The Schur multiplier of pairs of groups of order \( p^3 q \)

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Degenerate Representations of the Symplectic Groups II. The Noncompact Group \( \text{Sp}(p, q) \)
The Schur Multiplier of Pairs of Groups of Order $p^3q$

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Abstract. Let $(G, N)$ be a pair of groups in which $N$ is a normal subgroup of $G$. Then, the Schur multiplier of pairs $(G, N)$, denoted by $M(G, N)$, is an extension of the Schur multiplier of a group $G$, which is a functorial abelian group. In this research, the Schur multiplier of pairs of all groups of order $p^3q$ where $p$ is an odd prime and $p < q$ is determined.

INTRODUCTION

The Schur multiplier of a group $G$, denoted as $M(G)$, was introduced by Schur [1] while studying projective representations of groups in 1904. The Schur multiplier of a group $G$ is defined as the second cohomology group $H_2(G, \mathbb{C}^*)$ where the modular multiplication acts identically: $gc = c$ for $g \in G$; $c \in \mathbb{C}^*$ and $\mathbb{C}^*$ represents the nonzero complex numbers. In [2], Schur stated that for a group $G$ with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$, the Schur multiplier of $G$ is isomorphic to $\left( \frac{R \cap [F,F]}{[F,R]} \right)$ where $F$ is a free group, the group $R$ of relators is the kernel of the surjective homomorphism $F \rightarrow G$ and $[F,R]$ is the group generated by all elements of the form $fr^{-1}r^{-1}$ for $f \in F$ and $r \in R$.

Schur computed the Schur multiplier for many different kinds of group such as alternating groups, symmetric groups and dihedral groups. All results of his computations can be found in [3]. In [4], Rashid computed the Schur multiplier of nonabelian groups of order $p^3q$ for distinct primes $p$ and $q$ where $p < q$ by using the classification of nonabelian groups of order $p^3q$ given by Western in [5]. The result shows that the Schur multiplier of nonabelian groups of order $p^3q$ is either trivial, cyclic or elementary abelian.

In 1998, Ellis [6] defined the notion of the Schur multiplier of a pair of groups as follows:

**Definition 1** [6] Let $(G, N)$ be an arbitrary pair of finite groups where $N$ is a normal subgroup of $G$. Then the Schur multiplier of the pair, $M(G, N)$ is a functorial abelian group whose principal feature is a natural exact sequence
In which \( \eta \) denotes some finiteness-preserving functor from groups to abelian groups (to be precise, \( \eta \) is the third homology of a group with integer coefficients). The homomorphisms \( \eta, \mu, \alpha \) are those due to the functorial of \( H_3(-) \), \( M(-) \) and \((-)^{ab} \). He also gave a group theoretic definition of \( M(G, N) \). The theoretic definition is given in the following theorem.

**Theorem 1** [6] For any pair of groups \((G, N)\) there is an isomorphism \( M(G, N) \cong \ker(\partial) \) where \( \partial : N \ltimes G \to G \), a map from a nonabelian exterior product of \( N \) and \( G \) to the group \( G \).

In [6], Ellis also showed that the order Schur multiplier of \((G, N)\) is bounded by \( p^{\frac{1}{\log(2p+1)}} \) if \( G \) is a finite \( p \)-group with a normal subgroup \( N \) of order \( p^n \) and its quotient of order \( p^m \). So there exists a non-negative integer \( t(G, N) \) such that \( |M(G, N)| = p^{\frac{1}{\log(2p+1)}-t(G, N)} \). In [7], Moghaddam et al. determined all pairs of finite \( p \)-groups \((G, N)\), which satisfy the equality for \( t(G, N) = 0, 1, 2 \). Besides, Moghaddam et al. in [8] showed that if \( S \) is a normal subgroup of \( F \) such that \( N \cong S/R \) then \( M(G, N) \cong (R \cap [S, F]) /[F, R] \).

In our previous research, the commutator subgroup and centre of groups of order \( p^aq \), where \( p \) and \( q \) are distinct primes and \( p < q \), and the Schur multiplier of pairs of groups of order \( p^aq \) where \( p \) and \( q \) are prime numbers that have been determined in [9] and [10] respectively. In this research, the Schur multiplier of pairs of all groups of order \( p^aq \) where \( p \) is an odd prime and \( p < q \) is determined. Note that throughout this paper, we denote the trivial group as 1.

**PRELIMINARIES**

This section includes some preliminary results that are used in proving our main theorem. The definition of normal Hall subgroup is given below.

**Definition 2** [3] A normal subgroup \( N \) of \( G \) is called a normal Hall subgroup of \( G \) if the order of \( N \) is coprime to its index in \( G \).

The classification of nonabelian groups of order \( p^aq \), where \( p \) and \( q \) are distinct primes and \( p < q \), given in [5], are listed in the following.

**Theorem 2** [5] Let \( G \) be a nonabelian group of order \( p^aq \), where \( p \) and \( q \) are distinct primes and \( p < q \). Then exactly one of the following holds:

The case \( p = 2 \) (The first nine groups exist for all values of \( q (q > 2) \))

\[
\begin{align*}
G_1 & \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, bab = a^{-1}, ac = ca, bc = cb \rangle; \\
G_2 & \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, b^1ab = a^{-1}, ac = ca, bc = cb \rangle; \\
G_3 & \cong \langle a, b \mid a^2 = b^2 = 1, a^{-1}ba = b^{-1} \rangle; \\
G_4 & \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb, c^{-1} \rangle; \\
G_5 & \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = ba, a^{-1}ca = c^{-1}, bc = cb \rangle; \\
G_6 & \cong \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, ab = ba, ac = ca, bc = cb, ad = da, bd = db, cdc = d^{-1} \rangle; \\
G_7 & \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, bab = a^{-1}, ac = ca, c^{-1} \rangle; \\
G_8 & \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, bab = a^{-1}, a^{-1}ca = c^{-1}, bc = cb \rangle.
\end{align*}
\]
\( G_0 \cong \{a, b, c \mid a^4 = b^4 = c^4 = 1, b^{-1}ab = a^{-1}, ac = ca, b^{-1}cb = c^{-1}\} \).
\( G_0 \cong \{a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^m\} \) where \( m \) is any primitive root of \( m^4 \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{4} \).
\( G_1 \cong \{a, b, c \mid a^4 = b^4 = c^8 = 1, ab = ba, a^{-1}ca = c^m, bc = cb\} \), where \( m \) is any primitive root of \( m^4 \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{4} \).
\( G_5 \cong \{a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^m\} \) where \( m \) is any primitive root of \( m^8 \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{8} \).

The case \( p = 2 \ (q = 3) \)
\( G_{13} \cong \{a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, ab = ba, ac = ca, bc = cb, ad = da, d^{-1}bd = c, d^{-1}cd = bc\} \).
\( G_{14} \cong \{a, b, c \mid a^4 = b^4 = c^4 = 1, a^{-1}b^2, b^{-1}ab = a^{-1}, c^{-1}ac = b, c^{-1}bc = ab\} \).
\( G_{15} \cong \{a, b, c \mid a^4 = b^4 = c^3 = 1, bab = a^{-1}, c^{-1}a^2b = b, c^{-1}bc = a^2b, a^{-1}ca = c^2a^2b\} \).

The case \( p = 2 \ (q = 7) \)
\( G_{17} \cong \{a, b, c, d \mid a^2 = b^2 = c^2 = d^7 = 1, ab = ba, ac = ca, bc = cb, d^{-1}ad = b, d^{-1}bd = c, d^{-1}cd = ab\} \).

The case \( p \) is odd
\( G_{19} \cong \{a, b, c, d \mid a^2 = b^2 = c^2 = d^5 = 1, ab = ba, ac = ca, bc = cb, ad = da, d^{-1}bd = c, d^{-1}cd = dc\} \).
\( G_{20} \cong \{a, b \mid a^2 = b^8 = 1, a^{-1}ba = b^m\} \), where \( m \) is any primitive root of \( m^5 \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p} \).
\( G_{21} \cong \{a, b, c \mid a^2 = b^{p^r} = c^5 = 1, ab = ba, ac = ca, bc = cb\} \), where \( m \) is any primitive root of \( m^p \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p} \).
\( G_{22} \cong \{a, b, c, d \mid a^2 = b^{p^r} = c^q = d^5 = 1, ab = ba, ac = ca, bc = cb, d = da, bd = db, c^{-1}dc = d^{-1}c\} \), where \( m \) is any primitive root of \( m^p \equiv 1 \pmod{q} \) and \( p \equiv 1 \pmod{q} \).
\( G_{23} \cong \{a, b, c, d \mid a^2 = b^{p^r} = c^q = d^5 = 1, ab = ba, ac = ca, bc = cb, d = da, bd = db, c^{-1}dc = d^{-1}c\} \), where \( m \) is any primitive root of \( m^p \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p} \).
\( G_{24} \cong \{a, b, c, d \mid a^2 = b^{p^r} = c^q = d^5 = 1, ab = ba, ac = ca, ad = da, bd = db, c^{-1}bc = ab, c^{-1}dc = d^{-1}c\} \), where \( m \) is any primitive root of \( m^p \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p} \).
\( G_{25} \cong \{a, b \mid a^2 = b^8 = 1, a^{-1}ba = b^m\} \), where \( m \) is any primitive root of \( m^p \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p^2} \).
\( G_{26} \cong \{a, b, c \mid a^2 = b^{p^r} = c^q = 1, ab = ba, a^{-1}ca = c^m, bc = cb\} \), where \( m \) is any primitive root of \( m^{p^2} \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p^2} \).
\( G_{27} \cong \{a, b \mid a^2 = b^8 = 1, a^{-1}ba = b^m\} \), where \( m \) is any primitive root of \( m^{p^2} \equiv 1 \pmod{q} \) and \( q \equiv 1 \pmod{p^2} \).

Note that:
\[ G_{18} \cong \mathbb{Z}_2 \times \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) ; \]
\[ G_{20} \cong \mathbb{Z}_2 \times \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) ; \]
\[ G_{21} \cong \mathbb{Z}_2 \times \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) ; \]
\[ G_{22} \cong \mathbb{Z}_2 \times \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) ; \]
\[ G_{24} = \mathbb{Z}_2 \times \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) \times \mathbb{Z}_2 ; \]
\[ G_{26} = \mathbb{Z}_2 \times \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) . \]

The commutator subgroup and center of groups of order \( p^aq \) where \( p < q \) are given in the following theorem.

**Theorem 3** [4] Let \( G \) be a nonabelian group of order \( p^aq \) where \( p \) and \( q \) are primes and \( p < q \). Then

(i) for the commutator subgroup of \( G \) exactly one of the following holds:

\[
G' = \begin{cases} 
\mathbb{Z}_2, & \text{if } G \text{ is of type } G_1 \text{ and } G_2, \\
\mathbb{Z}_4, & \text{if } G \text{ is of type } G_3 \text{ to } G_{12}, G_{19} \text{ to } G_{22}, \text{ and } G_{25} \text{ to } G_{27}, \\
\mathbb{Z}_{24}, & \text{if } G \text{ is of type } G_7 \text{ to } G_9, \\
(\mathbb{Z}_2)^3, & \text{if } G \text{ is of type } G_{13}, \\
\mathbb{Q}_2, & \text{if } G \text{ is of type } G_{14}, \\
\mathbb{A}_4, & \text{if } G \text{ is of type } G_{15}, \\
(\mathbb{Z}_2)^3, & \text{if } G \text{ is of type } G_{16}, \\
\mathbb{Z}_p, & \text{if } G \text{ is of type } G_7, G_8, G_9, \text{ and } G_{18}, \\
\mathbb{Z}_{pq}, & \text{if } G \text{ is of type } G_{23} \text{ and } G_{24}.
\end{cases}
\]

(ii) for the center of \( G \) exactly one of the following holds:

\[
Z(G) = \begin{cases} 
\mathbb{Z}_2, & \text{if } G \text{ is of type } G_{12}, G_{13}, G_{16}, \text{ and } G_{27}, \\
\mathbb{Z}_2, & \text{if } G \text{ is of type } G_4 \text{ to } G_{11}, G_{15}, \text{ and } G_{14}, \\
\mathbb{Z}_p, & \text{if } G \text{ is of type } G_{23} \text{ to } G_{26}, \\
\mathbb{Z}_{14}, & \text{if } G \text{ is of type } G_1 \text{ and } G_2, \\
\mathbb{Z}_q, & \text{if } G \text{ is of type } G_3 \text{ and } G_4, \\
\mathbb{Z}_p, & \text{if } G \text{ is of type } G_{16} \text{ and } G_{20}, \\
\mathbb{Z}_{pq}, & \text{if } G \text{ is of type } G_{17} \text{ and } G_{18}, \\
\mathbb{Z}_p \times \mathbb{Z}_p, & \text{if } G \text{ is of type } G_{21} \text{ and } G_{22}, \\
\mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } G \text{ is of type } G_3 \text{ and } G_6.
\end{cases}
\]

Some results that are essential to compute the Schur multiplier and the nonabelian tensor product are stated below.

**Theorem 4** [11] The factor group \( G/G' \) is abelian. If \( K \) is a normal subgroup of \( G \) such that \( G/K \) is abelian, then \( G' \leq K \).

**Theorem 5** [12] Let \( G \cong \mathbb{Z}_n \) and \( H \cong \mathbb{Z}_m \) be cyclic groups that act trivially on each other. Then \( G \otimes H \cong \mathbb{Z}_{(m,n)} \).

**Theorem 6** [13] Let \( A, B, C \) be groups, with given actions of \( A \) on \( B \) and \( C \), and of \( B \) and \( C \) on \( A \). Suppose that the latter actions

(i) commute: \( b^a = a^b \), so that \( B \times C \) acts on \( A \),
(ii) induce the trivial action of $B$ on $A \otimes C$; \( (a \otimes c)^b = a \otimes c \), and

(iii) induce the trivial action of $C$ on $A \otimes B$; \( (a \otimes b)^c = a \otimes b \),

for all $a \in A, b \in B, c \in C$. Then $A \otimes (B \times C) \cong (A \otimes B) \times (A \otimes C)$.

**Theorem 7** [2] Let $G$ be a finite group and let $G \cong F/R$ where $F$ is a free group of rank $n$. Then

$$M(G) \cong \left( \frac{R \cap [F,F]}{[F,R]} \right) / \left[ F, R \right].$$

**Theorem 8** [6] Let $(G, N)$ be a pair of group such that $N$ has a complement in $G$ then

$$M(G, N) = \ker \left( \mu : M(G) \rightarrow M\left( \frac{G}{N} \right) \right).$$

As a consequence of Theorem 7 and Theorem 8, Moghaddam et al. [8] obtained the following result.

**Proposition 1** [8] Let $G$ be a group with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ and let $(G, N)$ be the pairs of groups such that $N$ has a complement in $G$. If $S$ is a normal subgroup of $F$ such that $N \cong S/R$, then

$$M(G, N) \cong \left( \frac{R \cap [S,F]}{[F,R]} \right) / \left[ F, R \right].$$

**Theorem 9** [3] Let $G$ be a finite group. Then

(i) $M(G)$ is a finite group whose elements have order dividing the order of $G$.

(ii) $M(G) = 1$ if $G$ is cyclic.

**Theorem 10** [3] If the Sylow $p$-subgroups of $G$ are cyclic for all $p$ divides $|G|$, then $M(G) = 1$.

**Theorem 11** [3] Let $N$ be a normal Hall subgroup of $G$ and $T$ be a complement of $N$ in $G$. Then

$$M(G) \cong M(T) \times M(N).$$

The Schur multiplier of groups of order $p^2 q$ is given below.

**Theorem 12** [14] Let $G$ be a group of order $p^2 q$ where $p$ and $q$ are distinct primes. Then $M(G) = 1$ or $\mathbb{Z}_p$.

Note that for groups of order $p^2 q$ where $p$ and $q$ are distinct primes, there are three cases to be considered.

Case 1: Let $P$ be a normal Sylow $p$-subgroup of $G$.

Since $|P[^G_P]| = 1$, thus $P$ is a normal Hall subgroup of $G$. Therefore, $G \cong P \rtimes T$, where $T$ is a subgroup of $G$ of order $q$ and $P \cong \mathbb{Z}_p$, or $\mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, by Theorem 12,

$$M(G) = M(T) \times M(P)^T = M(P)^T = \begin{cases} 1; & P = \mathbb{Z}_p^2, \\ \mathbb{Z}_p; & P = \mathbb{Z}_p \times \mathbb{Z}_p. \end{cases}$$

Case 2: Let $Q$ be a normal Sylow $q$-subgroup of $G$.

Since $|Q[^G_Q]| = 1$, thus $Q$ is a normal Hall subgroup of $G$. Therefore, $G \cong Q \rtimes T$ where $T$ is a subgroup of $G$ of order $p^2$. Therefore,

$$M(G) = M(T) \times M(Q)^T = M(T) = \begin{cases} 1; & T = \mathbb{Z}_p^2, \\ \mathbb{Z}_p; & T = \mathbb{Z}_p \times \mathbb{Z}_p. \end{cases}$$
Case 3: If $G \cong A_4$, then $M(G)$ has been computed in [13].

The Schur multiplier of nonabelian groups of order $p^3q$, where $p$ and $q$ are primes and $p < q$, is stated in the following theorem.

**Theorem 13** [4] Let $G$ be a nonabelian group of order $p^3q$ where $p$ and $q$ are primes and $p < q$. Then exactly one of the following holds:

\[
M(G) = \begin{cases} 
1; & G \text{ is of type } G_2, G_3, G_6, G_{10}, G_{12}, G_{14}, G_{16}, G_{17}, G_{19}, G_{23}, G_{25} \text{ or } G_{27}, \\
\mathbb{Z}_2; & G \text{ is of type } G_4, G_5, G_7, G_8, G_{11}, G_{13} \text{ or } G_{15}, \\
\left(\mathbb{Z}_2\right)^3; & G \text{ is of type } G_9, \\
\mathbb{Z}_p; & G \text{ is of type } G_{20}, G_{21} \text{ or } G_{26}, \\
\mathbb{Z}_p \times \mathbb{Z}_p; & G \text{ is of type } G_{18} \text{ or } G_{24}, \\
\left(\mathbb{Z}_p\right)^3; & G \text{ is of type } G_{22}. 
\end{cases}
\]

The following theorems are some of the basic results of the Schur multiplier of a pair deduced by Ellis [6].

**Theorem 14** [6] Let $N = 1$, then $M(G, N) = 1$.

**Theorem 15** [6] Let $N = G$, then $M(G, G) = M(G)$.

**Theorem 16** [6] Suppose that $G$ is a finite group. Let the order of the normal subgroup $N$ be coprime to its index in $G$ and $T$ a complement of $N$ in $G$. Then $G \cong N_TG$ and $M(G, N) \cong M(N)^T$.

The structure for the Schur multiplier of a direct product of finite groups given by Karpilovsky in [3] is shown as follows:

**Theorem 17** [3] If $G_1$ and $G_2$ are finite groups, then $M(G_1 \times G_2) = M(G_1) \times M(G_2) \times \left(G_1^{ab} \otimes G_2^{ab}\right)$.

As a consequence of the above fact, Mohammadzadeh et al. [13] gave the following result.

**Theorem 18** [15] Let $(G, N)$ be a pair of groups and $K$ be the complement of $N$ in $G$. Then $|M(G, N)| = |M(N)|^{|N^{ab} \otimes K^{ab}|}$.

**MAIN RESULTS**

In the following theorem, the Schur multiplier of pairs of groups of order $p^3q$, where $p$ and $q$ are distinct odd primes and $p < q$, is stated and proved.

**Theorem 19** Let $G$ be a nonabelian group of order $p^3q$, where $p$ and $q$ are distinct odd primes and $p < q$. Then exactly one of the following holds:
\[
M(G, N) = \begin{cases} 
1; & \text{if } (G = G_{17}, G_{19}, G_{23}, G_{25}, G_{27}) \text{ or } (G = G_{15}, G_{20}, G_{21}, G_{22}, G_{24}, G_{26} \text{ when } N = 1, \mathbb{Z}_q), \\
\mathbb{Z}_p; & \text{if } (G = G_{20} \text{ when } N = \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_q \times \mathbb{Z}_p, \mathbb{Z}_q \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q, G). \\
\left(\mathbb{Z}_p^n\right)^2; & \text{if } (G = G_{21} \text{ when } N = \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_q \times \mathbb{Z}_p, \mathbb{Z}_q \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q, G) \\
\left(\mathbb{Z}_p^n\right)^3; & \text{if } (G = G_{22} \text{ when } N = \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_q \times \mathbb{Z}_p, \mathbb{Z}_q \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q, G). \\
\end{cases}
\]

**Proof**

Let $G$ be a nonabelian group of order $p^iq$ where $p$ and $q$ are distinct odd primes and $p < q$. Suppose $N \triangleleft G$, then the Schur multiplier of pairs of $G$ is computed below by using the classification in Theorem 2. First, we have the following:

(i) If $N = 1$, then by Theorem 14, $M(G, N) = 1$.

(ii) If $N = G$, then by Theorem 15, $M(G, N) = M(G)$.

Next, we have the following cases:

**Case 1:** Let $G = G_{17}, G_{19}, G_{23}, G_{25}$ or $G_{27}$. Then by Theorem 13, $M(G) = 1$. Since we have

\[
M(G) \equiv \frac{(R \cap [F, F])}{[F, R]} \text{ and } M(G, N) \equiv \frac{(R \cap [S, F])}{[F, R]}
\]

where $S$ is a normal subgroup of $F$ such that $N \equiv S/R$ in Theorem 7 and Proposition 1 respectively, thus $M(G, N) \leq M(G)$. So for all normal subgroups $N$ of $G$, $M(G, N) \leq M(G) = 1$.

Therefore, for $G = G_{17}, G_{19}, G_{23}, G_{25}$ or $G_{27}$, $M(G, N) = 1$.

**Case 2:** Let $G = G_{i8}$ which is $G \equiv \mathbb{Z}_p \times \left(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\right)$. Then by Theorem 3 and Theorem 13, $G' = \mathbb{Z}_p$, $Z(G) = \mathbb{Z}_{p^2}$ and $M(G) = \mathbb{Z}_p \times \mathbb{Z}_p$.

(i) If $N = \mathbb{Z}_p \triangleleft G$ then $G/N \equiv G/\mathbb{Z}_p \equiv p^2q$ which implies $G/N \equiv \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Then we have the complement of $N$, $K \equiv G/N \equiv \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

\[
\begin{align*}
\left|M(G, N)\right| &= \left|M\left(\mathbb{Z}_p\right)\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_p\right)\right| \\
&= \left(1\left[\mathbb{Z}_p \otimes \mathbb{Z}_{p^2} \times \mathbb{Z}_p\right]\right) \\
&= \left[\mathbb{Z}_p \otimes \mathbb{Z}_{p^2} \times \mathbb{Z}_p\right] \\
&= \left[\mathbb{Z}_{(p, p^2)} \times \mathbb{Z}_{(p, p)}\right] \\
&= \mathbb{Z}_p \times \mathbb{Z}_p.
\end{align*}
\]

Therefore, $M(G, N) \equiv \mathbb{Z}_p \times \mathbb{Z}_p$.

(ii) If $N = \mathbb{Z}_p \times \mathbb{Z}_p$ then $G/N \equiv pq$ which implies $G/N \equiv \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, or $Z_p \times \mathbb{Z}_q$. 

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a) If \( \frac{G}{N} \cong \mathbb{Z}_{p^n} \) then by Theorem 9, \( M\left(\frac{G}{N}\right) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M\left(\frac{G}{N}\right) = 1 \) shows that \( \frac{M(G,N)}{\kappa} \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \).

b) If \( \frac{G}{N} \cong \mathbb{Z}_p \times \mathbb{Z}_q \) then by Sylow’s theorems, \( \mathbb{Z}_p \) is the Sylow \( p \)-subgroup of \( \frac{G}{N} \cong \mathbb{Z}_p \times \mathbb{Z}_q \) and \( \mathbb{Z}_q \) is the Sylow \( q \)-subgroup of \( \frac{G}{N} \cong \mathbb{Z}_p \times \mathbb{Z}_q \). Since \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \) are cyclic then Theorem 10, \( M\left(\frac{G}{N}\right) \cong 1 \).

Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M\left(\frac{G}{N}\right) = 1 \) shows that \( \frac{M(G,N)}{\kappa} \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \).

(iii) If \( N = \left(\mathbb{Z}_p \times \mathbb{Z}_p\right) \times \mathbb{Z}_p \), then \( \frac{G}{N} \cong q \) which implies \( \frac{G}{N} \cong \mathbb{Z}_q \). By Theorem 9, \( M\left(\frac{G}{N}\right) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M\left(\frac{G}{N}\right) = 1 \) shows that \( \frac{M(G,N)}{\kappa} \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \).

(iv) If \( N = \mathbb{Z}_q \) then by Definition 2, \( N \) is a normal Hall subgroup of \( G \) since \( |N| = q \) and \( \frac{G}{N} = p^r \) are coprime. By Theorem 16, \( M(G,N) = M\left(\frac{G}{N}\right) = 1 \) since \( M\left(\frac{G}{N}\right) = 1 \).

(v) If \( N = \mathbb{Z}_{p^n} = Z(G) \) then \( \frac{G}{N} \cong \frac{G}{Z(G)} = p^r \) which implies \( \frac{G}{N} \cong \mathbb{Z}_r \). Then we have the complement of \( N \), \( K \cong \frac{G}{N} \cong \mathbb{Z}_r \). By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,
\[
\left| M(G,N) \right| = \left| M\left(\frac{G}{N}\right) \right| = \left| \left(\mathbb{Z}_r \times \mathbb{Z}_r \right) \otimes \left(\mathbb{Z}_p \times \mathbb{Z}_p \right) \right| = 1 \left(\mathbb{Z}_r \otimes \mathbb{Z}_r \right) \otimes \left(\mathbb{Z}_p \times \mathbb{Z}_p \right) = \left(\mathbb{Z}_r \otimes \mathbb{Z}_p \right) \otimes \left(\mathbb{Z}_r \otimes \mathbb{Z}_r \right) = \mathbb{Z}_r \times \mathbb{Z}_p .
\]

Therefore, \( M(G,N) \cong \mathbb{Z}_r \times \mathbb{Z}_p \).

(vi) If \( N = \mathbb{Z}_{p^n} \times \mathbb{Z}_p \), then \( \frac{G}{N} \cong p \) which implies \( \frac{G}{N} \cong \mathbb{Z}_p \). By Theorem 9, \( M\left(\frac{G}{N}\right) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M\left(\frac{G}{N}\right) = 1 \) shows that \( \frac{M(G,N)}{\kappa} \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \).

Case 3: Let \( G = G_{20} \) which is \( G \cong \mathbb{Z}_{p^n} \times \left(\mathbb{Z}_q \times \mathbb{Z}_p \right) \). Then by Theorem 13, \( G' = \mathbb{Z}_q \times \mathbb{Z}_p \), \( Z(G) = \mathbb{Z}_p \), and \( M(G) = \mathbb{Z}_p \).

(i) If \( N = \mathbb{Z}_p \), then \( \frac{G}{N} \cong p \) which implies \( \frac{G}{N} \cong \mathbb{Z}_p \) or \( \mathbb{Z}_p \times \mathbb{Z}_p \). (If \( \frac{G}{N} \) is abelian then by Theorem 4 \( G' \subseteq N \); that is \( \mathbb{Z}_p \subseteq \mathbb{Z}_p \) and this statement is a contradiction.) Then by Theorem 12, \( M\left(\frac{G}{N}\right) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M\left(\frac{G}{N}\right) = 1 \) shows that \( \frac{M(G,N)}{\kappa} \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \).
(ii) If \( N = Z_{pq} = Z(G) \) then \( \frac{G}{N} = \frac{G}{Z(G)} = pq \) which implies \( G/N = G/Z(G) \cong \mathbb{Z}_q \times \mathbb{Z}_p \). Then by Sylow's theorems, \( Z_p \) is the Sylow p-subgroup of \( G/N = \mathbb{Z}_q \times \mathbb{Z}_p \) and \( Z_q \) is the Sylow q-subgroup of \( G/N = \mathbb{Z}_q \times \mathbb{Z}_p \). Since \( Z_p \) and \( Z_q \) are cyclic then Theorem 10, \( M(G/N) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M(G/N) \cong 1 \) shows that \( M(G,N)/\kappa \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \).

(iii) If \( N = Z_{p^2} \times \mathbb{Z}_p \) or \( \mathbb{Z}_p \times \mathbb{Z}_p \) then \( \frac{G}{N} = q \) which implies \( G/N \cong \mathbb{Z}_q \). By Theorem 9, \( M(G/N) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M(G/N) \cong 1 \) shows that \( M(G,N)/\kappa \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \).

(iv) If \( N = Z_q = G' \) then by Definition 2, \( N \) is a normal Hall subgroup of \( G \) since \( |N| = q \) and \( |G/N| = p^3 \) are coprime. By Theorem 16, \( M(G,N) = M(Z_q)^{\alpha} = 1 \) since \( M(Z_q) = 1 \).

(v) If \( N = \mathbb{Z}_q \times \mathbb{Z}_p \) then \( \frac{G}{N} = p^2 \) which implies \( G/N \cong \mathbb{Z}_{p^2} \). By Theorem 9, \( M(G/N) \cong 1 \). Thus the exact sequence \( M(G,N) \rightarrow M(G) \rightarrow M(G/N) \cong 1 \) shows that \( M(G,N)/\kappa \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \rightarrow M(G) \). Thus, \( M(G,N) \cong \mathbb{Z}_p \).

Case 4: Let \( G = G_{21} \) which is \( G \cong \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_{p^2}) \). Then by Theorem 3 and Theorem 13, \( G' = Z_q \), \( Z(G) = \mathbb{Z}_q \times \mathbb{Z}_p \) and \( M(G) = \mathbb{Z}_p \).

(i) If \( N = \mathbb{Z}_p \) then \( \frac{G}{N} = \frac{G}{Z(G)} = p^2q \) which implies \( G/N \cong \mathbb{Z}_q \times \mathbb{Z}_{p^2} \) or \( \mathbb{Z}_p \times \mathbb{Z}_{p^2} \). Thus, by similar way as in case 3(i), \( M(G,N) \cong \mathbb{Z}_p \).

(ii) If \( N = \mathbb{Z}_q \times \mathbb{Z}_p = Z(G) \) then \( \frac{G}{N} = \frac{G}{Z(G)} = pq \) which implies \( G/N \cong \mathbb{Z}_q \times \mathbb{Z}_p \). Thus, by similar way as in case 3(ii), \( M(G,N) \cong \mathbb{Z}_p \).

(iii) If \( N = \mathbb{Z}_p \times \mathbb{Z}_{p^2} \) or \( \mathbb{Z}_p \times \mathbb{Z}_p \) then \( \frac{G}{N} = q \) which implies \( G/N \cong \mathbb{Z}_q \). Thus, by similar way as in case 3(iii), \( M(G,N) \cong \mathbb{Z}_p \).

(iv) If \( N = \mathbb{Z}_q = G' \) then by Definition 2, \( N \) is a normal Hall subgroup of \( G \) since \( |N| = q \) and \( |G/N| = p^3 \) are coprime. Thus, by similar way as in case 3(iv), \( M(G,N) \cong 1 \).

(v) If \( N = \mathbb{Z}_{pq} \) then \( \frac{G}{N} = \frac{G}{Z(G)} = p^2 \) which implies \( G/N \cong \mathbb{Z}_{p^2} \). Thus, by similar way as in case 3(v), \( M(G,N) \cong \mathbb{Z}_p \).
(vi) If \( N \cong \mathbb{Z}_q \times \mathbb{Z}_p \) or \( \mathbb{Z}_p \times \mathbb{Z}_q \), then \( \left| \frac{G}{N} \right| = p \) which implies \( \frac{G}{N} \cong \mathbb{Z}_p \). Thus, by similar way as in case 3(v), \( M(G,N) \cong \mathbb{Z}_p \).

Case 5: Let \( G = G_{22} \) which is \( G \cong \mathbb{Z}_q \times \mathbb{Z}_r \times (\mathbb{Z}_s \times \mathbb{Z}_p) \). Then by Theorem 3 and Theorem 13, \( G' = \mathbb{Z}_s \), \( Z(G) = \mathbb{Z}_p \times \mathbb{Z}_r \) and \( M(G) = (\mathbb{Z}_p)^3 \).

(i) If \( N = \mathbb{Z}_p \) then \( \left| \frac{G}{N} \right| = p'q \) which implies \( \frac{G}{N} \cong \mathbb{Z}_r \times (\mathbb{Z}_s \times \mathbb{Z}_p) \). Then we have the complement of \( N \), \( K \cong \frac{G}{N} \cong \mathbb{Z}_r \times (\mathbb{Z}_s \times \mathbb{Z}_p) \). By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

\[
\left| M(G,N) \right| = \left| M(\mathbb{Z}_p) \right| \left| (\mathbb{Z}_r)^{ab} \otimes (\mathbb{Z}_s \times (\mathbb{Z}_p \times \mathbb{Z}_p))^{ab} \right|
\]

\[
= (1) \left| (\mathbb{Z}_p) \otimes (\mathbb{Z}_r)^{ab} \times (\mathbb{Z}_s \times (\mathbb{Z}_p \times \mathbb{Z}_p))^{ab} \right|
\]

\[
= \left| (\mathbb{Z}_r) \times (\mathbb{Z}_s \otimes \mathbb{Z}_p) \right|
\]

\[
= \left| (\mathbb{Z}_r)^{ab} \times (\mathbb{Z}_s \otimes \mathbb{Z}_p) \right|
\]

\[
= |\mathbb{Z}_r \times \mathbb{Z}_p|.
\]

Therefore, \( M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_r \).

(ii) If \( N = \mathbb{Z}_p \times \mathbb{Z}_s = Z(G) \) then \( \left| \frac{G}{N} \right| = \left| Z(G) / N \right| = pq \) which implies \( \frac{G}{N} \cong \mathbb{Z}_q \times \mathbb{Z}_r \). Thus, by similar way as in case 3(ii), \( M(G,N) = (\mathbb{Z}_r)^3 \).

(iii) If \( N = (\mathbb{Z}_r \times \mathbb{Z}_s)^{ab} \times (\mathbb{Z}_p \times \mathbb{Z}_s) \) or \( \mathbb{Z}_p \times (\mathbb{Z}_s \times \mathbb{Z}_p) \), then \( \left| \frac{G}{N} \right| = q \) which implies \( \frac{G}{N} \cong \mathbb{Z}_s \). Thus, by similar way as in case 3(iii), \( M(G,N) = (\mathbb{Z}_r)^3 \).

(iv) If \( N = \mathbb{Z}_q = G' \) then by Definition 2, \( N \) is a normal Hall subgroup of \( G \) since \( |N| = q \) and \( \left| \frac{G}{N} \right| = p^3 \) are coprime. Thus, by similar way as in case 3(iv), \( M(G,N) = 1 \).

(v) If \( N = \mathbb{Z}_r \times \mathbb{Z}_p \) then \( \left| \frac{G}{N} \right| = p^2 \) which implies \( \frac{G}{N} \cong \mathbb{Z}_r \times \mathbb{Z}_p \). Then we have the complement of \( N \), \( K \cong \frac{G}{N} \cong \mathbb{Z}_r \times \mathbb{Z}_p \). By Theorem 18, Theorem 10, Theorem 6 and Theorem 5,

\[
\left| M(G,N) \right| = \left| M(\mathbb{Z}_r \times \mathbb{Z}_p) \right| \left| (\mathbb{Z}_r \times \mathbb{Z}_p)^{ab} \otimes (\mathbb{Z}_s \times (\mathbb{Z}_p \times \mathbb{Z}_p))^{ab} \right|
\]

\[
= (1) \left| (\mathbb{Z}_r \times \mathbb{Z}_p) \otimes (\mathbb{Z}_r \times (\mathbb{Z}_p \times \mathbb{Z}_p))^{ab} \right|
\]

\[
= \left| (\mathbb{Z}_r \times \mathbb{Z}_p) \times (\mathbb{Z}_r \times (\mathbb{Z}_p \times \mathbb{Z}_p))^{ab} \right|
\]

\[
= |\mathbb{Z}_r \times \mathbb{Z}_p|.
\]

Therefore, \( M(G,N) \cong \mathbb{Z}_r \times \mathbb{Z}_p \).

(vi) If \( N \cong \mathbb{Z}_q \times (\mathbb{Z}_r \times \mathbb{Z}_p) \) or \( \mathbb{Z}_p \times \mathbb{Z}_r \), then \( \left| \frac{G}{N} \right| = p \) which implies \( \frac{G}{N} \cong \mathbb{Z}_p \). By Theorem 9, \( M\left( \frac{G}{N} \right) \cong 1 \). Thus the exact sequence \( M(G,N) \to M(G) \to M\left( \frac{G}{N} \right) = 1 \) shows that \( M(G,N)/\kappa \cong 1 \) where \( \kappa \) is the kernel of homomorphism \( M(G,N) \to M(G) \). Thus, \( M(G,N) = (\mathbb{Z}_p)^3 \).
Case 6: Let $G = G_{x}$ which is $G \equiv (Z_{m} \times Z_{p}) \times Z_{p}$. Then by Theorem 3 and Theorem 13, $G' = \tilde{Z}_{m}$, $Z (G) = Z_{p}$ and $M (G) = Z_{p} \times Z_{p}$.

(i) If $N = Z_{p} = Z (G)$ then $[G' \mod N] = [G' \mod Z (G)] = p^{2}q$ which implies $G' \mod N = G' \mod Z (G) \equiv Z_{q} \times (Z_{m} \times Z_{p})$. Therefore, by similar way as in case 5(i), $M (G, N) \equiv Z_{p} \times Z_{p}$.

(ii) If $N = Z_{p} \times Z_{p}$ then $|G' \mod N| = pq$ which implies $G' \mod N \equiv Z_{q} \times Z_{p}$. (If $G' \mod N$ is abelian then by Theorem 4 $G' \subseteq N$; that is $Z_{m} \subseteq Z_{p} \times Z_{p}$ and this statement is a contradiction.) Thus, by similar way as in case 3(ii), $M (G, N) = Z_{p} \times Z_{p}$.

(iii) If $N = (Z_{p} \times Z_{p}) \times Z_{p}$ then $|G' \mod N| = q$ which implies $G' \mod N \equiv Z_{q}$. By Theorem 9, $M (G' \mod N) \equiv 1$. Thus the exact sequence $M (G, N) \rightarrow M (G) \rightarrow M (G' \mod N) \equiv 1$ shows that $M (G, N) \mod \kappa \equiv 1$ where $\kappa$ is the kernel of homomorphism $M (G, N)$ to $M (G)$. Thus, $M (G, N) = Z_{p} \times Z_{p}$.

(iv) If $N = Z_{q}$ then by Definition 2, $N$ is a normal Hall subgroup of $G$ since $|N| = q$ and $|G' \mod N| = p^{2}$ are coprime. Thus, by similar way as in case 3(iv), $M (G, N) = 1$.

If $N = Z_{m}$ then $|G' \mod N| = p^{2}$ which implies $G' \mod N = G' \mod Z_{m} \equiv Z_{p} \times Z_{q}$. Therefore, by similar way as in case 5(v), $M (G, N) = Z_{p} \times Z_{p}$.

If $N \equiv Z_{q} \times (Z_{m} \times Z_{p})$ or $Z_{m} \times Z_{q}$, then $|G' \mod N| = q$ which implies $G' \mod N \equiv Z_{q}$. Thus, by similar way as in case 5(vi), $M (G, N) = Z_{p} \times Z_{p}$.

Case 7: Let $G = G_{x}$ which is $G \equiv Z_{n} \times (Z_{m} \times Z_{p})$. Then by Theorem 3 and Theorem 13, $G' = Z_{q}$, $Z (G) = Z_{p}$ and $M (G) = Z_{p}$.

(i) If $N = Z_{p} = Z (G)$ then $|G' \mod N| = |G' \mod Z (G)| = p^{2}q$ which implies $G' \mod N = G' \mod Z (G) \equiv Z_{q} \times Z_{p}$. Then by Theorem 12, $M (G' \mod N) \equiv 1$. The exact sequence $M (G, N) \rightarrow M (G) \rightarrow M (G' \mod N) \equiv 1$ shows that $M (G, N) \mod \kappa \equiv 1$ where $\kappa$ is the kernel of homomorphism $M (G, N)$ to $M (G)$. Thus, $M (G, N) \equiv Z_{p}$.

(ii) If $N = Z_{p} \times Z_{p}$ then $|G' \mod N| = pq$ which implies $G' \mod N$ is a nonabelian group of order $pq$. (If $G' \mod N$ is abelian then by Theorem 4 $G' \subseteq N$; that is $Z_{m} \subseteq Z_{p} \times Z_{p}$ and this statement is a contradiction.) By Theorem 10, $M (G' \mod N) \equiv 1$. Thus the exact sequence $M (G, N) \rightarrow M (G) \rightarrow M (G' \mod N) \equiv 1$ shows that $M (G, N) \mod \kappa \equiv 1$ where $\kappa$ is the kernel of homomorphism $M (G, N)$ to $M (G)$. Thus, $M (G, N) \equiv Z_{p}$.

(iii) If $N = Z_{p} \times Z_{p}$ or $Z_{p} \times Z_{p}$ then $|G' \mod N| = q$ which implies $G' \mod N \equiv Z_{q}$. By Theorem 9, $M (G' \mod N) \equiv 1$. Thus the exact sequence $M (G, N) \rightarrow M (G) \rightarrow M (G' \mod N) \equiv 1$ shows that $M (G, N) \mod \kappa \equiv 1$ where $\kappa$ is the kernel of homomorphism $M (G, N)$ to $M (G)$. Thus, $M (G, N) \equiv Z_{p}$.

(iv) If $N = Z_{q} = G'$ then by Definition 2, $N$ is a normal Hall subgroup of $G$ since $|N| = q$ and $|G' \mod N| = p^{2}$ are coprime. By Theorem 16, $M (G, N) = M (Z_{q}) \mod \kappa = 1$ since $M (Z_{q}) \mod \kappa = 1$.
(v) If $N = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ then $|G_N| = p^3$ which implies $G_N \cong \mathbb{Z}_p^3$. By Theorem 9, $M(G_N) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G_N) = 1$ shows that $M(G,N)/\kappa = 1$ where $\kappa$ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

(vi) If $N = \mathbb{Z}_{pq}$ then $|G_N| = p^2$ which implies $G_N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then we have the complement of $N$, $K \cong G_N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

$$|M(G,N)| = |M(\mathbb{Z}_{pq})||Z_{pq}^\times|Z_{pq} \times Z_{pq}|^{ab} = (1)(\mathbb{Z}_{pq} \otimes (\mathbb{Z}_p \times \mathbb{Z}_p) = (\mathbb{Z}_{pq} \otimes (\mathbb{Z}_{pq} \otimes \mathbb{Z}_p) = (\mathbb{Z}_{pq} \otimes (\mathbb{Z}_{pq} \otimes \mathbb{Z}_p) = \mathbb{Z}_p \times \mathbb{Z}_p.$$

Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(vii) If $N \cong \mathbb{Z}_p \ltimes (\mathbb{Z}_q \times \mathbb{Z}_p)$ or $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, then $|G_N| = p$ which implies $G_N \cong \mathbb{Z}_p$. By Theorem 9, $M(G_N) = 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G_N) = 1$ shows that $M(G,N)/\kappa = 1$ where $\kappa$ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

**CONCLUSION**

There are twenty seven nonabelian groups of order $p^3$ where $p$ and $q$ are distinct primes and $p < q$. In this paper, we focus only on the eleven nonabelian groups of order $p^3$ where $p$ and $q$ are distinct odd primes and $p < q$ and we determined the Schur multiplier of pairs of groups of the groups mentioned. Our proofs show that $M(G,N)$ for those groups are equal to $1, \mathbb{Z}_p(\mathbb{Z}_p)^3$ or $(\mathbb{Z}_p)^3$ depending on their normal subgroups.

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