j}ab \\
= a^{-2}a^{-1}ba^{2j+2}ab \\
= a^{-3}ba^{2j-2}ab \\
\ldots \\
\ldots \\
= a^{-j-1}baa^{2(j-j)}b \\
= a^{-2j}bb \\
= a^{-2j}b^2 \\
= \left(a^{2j}\right)^{-1} \in H.
\]

If \(x = a^3b\) and \(y = a^{2j+1}b\) then,

\[
xy = a^3b \cdot a^{2j+1}b \\
= a^3ba^{2j}ab \\
= a^3ba^2a^2 \ldots a^2 ab \\
= a^3a^{-1}baa^{2j-1}ab \\
= a^2a^{-3}ba^{2j-1}ab \\
= aa^{-1}baa^{2j-2}ab \\
= a^{-1}ba^{2j-2}ab \\
= a^{-1}a^{-1}baa^{2j}ab \\
= a^{-2}a^{-1}ba^{2j+2}ab \\
= a^{-3}ba^{2j-2}ab \\
\ldots \\
\ldots \\
= a^{-j-1}baa^{2(j-j)}b \\
= a^{-2j}bb \\
= a^{-2j}b^2 \\
= \left(a^{2j}\right)^{-1} \in H.
\]

Similarly, we have the following:

If \(x = a^{n-1}b\), \(y = a^{2j+1}b\) then
Here, the possible number of pair of \( x, y \in A \) is \( |H| \cdot n \) implies \( \left( \frac{n}{2} \right) \cdot n \).

Therefore the possible number of pair of \( x, y \in A \) is \( |A| = \left( \frac{n}{2} \right)^2 + \left( \frac{n}{2} \right) \cdot n = \frac{3n^2}{4} \). Thus, by Definition 4,

\[
P_{xy}(D_n) = \frac{\left( \frac{3n^2}{4} \right)}{(2n)^2} = \frac{3}{16}.
\]

Remark: This calculation involves multiplying the element of \( x \) and \( y \) where \( x \) and \( y \) either both rotations or both reflections. \( D_4 \) is not included in this calculation because for \( x, y \) in \( A \), \( x \) and \( y \) can be both rotations, both reflections, one of them rotation and another one reflection and vice versa.

The following is an example that explains how the multiplicative degree of a cyclic subgroup of order \( \frac{n}{2} \) can be computed.

**Example 2:**

Let \( D_8 \) be a dihedral group of order 16. Then we can write

\[
D_8 = \{1, a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}.
\]

Let \( H \) be a subgroup of \( D_8 \), \( H = \langle a^2 \rangle = \{1, a^2, a^4, a^6\} \). We have \( A = \{x, y \in D_8 \setminus H \text{ but } xy \in H \} \). Therefore

\[
|A| = \left( \frac{n}{2} \right)^2 + \left( \frac{n}{2} \right) = \left( \frac{4}{2} \right)^2 + (4 \times 8) = 48.\]

Thus by Definition 4, \( P_{xy}(D_8) = \frac{|A|}{|D_k|^2} = \frac{48}{16^2} = \frac{3}{16} \).

**CONCLUSION**

In this research, the multiplicative degree of some of dihedral groups is found. For every \( x, y \in H \) we have \( xy \in H \) then the multiplicative degree of a dihedral group is equal to \( \left( \frac{|H|}{|D_k|^2} \right)^2 \). Meanwhile, for every \( x, y \in D_n \setminus H \) but \( xy \in H \) and \( H \) is a cyclic subgroup of \( D_n \) of order \( n \) and \( \frac{n}{2} \) then the multiplicative degree of a dihedral group is equal to \( \frac{1}{4} \) and \( \frac{3}{16} \) respectively.
ACKNOWLEDGMENT

The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) for the financial funding through the Research University Grant (RUG) Vote No. 10J68 and Ministry of Higher Education (MOHE) Malaysia for their support. The first author would also like to thank Universiti Teknologi MARA (UiTM) for the fellowship scheme.

REFERENCES