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The Central Subgroup of the Nonabelian Tensor Square of a Torsion Free Space Group

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Abstract. Space groups of a crystal expound its symmetry properties. One of the symmetry properties is the central subgroup of the nonabelian tensor square of a group. It is a normal subgroup of the group which can be ascertained by finding the abelianisation of the group and the nonabelian tensor square of the abelianisation group. In this research, our focus is to explicate the central subgroup of the nonabelian tensor square of the torsion free space groups of a crystal which are called the Bieberbach groups.

INTRODUCTION

Bieberbach groups are torsion free space groups. From Crystallographic Algorithms and Tables (CARAT) package [1], there are four Bieberbach groups with quaternion point group of order eight, denoted as $Q_n(6)$, where $n = 1, 2, 3, 4$. All of them are of dimension six. In this research, we want to expound their symmetry property which is the central subgroup of the nonabelian tensor square of the group. The nonabelian tensor square, denoted as $G \otimes G$, is generated by the symbols $g \otimes h$ for all $g, h \in G$, subject to relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^k \otimes h^k)$ for all $g, h, k \in G$ where $g^h = h^{-1}gh$. The central subgroup of the nonabelian tensor square of a group G, denoted by $\nabla(G)$, is a normal subgroup generated by the element $g \otimes g$ for all $g \in G$. It is a normal subgroup of the group which can be ascertained by finding the abelianisation of the group, denoted as G^{ab}, and the nonabelian tensor square of the abelianisation group, denoted as $G^{ab} \otimes G^{ab}$. The abelianisation of the group, G^{ab} is the quotient group G/G' where G' is the derived subgroup of G.

Throughout the years, researches involving the computation of the central subgroup have been conducted. In [2], Masri computed $\nabla(G)$ of some Bieberbach groups with cyclic point group of order two. In 2015, Mohd Idrus [3] did the same work but on a nonabelian point group namely the Bieberbach group of dimension five with dihedral point group of order eight. Furthermore, Mat Hassim [4] focused on the central subgroup of the Bieberbach groups with cyclic point groups of order three and five. In this research, the central subgroup of the nonabelian tensor square of the first Bieberbach group of dimension six with quaternion point group of order eight where $n = 1$, denoted as $Q_1(6)$, is computed. This paper is structured as follows: The first section provides the preliminaries that are used throughout the research followed by the second section which are the main results for this research.

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PRELIMINARIES

This section provides some basic definitions and structural results which are used throughout this research. We start with the definition of $\nu(G)$.

Definition 1 [5]
Let G be a group with presentation $\langle G \mid R\rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \langle G, G^{\varphi} \mid R, R^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x\varphi}[g, h^{\varphi}], \forall x, g, h \in G\rangle.$$

Rocco in [5] and Ellis and Leonard in [6] have shown that the subgroup of $\nu(G)$ is isomorphic to the nonabelian tensor square of the group G, as given in the following theorem:

Theorem 1 [6]
Let G be a group. The map $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.
Next, Proposition 1 gives some properties of $\nu(G)$ that will be used in proving our main results in the next section.

Proposition 1 [5, 7]
Let G be a group. The following relations hold in $\nu(G)$:

(i) $[g_1, g_2^{\varphi}]^{[g_3, g_4^{\varphi}]} = [g_1, g_2{}^{\varphi}]^{[g_3, g_4]}$ for all g_1, g_2, g_3, g_4 in G;

(ii) $[g_1, g_2^{\varphi}, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$ and
$[g_1^{\varphi}, g_2, g_3] = [g_1^{\varphi}, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2^{\varphi}, g_3]$ for all g_1, g_2, g_3 in G;

(iii) $[g_1, [g_2, g_3]^{\varphi}] = [g_2, g_3, g_1^{\varphi}]^{-1}$ for all g_1, g_2, g_3 in G;

(iv) $[g, g^{\varphi}]$ is central in $\nu(G)$ for all g in G;

(v) $[g_1, g_2{}^{\varphi}][g_2, g_1{}^{\varphi}]$ is central in $\nu(G)$ for all g_1, g_2 in G;

(vi) $[g, g^{\varphi}] = 1$ for all g in G'.

The first Bieberbach group of dimension six with quaternion point group denoted as $Q_1(6)$ is a polycyclic group and has a presentation as given in the following:

$$\begin{aligned} Q_1(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \mid\ & a^2 = cl_6,\ b^2 = cl_5 l_6^{-1},\ b^a = bcl_5^{-2} l_6^2,\ c^2 = l_5 l_6^{-1},\ c^a = cl_5^{-1} l_6,\ c^b = c,\ l_1^a = l_1^{-1} l_2 l_4^{-1},\ l_1^b = l_3^{-1},\ l_1^c = l_1^{-1}, \\ & l_2^a = l_1^{-1} l_2 l_3,\ l_2^b = l_1^{-1} l_3^{-1} l_4^{-1},\ l_2^c = l_2^{-1},\ l_3^a = l_2^{-1} l_3^{-1} l_4^{-1},\ l_3^b = l_3,\ l_3^c = l_3^{-1},\ l_4^a = l_1 l_3 l_4,\ l_4^b = l_1^{-1} l_2 l_3, l_4^c = l_4^{-1}, l_5^a = l_6, l_5^b = l_5, \\ & l_5^c = l_5,\ l_6^a = l_5,\ l_6^b = l_6,\ l_6^c = l_6,\ l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \le i, j \le 6\rangle \end{aligned} \tag{1}$$

The method developed by Blyth and Morse [7], which is aided by the following proposition, is used to compute the central subgroup of its nonabelian tensor square.

Proposition 2 [7]. Let G be a polycyclic group with a polycyclic generating sequence $g_1, \ldots\ldots, g_k$. Then $\left[G, G^{\varphi}\right]$ is a subgroup of $\nu(G)$ and is generated by

$$\left[G, G^{\varphi}\right] = \left\langle \left[g_i, g_i{}^{\varphi}\right],\ \left[g_i{}^{\partial}, (g_j{}^{\varphi})^{\varepsilon}\right],\ \left[g_i, g_j{}^{\varphi}\right]\left[g_j, g_i{}^{\varphi}\right] \right\rangle$$

and $\left[G, G^{\varphi}\right]_{\tau(G)}$ a subgroup $\tau(G)$ is generated by

$$\left[G,G^{\varphi}\right]_{\tau(G)}=\left\langle\left[g_i^{\partial},(g_j^{\varphi})^{\varepsilon}\right],\ \left[g_j^{\varepsilon},(g_i^{\varphi})^{\partial}\right]\right\rangle \text{ for } 1\le i<j\le k, \text{ where}$$

$$\partial=\begin{cases}1 & \text{if } |g_i|<\infty\\ \pm1 & \text{if } |g_i|=\infty\end{cases} \quad \text{and} \quad \varepsilon=\begin{cases}1 & \text{if } |g_j^{\varphi}|<\infty\\ \pm1 & \text{if } |g_j^{\varphi}|=\infty\end{cases}.$$

In [8], Blyth *et al*. showed that the structure of $\nabla(G)$ essentially depends on G^{ab} as given in the following proposition.

Proposition 3 [8]
Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_iG'\rangle$, for $i=1,......,s$ and set $E(G)$ to be $\langle[x_i,x_j^{\varphi}]\,|\,i<j\rangle\left[G,G'^{\varphi}\right]$. Then, the following hold:

(i) $\nabla(G)$ is generated by the elements of the set $\left\{\left[x_i,x_i^{\varphi}\right],\left[x_i,x_j^{\varphi}\right]\left[x_j,x_i^{\varphi}\right]\,\middle|\,1\le i<j\le s\right\}$;

(ii) $\left[G,G^{\varphi}\right]=\nabla(G)E(G)$.

Proposition 4 gives the properties of the ordinary tensor product of abelian groups, Proposition 5 gives the properties of the order of homomorphism while Theorem 2 gives the epimorphism mapping of the nonabelian tensor square.

Proposition 4 [9]
Let A, B and C be abelian groups and C_0 is the infinite cyclic group. Consider the ordinary tensor product of two abelian groups. Then,

(i) $C_0\otimes A\cong A$,
(ii) $C_0\otimes C_0\cong C_0$,
(iii) $C_n\otimes C_m\cong C_{\gcd(n,m)}$ for $n,m\in\mathbb{Z}$, and
(iv) $A\otimes(B\times C)=(A\otimes B)\times(A\otimes C)$.

Proposition 5 [2]
Let G and H be groups and let $g\in G$. Suppose ϕ is a homomorphism from G to H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of $\phi(g)$ equals the order of g.

Theorem 2 [10]
Let G and H be groups such that there is an epimorphism $\eta: G\to H$. Then there is an epimorphism $\alpha: G\otimes G\to H\otimes H$ defined by $\alpha(g\otimes h)=\eta(g)\otimes\eta(h)$.

MAIN RESULT

In order to compute $\nabla(Q_1(6))$, we first need to find the abelianisation of $Q_1(6)$. Lemma 1 gives us the abelianisation of $Q_1(6)$.

Lemma 1
The derived subgroup of $Q_1(6)$ is

$$(Q_1(6))'=\left\langle b^2,l_1^2l_2^{-1}l_4,l_1l_3^{-1},l_2l_3^2l_4,l_1l_3\right\rangle$$

while the abelianisation of $Q_1(6)$ is

$$Q_1(6)^{ab} = Q_1(6) \big/ \left(Q_1(6)\right)' = \left\langle a\left(Q_1(6)\right)',\ b\left(Q_1(6)\right)',\ l_1\left(Q_1(6)\right)',\ l_2\left(Q_1(6)\right)' \right\rangle \cong C_0 \times C_2^3.$$

Proof. The Bieberbach group $Q_1(6)$ is generated by $a,b,c,l_1,l_2,l_3,l_4,l_5$ and l_6 where $[a,b]=cl_5l_6^{-1}=b^2$ by relation of $Q_1(6)$ in (1), $[a,c]=[a,l_5]=l_5l_6^{-1}=c^2$, $[a,l_1]=l_1^{-1}l_2^{-1}l_4$, $[a,l_2]=l_1l_3^{-1}$, $[a,l_3]=l_2l_3^2l_4$, $[a,l_4]=l_1^{-1}l_3^{-1}$, $[a,l_6]=l_5^{-1}l_6$, $[b,c]=[b,l_5]=[b,l_6]=[c,l_5]=[c,l_6]=1$, $[b,l_1]=l_1l_3$, $[b,l_2]=l_1l_2l_3l_4$, $[b,l_3]=l_1^{-1}l_3$, $[b,l_4]=l_1l_2^{-1}l_3^{-1}l_4$, $[c,l_1]=l_1^2$, $[c,l_2]=l_2^2$, $[c,l_3]=l_3^2$, $[c,l_4]=l_4^2$ and $[l_i,l_j]=1$ for all $1 \le i < j \le 6$. Then,

$$\left(Q_1(6)\right)' = \left\langle b^2, c^2, l_1^2l_2^{-1}l_4, l_1l_3^{-1}, l_2l_3^2l_4, l_1^{-1}l_3^{-1}, l_5^{-1}l_6, l_1l_3, l_1l_2l_3l_4, l_1^{-1}l_3, l_1l_2^{-1}l_3^{-1}l_4, l_1^2, l_2^2, l_3^2, l_4^2 \right\rangle.$$

However this can be reduced to its minimum independent generators. As

$l_3^2 = (l_1l_3)(l_1^{-1}l_3)$;

$l_5^{-1}l_6 = \left(l_5l_6^{-1}\right)^{-1}$;

$l_1^2 = \left(l_1l_3^{-1}\right)(l_1l_3)$;

$l_4^2 = \left(l_1l_2^{-1}l_3^{-1}l_4\right)\left(l_2l_3^2l_4\right)(l_1l_3)^{-1}$;

$l_2^2 = \left(l_1^2l_2^{-1}l_4\right)^{-1}\left(l_2l_3^2l_4\right)\left(l_1l_3^{-1}\right)\left(l_1^{-1}l_3\right)^{-1}$;

$l_1^{-1}l_3^{-1} = (l_1l_3)^{-1}$; $l_1l_2l_3l_4 = \left(l_1^{-1}l_3\right)^{-1}\left(l_2l_3^2l_4\right)$; $l_1l_2^{-1}l_3^{-1}l_4 = \left(l_1^2l_2^{-1}l_4\right)(l_1l_3)^{-1}$; $l_1^{-1}l_3 = \left(l_1l_3^{-1}\right)^{-1}$. As $cl_5l_6^{-1} = b^2$ and $l_5l_6^{-1} = c^2$ by relation in $Q_1(6)$, then, $b^2 = c\left(c^2\right) = c^3$. Therefore, the minimum generators for derived subgroup of $Q_1(6)$ is:

$$\left(Q_1(6)\right)' = \left\langle b^2, l_1^2l_2^{-1}l_4, l_1l_3^{-1}, l_2l_3^2l_4, l_1l_3 \right\rangle$$

Since the abelianisation $Q_1(6)^{ab}$ is defined as the factor group $Q_1(6) \big/ \left(Q_1(6)'\right)$, hence it is generated by $a\left(Q_1(6)\right)'$, $b\left(Q_1(6)\right)'$, $l_1\left(Q_1(6)\right)'$, $l_2\left(Q_1(6)\right)'$, $l_3\left(Q_1(6)\right)'$, $l_4\left(Q_1(6)\right)'$, $l_5\left(Q_1(6)\right)'$ and $l_6\left(Q_1(6)\right)'$. However, some of the generators can be written in terms of the other generators. Using the relations in $Q_1(6)$, $b^2 = cl_5l_6^{-1}$. This implies that $\left(b\left(Q_1(6)\right)'\right)^2 = \left(cl_5l_6^{-1}\right)\left(Q_1(6)\right)'$. By properties of a factor group $\left(b\left(Q_1(6)\right)'\right)^2 = c\left(Q_1(6)\right)'\left(l_5l_6^{-1}\right)\left(Q_1(6)\right)'$. Then, since by one of the relations on $Q_1(6)$, $c^2 = l_5l_6^{-1}$, it is shown that $\left(b\left(Q_1(6)\right)'\right)^2 = c\left(Q_1(6)\right)'\left(c\left(Q_1(6)\right)'\right)^2$. Therefore, $\left(b\left(Q_1(6)\right)'\right)^2 = \left(c\left(Q_1(6)\right)'\right)^3$. Next, $l_1\left(Q_1(6)\right)' = l_3\left(Q_1(6)\right)'$ and $l_5\left(Q_1(6)\right)' = l_6\left(Q_1(6)\right)'$ since $l_1l_3^{-1}, l_5l_6^{-1} \in \left(Q_1(6)\right)'$. Moreover, by relations of $Q_1(6)$, $c^2 = l_5l_6^{-1}$. This implies that $l_5 = c^2c^{-1}a^2$, because $l_6 = c^{-1}a^2$ by $a^2 = cl_6$. It gives $l_5 = ca^2$. Since c is commute with b, therefore $l_5\left(Q_1(6)\right)' = b\left(Q_1(6)\right)'\left(a\left(Q_1(6)\right)'\right)^2$. Next, $l_1^2l_2^{-1}l_4$ is also in $\left(Q_1(6)\right)'$, thus,

$l_4(Q_1(6))' = (l_1^{-2}l_2)(Q_1(6))'$. Then, by properties of a factor group $l_4(Q_1(6))' = \left(l_1(Q_1(6))'\right)^{-2} l_2(Q_1(6))'$. Therefore, the independent generators of $Q_1(6)^{ab}$ are $a(Q_1(6))'$, $b(Q_1(6))'$, $l_1(Q_1(6))'$ and $l_2(Q_1(6))'$. Next, the order of $a(Q_1(6))'$, $b(Q_1(6))'$, $l_1(Q_1(6))'$ and $l_2(Q_1(6))'$ are being determined. By the relations of $Q_1(6)$, it can be concluded that $b(Q_1(6))'$, $l_1(Q_1(6))'$ and $l_2(Q_1(6))'$ are of order two since $b^2, l_1^2, l_2^2 \in (Q_1(6))'$. However, $a(Q_1(6))' \cap c(Q_1(6))' \cap l_6(Q_1(6))'$ is not trivial since $\left(a(Q_1(6))'\right)^2 = (cl_6)(Q_1(6))'$ that implies $\left(a(Q_1(6))'\right)^2 = c(Q_1(6))' l_6(Q_1(6))'$ by the relations of $Q_1(6)$. Hence, $a(Q_1(6))'$ has infinite order.
Thus,

$$Q_1(6)^{ab} = {Q_1(6)}\big/{(Q_1(6))'} = \left\langle a(Q_1(6))', b(Q_1(6))', l_1(Q_1(6))', l_2(Q_1(6))' \right\rangle \cong C_0 \times C_2^3.$$

Thus, $Q_1(6)^{ab}$ is finitely generated. □

Next, the computation of the central subgroup of the nonabelian tensor square for the first Bieberbach group of dimension six with quaternion point group of order eight denoted as $\nabla(Q_1(6))$ is done in the following theorem.

Theorem 3

Let $Q_1(6)$ be the first Bieberbach group of dimension six with quaternion point group of order eight and has polycyclic presentation as in (1). Then, the subgroup $\nabla(Q_1(6))$ is given as

$$\begin{aligned}\nabla(Q_1(6)) = \langle & a\otimes a,\ b\otimes b,\ l_1\otimes l_1,\ l_2\otimes l_2,\ (a\otimes b)(b\otimes a),\ (a\otimes l_1)(l_1\otimes a),\\ & (a\otimes l_2)(l_2\otimes a),\ (b\otimes l_1)(l_1\otimes b),\ (b\otimes l_2)(l_2\otimes b),\ (l_1\otimes l_2)(l_2\otimes l_1)\rangle \\ \cong\ & C_0\times C_4\times C_2\times C_2\times C_2\times C_2\times C_2\times C_2\times C_2\times C_2\\ =\ & C_0\times C_2^8\times C_4.\end{aligned}$$

Proof. By Lemma 1, $Q_1(6)^{ab}$ is generated by $a(Q_1(6))'$ which is of infinite order and $b(Q_1(6))'$, $l_1(Q_1(6))'$ and $l_2(Q_1(6))'$ which are of order two. Then based on Proposition 3(i) $\nabla(Q_1(6))$ is generated by $[a,a^\varphi]$, $[b,b^\varphi]$, $[l_1,l_1^\varphi]$, $[l_2,l_2^\varphi]$, $[a,b^\varphi][b,a^\varphi]$, $[a,l_1^\varphi][l_1,a^\varphi]$, $[a,l_2^\varphi][l_2,a^\varphi]$, $[b,l_1^\varphi][l_1,b^\varphi]$, $[b,l_2^\varphi][l_2,b^\varphi]$ and $[l_1,l_2^\varphi][l_2,l_1^\varphi]$. Since $\nabla(Q_1(6))$ is a subgroup of $Q_1(6)\otimes Q_1(6)$ and the mapping $\alpha : Q_1(6)\otimes Q_1(6) \to \left[Q_1(6),(Q_1(6))'\right]$ defined by $\alpha(g\otimes h) = [g\ h^\varphi]$ for all $g,h\in Q_1(6)$ is an isomorphism, thus

$$\begin{aligned}\nabla(Q_1(6)) = \langle & a\otimes a,\ b\otimes b,\ l_1\otimes l_1,\ l_2\otimes l_2,\ (a\otimes b)(b\otimes a),\ (a\otimes l_1)(l_1\otimes a),\\ & (a\otimes l_2)(l_2\otimes a),\ (b\otimes l_1)(l_1\otimes b),\ (b\otimes l_2)(l_2\otimes b),\ (l_1\otimes l_2)(l_2\otimes l_1)\rangle\end{aligned}$$

Next, the order of each generator of $\nabla(Q_1(6))$ is computed. The abelianisation of $Q_1(6)$ is with natural homomorphism $\eta : Q_1(6) \to Q_1(6)^{ab}$. Since $Q_1(6)^{ab}$ is finitely generated by Lemma 1, then its nonabelian tensor square is just an ordinary tensor product of $Q_1(6)^{ab} \cong C_0 \times C_2^3$. By Proposition 4,

$$
\begin{aligned}
&Q_1(6)^{ab} \otimes Q_1(6)^{ab} \\
&\cong (C_0 \times C_2 \times C_2 \times C_2) \otimes (C_0 \times C_2 \times C_2 \times C_2) \\
&= (C_0 \otimes (C_0 \times C_2 \times C_2 \times C_2)) \times (C_2 \otimes (C_0 \times C_2 \times C_2 \times C_2)) \times (C_2 \otimes (C_0 \times C_2 \times C_2 \times C_2)) \times (C_2 \otimes (C_0 \times C_2 \times C_2 \times C_2)) \\
&= (C_0 \otimes C_0) \times (C_0 \otimes C_2) \times (C_0 \otimes C_2) \times (C_0 \otimes C_2) \times (C_2 \otimes C_0) \times (C_2 \otimes C_2) \times (C_2 \otimes C_2) \times (C_2 \otimes C_2) \\
&\qquad \times (C_2 \otimes C_0) \times (C_2 \otimes C_2) \times (C_2 \otimes C_2) \times (C_2 \otimes C_2) \times (C_2 \otimes C_0) \times (C_2 \otimes C_2) \times (C_2 \otimes C_2) \times (C_2 \otimes C_2) \\
&\cong C_0 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \\
&= C_0 \times C_2^{15}.
\end{aligned}
$$

Then, Lemma 1 provides that $Q_1(6)^{ab}$ is generated by $\eta(a)$ of infinite order and $\eta(b), \eta(l_1), \eta(l_2)$ of order 2. Again by Proposition 4, $\langle \eta(a) \otimes \eta(a) \rangle \cong C_0$. By Theorem 2, there is a natural epimorphism $\alpha : Q_1(6) \otimes Q_1(6) \to Q_1(6)^{ab} \otimes Q_1(6)^{ab}$. Therefore, the image $\alpha(a \otimes a) = \eta(a) \otimes \eta(a)$ has infinite order. Hence, by Proposition 5, $a \otimes a$ has also infinite order.

Next, $[b, b^{\varphi}]$ will be shown to has order four. By $\langle \eta(b) \otimes \eta(b) \rangle \cong C_2$, it gives the order of $[b, b^{\varphi}]$ is at least two. If the order of $[b, b^{\varphi}]$ is two then, $[b, b^{\varphi}]^2 = [b^2, b^{\varphi}] = 1$. But, there is no relation in $Q_1(6)$ to allow b^2 and b^{φ} to commute. Therefore, the order of $[b, b^{\varphi}]$ is four since $[b, b^{\varphi}]^4 = [b^2, b^{2\varphi}] = 1$ by Proposition 1(vi) and it concludes that $b \otimes b$ is also of order four by Proposition 5. Next,

$$
\begin{gathered}
[l_1, l_1^{\varphi}],\ [l_2, l_2^{\varphi}],\ [a, b^{\varphi}][b, a^{\varphi}],\ [a, l_1^{\varphi}][l_1, a^{\varphi}],\ [a, l_2^{\varphi}][l_2, a^{\varphi}], \\
[b, l_1^{\varphi}][l_1, b^{\varphi}], [b, l_2^{\varphi}][l_2, b^{\varphi}] \text{ and } [l_1, l_2^{\varphi}][l_2, l_1^{\varphi}]
\end{gathered}
$$

will be shown to have order two.

By homomorphism in Proposition 4, we have

$$
\begin{gathered}
\langle \eta(l_1) \otimes \eta(l_1) \rangle \cong C_2,\ \langle \eta(l_2) \otimes \eta(l_2) \rangle \cong C_2,\ \langle \eta(a) \otimes \eta(b) \rangle \cong C_2,\ \langle \eta(b) \otimes \eta(a) \rangle \cong C_2,\ \langle \eta(a) \otimes \eta(l_1) \rangle \cong C_2, \\
\langle \eta(l_1) \otimes \eta(a) \rangle \cong C_2,\ \langle \eta(a) \otimes \eta(l_2) \rangle \cong C_2,\ \langle \eta(l_2) \otimes \eta(a) \rangle \cong C_2,\ \langle \eta(b) \otimes \eta(l_1) \rangle \cong C_2,\ \langle \eta(l_1) \otimes \eta(b) \rangle \cong C_2, \\
\langle \eta(b) \otimes \eta(l_2) \rangle \cong C_2,\ \langle \eta(l_2) \otimes \eta(b) \rangle \cong C_2,\ \langle \eta(l_1) \otimes \eta(l_2) \rangle \cong C_2,\ \langle \eta(l_2) \otimes \eta(l_1) \rangle \cong C_2.
\end{gathered}
$$

Therefore, the order of $l_1 \otimes l_1$, $l_2 \otimes l_2$, $(a \otimes b)(b \otimes a)$, $(a \otimes l_1)(l_1 \otimes a)$, $(a \otimes l_2)(l_2 \otimes a)$, $(b \otimes l_1)(l_1 \otimes b)$, $(b \otimes l_2)(l_2 \otimes b)$ and $(l_1 \otimes l_2)(l_2 \otimes l_1)$ are of order two. Hence,

$$
\begin{aligned}
\nabla(Q_1(6)) &= \langle a \otimes a,\ b \otimes b,\ l_1 \otimes l_1,\ l_2 \otimes l_2,\ (a \otimes b)(b \otimes a),\ (a \otimes l_1)(l_1 \otimes a), \\
&\qquad (a \otimes l_2)(l_2 \otimes a),\ (b \otimes l_1)(l_1 \otimes b),\ (b \otimes l_2)(l_2 \otimes b),\ (l_1 \otimes l_2)(l_2 \otimes l_1) \rangle \\
&\cong C_0 \times C_4 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2 \\
&= C_0 \times C_2^8 \times C_4.
\end{aligned}
$$

□

CONCLUSION

In this paper, the central subgroup of the nonabelian tensor square of the first Bieberbach group with quaternion point group of order eight, $\nabla(Q_1(6))$ is computed. The result shows that this central subgroup is elementary abelian.

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REFERENCES

1. Torsion Free Space Groups. (http://wwwb.math.rwth-aachen.de/carat/bieberbach.html).
2. R. Masri, "The Nonabelian Tensor Squares of Certain Bieberbach Groups with Cyclic Point Group of Order Two," Ph.D. thesis, Universiti Teknologi Malaysia, 2009.
3. N. Mohd Idrus, W.N.F. Wan Mohd Fauzi, R. Masri, Y.T. Tan, N.H. Sarmin, and H.I. Mat Hassim, International Journal of Applied Mathematics and Statistics **53**(4), 104-109 (2015).
4. H.I. Mat Hassim, "The Homological Functors of Bieberbach Groups with Cyclic Point Groups of Order Two, Three and Five," Ph.D. thesis, Universiti Teknologi Malaysia, 2014.
5. N. R. Rocco, Bol. Soc. Brasil. Mat. (N. S.) **22**(1), 63-70 (1991).
6. G. Ellis and F. Leonard, Proc. Roy. Irish Acad. Sect. A **95**(2), 137-147 (1995).
7. R. D. Blyth and R. F. Morse, Journal of Algebra **321**, 2139-2148 (2009).
8. R. D. Blyth, F. Fumagalli and M. Morigi, Journal of Group Theory **13**, 83-94 (2010).
9. A. J. Zomorodian, *Topology for Computing* (Cambridge University Press, United Kingdom, 2005), pp. 79-80.
10. R. Brown, D.L. Johnson and E.F. Robertson, Journal of Algebra **111**, 177-202 (1987).