The cubed commutativity degree of dihedral groups

Muhanizah Abdul Hamid, Nor Muhainiah Mohd Ali, Nor Haniza Sarmin, and Ahmad Erfanian

Citation: AIP Conference Proceedings **1750**, 050004 (2016); doi: 10.1063/1.4954592
View online: http://dx.doi.org/10.1063/1.4954592
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1750?ver=pdfcov
Published by the AIP Publishing

The Cubed Commutativity Degree of Dihedral Groups

Muhanizah Abdul Hamid[1, a)], Nor Muhainiah Mohd Ali[1, b)],
Nor Haniza Sarmin[1, c)] and Ahmad Erfanian[2, d)]

[1]*Department of Mathematical Sciences,*
Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Malaysia
[2]*Department of Pure Mathematics, Faculty of Mathematical Sciences and Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran*

[b)]*normuhainiah@utm.my*
[a)]*muhanizah.maths@gmail.com*
[c)]*nhs@utm.my*
[d)]*erfanian@um.ac.ir*

Abstract. Let G be a finite group. The commutativity degree of a group is the probability that a random pair of elements in the group commute. Furthermore, the n-th power commutativity degree of a group is a generalization of the commutativity degree of a group which is defined as the probability that the n-th power of a random pair of elements in the group commute. In this research, the n-th power commutativity degree for some dihedral groups is computed for the case n equal to 3, called the cubed commutativity degree.

INTRODUCTION

Commutativity degree is the term that is used to determine the abelianness of groups. If G is a finite group, then the commutativity degree of G, denoted by $P(G)$, is the probability that two randomly chosen elements of G commute. The first appearance of this concept was in 1944 (see [1]). After a few years, the idea to compute $P(G)$ for symmetric groups has been introduced by Erdos and Turan [2] in 1968.

Mohd Ali and Sarmin [3] in 2006 extended the definition of commutativity degree of a group and defined a new generalization of this degree which is called the n-th commutativity degree of a group G, $P_n(G)$ where it is equal to the probability that the n-th power of a random element commutes with another random element from the same group. They also determined $P_n(G)$ for 2 generator 2-groups of nilpotency class two.

A few years later, Erfanian *et al*. [4] gave the relative case of n-th commutativity degree. They identify the probability that the n-th power of a random element of a subgroup, H commutes with another random element of a group G, denoted as $P_n(H,G)$.

In this research, the commutativity degree is further extended by defining a concept called the probability that the n-th power of a random pair of elements in the group commute, denoted as $P^n(G)$. However, the focus of this research is only for the determination of $P^n(G)$, where $n=3$ and G is a Dihedral group. Here $P^3(G)$ is called the cubed commutativity degree.

PRELIMINARIES

In this section, some important definitions which include the notion of commutativity degree and its generalizations are stated.

Advances in Industrial and Applied Mathematics
AIP Conf. Proc. 1750, 050004-1–050004-7; doi: 10.1063/1.4954592
Published by AIP Publishing. 978-0-7354-1407-5/$30.00

Definition 1 [2] **The Commutativity Degree of a Group *G***

Let G be a finite group. The commutativity degree of a group G is given as:

$$P(G)=\frac{\text{Number of ordered pairs }(x,y)\in G\times G \ni xy=yx}{\text{Total number of ordered pairs }(x,y)\in G\times G}$$
$$=\frac{\left|\left\{(x,y)\in G\times G \mid xy=yx\right\}\right|}{|G|^2}.$$

Definition 2 [3] **The *n*-th Commutativity Degree of a Group *G***

Let G be a finite group. The n-th commutativity degree of a group G is given as:

$$P_n(G)=\frac{\left|\left\{(x,y)\in G\times G \mid x^n y=yx^n\right\}\right|}{|G|^2}.$$

Definition 3 [5] **Dihedral Groups of Degree *n***

For $n\geq 3$, D_n is denoted as the set of symmetries of a regular n-gon. Furthermore, the order of D_n is $2n$ or equivalently, $|D_n|=2n$. Dihedral groups, D_n can be represented in a form of generators and relations given in the following representation:

$$D_n=\left\langle a,b \mid a^n=b^2=1, ba=a^{-1}b\right\rangle.$$

Definition 4 [6] **The *n*-th Centralizer of *a* in *G***

Let a be a fixed element of a group G. The n-th centralizer of a in G, $C_G^n(a)$ is the set of all elements in G that commute with a^n. In symbols,

$$C_G^n(a)=\left\{g\in G \middle| ga^n=a^n g\right\}=C_G\left(a^n\right).$$

Then $C_G^n(a)$ is a subgroup of G and $\bigcap_{a\in G} C_G^n(a)=C_G\left(G^n\right)$, where $G^n=\left\{a^n \middle| a\in G\right\}$.

Now define $T_G^n(a)=\left\{g\in G \middle| (ga)^n=(ag)^n\right\}$ and $T_G^n(G)=\bigcap_{a\in G} T_G^n(a)$. It is easy to see that $T_G^n(a)$ may not be a subgroup of G. But it can be seen easily that $T_G^n(G)=C_G\left(G^n\right)$ and so $T_G^n(G)$ is a normal subgroup of G. To prove $T_G^n(G)\subseteq C_G\left(G^n\right)$, let $a\in T_G^n(G)$. Then for all $g\in G$, $(ag)^n=(ga)^n$. Therefore $\left(a\left(a^{-1}g\right)\right)^n=\left(\left(a^{-1}g\right)a\right)^n$ and so $g^n=a^{-1}g^n a$. Hence $ag^n=g^n a$ and $a\in C_G\left(G^n\right)$. To see $C_G\left(G^n\right)\subseteq T_G^n(G)$, let $a\in C_G\left(G^n\right)$. Then for all $g\in G$, $ag^n=g^n a$. Therefore $a(ag)^n=(ag)^n a$ and so $(ag)^n=a^{-1}(ag)^n a$. Hence $(ag)^n(ga)^n$ and $a\in T_G^n(G)$.

Definition 5 The *n*-th Center of a Group

The n-th center $Z^n(G)$ of a group G is the set of elements in G given as the following:

$$Z^n(G)=\{a\in G \mid (ax)^n=(xa)^n \text{ for all } x \text{ in } G\}$$

RESULTS AND DISCUSSIONS

In this section, the results of the cubed commutativity degree are given. Before that, the new definition which is the n-th power commutativity degree of a group is given as below:

Definition 6 The *n*-th Power Commutativity Degree of a Group *G*
Let G be a finite group. The n-th power commutativity degree of a group G is given as:

$$P^n(G)=\frac{\left| x,y \in G\times G:\ xy^{\,n} = yx^{\,n} \right|}{|G|^2}.$$

If we replace $n=3$ in Definition 6, then $P^3(G)$ is called the cubed commutativity degree of a group and will be used in the main theorems, given as in the following:

$$\begin{aligned}P^3(G)&=\frac{\left|\left\{(x,y)\in G\times G:(xy)^3=(yx)^3\right\}\right|}{|G|^2}\\&=\frac{1}{|G|^2}\sum_{x\in G}\left|\left\{y\in G\middle|(xy)^3=(yx)^3\right\}\right|\\&=\frac{1}{|G|^2}\sum_{x\in G}\left|T_G^3(x)\right|\end{aligned}$$

Next, the following lemma is provided which is used in the proof of the propositions following it.

Lemma 1
If G is a Dihedral group then $Z^3(G)=Z(G)$.
Proof
Let G be a Dihedral group, D_n. Suppose $a,y\in D_n$ then we have $ya=a^{-1}y$, $a^n=1$, $a^{-1}=a^{n-1}$, and $y^2=1$. Note that $ay=ya^{-1}$ since $ya=a^{-1}y$ implies $ayaa^{-1}=aa^{-1}ya^{-1}$. Thus $ay=ya^{-1}$. To show $Z(G)\subseteq Z^3(G)$ is trivial since $ay=ya$ implies $(ay)^3=(ya)^3$. Now we are going to show that $Z^3(G)\subseteq Z(G)$, *i.e* $(ay)^3=(ya)^3$ implies $ay=ya$. Suppose $(ay)^3=(ya)^3$. Then $(ay)(ay)(ay)=(ya)(ya)(ya)$, which gives $(ay)(ay)(ay)=(a^{-1}y)(a^{-1}y)(a^{-1}y)$. Using the associativity of D_n and its property, $a(ya)(ya)y=a^{n-1}(ya^{-1})(ya^{-1})y$ which leads to $a(a^{-1}y)(a^{-1}y)y=a^{n-1}(ay)(ay)y$. This gives $ya^{-1}=a^n yay^2$, which leads to $ay=ya$.

Next, the following propositions are given which play an important role in the proof of the main results.

Proposition 1
Let G be a Dihedral group of order $2n$ where $n\geq 5$. For $x\notin Z^3(G)$,

$$\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right|=\begin{cases}n^2+n, & n \text{ is a prime},\\ n^2+3n, & n \text{ is not a prime},\end{cases}$$

where $T_G^3(x)=\{g\in G:(gx)^3=(xg)^3\quad \forall x\in G\}$ and $Z^3(G)=\left\{a\in G\,|\,(ay)^3=(ya)^3\quad \forall y\in G\right\}$.
Proof
Let $G=D_n$ where $|D_n|=2n$ and n is odd. Then by Definition 5 for $n=3$ and Lemma 1, $Z^3(G)=\{e\}$. Suppose $A=\left\{e,a,a^2,\ldots,a^{n-1}\right\}$ and $B=\left\{b,ab,a^2b,\ldots,a^{n-1}b\right\}$ then we have $\left|T_G^3(e)\right|=\left|T_G^3(a)\right|=\ldots=\left|T_G^3(a^{n-1})\right|=n$ since for any $y\in A$ and for all $z\in A$ we also have $(yz)^3=(zy)^3$ but for any $y\in A$ and for all $z\in B$ we have $(yz)^3\neq(zy)^3$.

Therefore for all $y,z \in A$, $\sum_{x\in A}\left|T_G^3(x)\right| = n(n-1)$ in which $x \notin Z^3(G)$. The proof for the part for any $y \in B$ and for all $z \in B$ is divided into two cases.

Case 1 (n is a prime):
We have $T_G^3(a^ib)=\left\{e,a^ib\right\}$ in which $\left|T_G^3(a^ib)\right|=2$ for $0\le i\le n-1$ since for any $y\in B$ and for all $z\in B$, there are two pairs of elements that satisfy $(yz)^3=(zy)^3$ which are the identity and the element itself. This implies that $\sum_{x\in B}\left|T_G^3(x)\right|=2n$. Hence,

$$\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right| = \sum_{x\in A\setminus Z^3(G)}\left|T_G^3(x)\right| + \sum_{x\in B}\left|T_G^3(x)\right| = n(n-1)+2n = n^2+n..$$

Case 2 (n is not a prime):
We have $T_G^3(a^ib)=\left\{e,a^ib,a^{\frac{n}{3}+i}b,a^{\frac{2n}{3}+i}b\right\}$ in which $\left|T_G^3(a^ib)\right|=4$ for $0\le i\le n-1$ since for any $y\in B$ and for all $z\in B$, there are four pairs of elements that satisfy $(yz)^3=(zy)^3$. This implies that $\sum_{x\in B}\left|T_G^3(x)\right|=4n$. Hence,

$$\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right| = \sum_{x\in A\setminus Z^3(G)}\left|T_G^3(x)\right| + \sum_{x\in B}\left|T_G^3(x)\right| = n(n-1)+4n = n^2+3n. \quad \square$$

Remark:- For the case $n=3$, namely for Dihedral Group of order 6, can be refered to the case when n is not a prime.

Proposition 2
Let G be a Dihedral group of order $2n$ where $n\ge 4$. For $x\notin Z^3(G)$ and n is even where $k\ge 0$,

$$\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right| = \begin{cases} n^2+6n, & n=6+6k, \\ n^2+2n, & n=4+6k \text{ and } n=8+6k, \end{cases}$$

where $T_G^3(x)=\{g\in G:(gx)^3=(xg)^3\}$ for all x in G and $Z^3(G)=\left\{a\in G\,|\,(ay)^3=(ya)^3\right\}$ for all y in G.

Proof

Let $G=D_n$ where $\left|D_n\right|=2n$ and n is even where $n\ge 4$. By Definition 5 for $n=3$ and Lemma 1, $Z^3(G)=\{e,a^{\frac{n}{2}}\}$. Suppose $A=\left\{e,a,a^2,...,a^{n-1}\right\}$ and $B=\left\{b,ab,a^2b,...,a^{n-1}b\right\}$ then we have $\left|T_G^3(e)\right|=\left|T_G^3(a)\right|=...=\left|T_G^3(a^{n-1})\right|=n$ since for any $y\in A$ and for all $z\in A$ we also have $(yz)^3=(zy)^3$ but for any $y\in A$ and for all $z\in B$ we have $(yz)^3\ne(zy)^3$. Therefore for all $y,z\in A$ implies that $\sum_{x\in A}\left|T_G^3(x)\right|=n(n-2)$ in which $x\notin Z^3(G)$. The proof for the part for any $y\in B$ and for all $z\in B$ is divided into two cases.

Case 1 $\left(n=6+6k,\ k\ge 0\right)$:
We have $T_G^3(a^ib)=\left\{e,a^{\frac{n}{2}},a^ib,a^{\frac{n}{6}+i}b,a^{\frac{n}{3}+i}b,a^{\frac{n}{2}+i}b,a^{\frac{2n}{3}+i},a^{n-\frac{n}{6}+i}b\right\}$ in which $\left|T_G^3(a^ib)\right|=8$ for $0\le i\le n-1$ since for any $y\in B$ and for all $z\in B$, there are eight pairs of elements that satisfy $(yz)^3=(zy)^3$. This implies that $\sum_{x\in B}\left|T_G^3(x)\right|=8n$. Hence,

$$\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right| = \sum_{x\in A\setminus Z^3(G)}\left|T_G^3(x)\right| + \sum_{x\in B}\left|T_G^3(x)\right| = n(n-2)+8n = n^2+6n.$$

Case 2 $\left(n=4+6k \text{ and } n=8+6k,\ k\geq 0\right)$:

We have $T_G^3(a^i b)=\left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}$ in which $\left|T_G^3(a^i b)\right|=4$ for $0\leq i\leq n-1$ since for any $y\in B$ and for all $z\in B$ there are four pairs of elements that satisfy $(yz)^3=(zy)^3$. This implies that $\sum_{x\in B}\left|T_G^3(x)\right|=4n$. Hence,

$$\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right|=\sum_{x\in A\setminus Z^3(G)}\left|T_G^3(x)\right|+\sum_{x\in B}\left|T_G^3(x)\right|=n(n-2)+4n=n^2+2n.. \quad \square$$

The main results of this research are stated in the following two theorems.

Theorem 1
Let G be Dihedral groups of order $2n$ where $n\geq 5$ and n is odd.

i. If n is prime then $P^3(G)=\dfrac{n+3}{4n}$.

ii. If n is not prime then $P^3(G)=\dfrac{n+5}{4n}$.

Proof
By Definition 6, we have

$$\begin{aligned}
P^3(G)&=\frac{\left|\left\{(x,y)\in G\times G:(xy)^3=(yx)^3\right\}\right|}{|G|^2}\\
&=\frac{1}{|G|^2}\sum_{x\in G}\left|\left\{y\in G\,\middle|\,(xy)^3=(yx)^3\right\}\right|\\
&=\frac{1}{|G|^2}\sum_{x\in G}\left|T_G^3(x)\right|\\
&=\frac{1}{|G|^2}\left[\sum_{x\in Z^3(G)}\left|T_G^3(x)\right|+\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right|\right]\\
&=\frac{1}{|G|^2}\left[\left|Z^3(G)\right||G|+\sum_{x\notin Z^3(G)}\left|T_G^3(x)\right|\right]
\end{aligned}$$

Note that $\left|Z^3(G)\right|=\left|Z^3(D_n)\right|=1$ for n is odd.

(i) By Proposition 1 (for n is a prime):

$$\begin{aligned}
P^3(G)&=\frac{1}{(2n)^2}\left[(1)|G|+\sum_{x\notin Z^3(D_n)}\left|T_{D_n}^3(x)\right|\right]\\
&=\frac{1}{4n^2}\left[2n+n^2+n\right]\\
&=\frac{1}{4n^2}\left[n^2+3n\right]\\
&=\frac{1}{4n}\left[n+3\right]\\
&=\frac{n+3}{4n}. \quad \square
\end{aligned}$$

(ii) By Proposition 1 (for n is not a prime):

$$P^3(G)=\frac{1}{(2n)^2}\left[(1)|G|+\sum_{x\notin Z^3(D_n)}\left|T_{D_n}^3(x)\right|\right]$$
$$=\frac{1}{4n^2}\left[2n+n^2+3n\right]$$
$$=\frac{1}{4n^2}\left[n^2+5n\right]$$
$$=\frac{1}{4n}\left[n+5\right]$$
$$=\frac{n+5}{4n}. \qquad \square$$

Theorem 2

Let G be Dihedral groups of order $2n$ where $n\geq 4$ and n is even.

i. If $n=6+6k$ for $k\geq 0$ then $P^3(G)=\frac{n+10}{4n}$.

ii. If $n=4+6k$ and $n=8+6k$ for $k\geq 0$ then $P^3(G)=\frac{n+6}{4n}$.

Note that $\left|Z^3(G)\right|=\left|Z^3(D_n)\right|=2$ for n is even.

(i) By Proposition 2 (Case 1):

$$P^3(G)=\frac{1}{(2n)^2}\left[(2)|G|+\sum_{x\notin Z^3(D_n)}\left|T_{D_n}^3(x)\right|\right]$$
$$=\frac{1}{4n^2}\left[4n+n^2+6n\right]$$
$$=\frac{1}{4n^2}\left[n^2+10n\right]$$
$$=\frac{n+10}{4n}. \qquad \square$$

(ii) By Proposition 2 (Case 2):

$$P^3(G)=\frac{1}{(2n)^2}\left[(2)|G|+\sum_{x\notin Z^3(D_n)}\left|T_{D_n}^3(x)\right|\right]$$
$$=\frac{1}{4n^2}\left[4n+n^2+2n\right]$$
$$=\frac{1}{4n^2}\left[n^2+6n\right]$$
$$=\frac{n+6}{4n}. \qquad \square$$

CONCLUSION

In this research, the cubed commutativity degree of Dihedral groups has been determined. The results are found for n even and n odd. However, for n even, the results are divided into two cases, namely when $n=6+6k$, $n=4+6k$, and $n = 8 + 6k$ for $k \geq 0$. Meanwhile for n odd, the results are divided into two cases, namely when n is a prime and n is not a prime.

ACKNOWLEDGMENTS

The authors would like to acknowledge Ministry of Higher Education (MOHE) Malaysia and Research Management Centre (RMC), Universiti Teknologi Malaysia for the financial funding through the Research University Grant (RUG) Vote No 08H07 and No 06H18. The first author would also like to thank Ministry of Higher Education (MOHE) for her MyPhD scholarship.

REFERENCES

1. G. A. Miller, "A Relative Number of Non-Invariant Operators in a Group", Proc. Nat. Acad. Sci.USA, 1944, pp. 25-28.
2. P. Erdos and P. Turan, IV Acta Math. Acad Sci. Hungaricae **19**, 413-435 (1968).
3. M. Mohd Ali and N. H Sarmin, Menemui Matematik (Discovering Mathematics) 32(2), 35-41 (2010), ISSN 2231-7023.
4. A. Erfanian, B. Tolue and N. H. Sarmin, Ars Combinatorial Journal **3**, 495-506 (2011).
5. D. S. Dummit and R. M. Foote*, Abstract Algebra, Third Edition*. John Wiley and Son, USA. *2004*, pp. 23-26.k
6. M. Mashkouri and B. Taeri, "On a Graph Associated to Groups," Bulletin of the Malaysian Mathematical Sciences Society (2) 34(3) 553-560 (2011).