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The Probability that an Element of a Metacyclic 3-Group of Negative Type Fixes a Set and Its Orbit Graph

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Abstract. In this paper, let G be a metacyclic 3-group of negative type of nilpotency class at least three. Let Ω be the set of all subsets of commuting elements of G of size three in the form of (a,b), where a and b commute and $lcm(|a|,|b|)=3$. The probability that an element of a group G fixes a set Ω is considered as one of the extensions of the commutativity degree that can be obtained under group actions on a set. In this paper, we compute the probability that an element of G fixes a set Ω in which G acts on Ω by conjugation. The results are then applied to graph theory, more precisely to orbit graph.

INTRODUCTION

Commutativity degree is the probability that two random elements in G commute, which was first introduced by Miller [1]. This concept was also crucial in finding the abelianness of a group. The definition of commutativity degree given by Miller is given in the following.

Definition 1 [1] Let G be a finite group. The commutativity degree is the probability that two random elements (x, y) in G commute, defined as follows:

$$P(G)=\frac{\left|\{(x,y)\in G\times G \mid xy=yx\}\right|}{|G|^2}.$$

A few years later, this concept was further reviewed by many researchers which has led to numerous results. The commutativity degree of symmetric groups was investigated by Erdos and Turan [2]. Few years later, Gustafson [3] and MacHale [4] proved that this probability is less than or equal to 5/8. The above probability has been generalized and extended by several authors. In this paper, we used one of these extensions, namely the probability that a group element fixes a set denoted as $P_G(\Omega)$. This probability was first introduced by Omer *et al.* [5] in 2013.

Basically, this research focuses on the probability that a group element fixes a set for metacyclic 3-groups of negative type which is found by using a group action namely the conjugate action. The computation is done by first

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finding the number of orbits under the same group action on the set. Afterward, some concepts of graph theory are introduced in which they will be applied to our results, particularly the orbit graph.

Next, we state some basic concepts that are needed in this paper.

Definition 2 [6] A group G is called metacyclic if it has a cyclic normal subgroup H such that the quotient group G/H is also cyclic.

Definition 3 [7] Let G be a finite group. A group G acts on itself if there is a function $G \times G \to G$ such that
i. $(gh)x = g(hx), \forall\, g,h,x \in G.$
ii. $1_G x = x, \forall x \in G.$

Definition 4 [8] Let G be any finite group and X be a set. A group G acts on X if there is a function $G \times X \to X$, such that
i. $(gh)x = g(hx), \forall\, g,h \in G, x \in X.$
ii. $1_G x = x, \forall x \in G.$

In the following, some concepts related to metacyclic *p*-groups are provided.

In 2005, Beuerle [9] gives several classifications of finite metacyclic *p*-groups of class at least three. In this paper, we used the classification of metacyclic *p*-groups of class at least three for odd prime, as given in the following theorem.

Theorem 1 [9] Let p be an odd prime and G a metacyclic p-group of nilpotency class at least three. Then G is isomorphic to exactly one group in the following list:

i. $G \cong \left\langle a,b \,\middle|\, a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\delta}} \right\rangle$, where $\alpha,\beta,\delta \in \mathbb{N}, \delta - 1 \le \alpha < 2\delta, \delta \le \beta;$

ii. $G \cong \left\langle a,b \,\middle|\, a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\varepsilon}}, [b,a] = a^{p^{\alpha-\delta}} \right\rangle$, where $\alpha,\beta,\delta,\varepsilon \in \mathbb{N}, \delta - 1 \le \alpha < 2\delta, \delta \le \beta, \alpha < \beta + \varepsilon.$

However, throughout this paper, we considered only on the first classification where *p* is an odd prime number, i.e *p* is equal to three with the first condition which is $\alpha = 1, \beta = 1, \delta = 1.$

Next is the definition of orbit in a group action, denoted as $O(x)$.

Definition 5 [10] Let G act on a set S, and $x \in S.$ If a group G acts on itself by conjugation, the orbit $O(x)$ is: $y \in G : y = axa^{-1}$ for some $a \in G.$ The orbit is also called the conjugacy classes of x in G.

Next, some fundamental concepts in graph theory that are needed in this paper are given. These concepts can be found in one of the references ([11] and [12]).

A graph is a representation of a set of objects or vertices which are connected by links or edges. Particularly, a graph Γ is a mathematical structure containing two sets, which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A graph Γ is connected if it there exists a path between every pair of distinct vertices, and is disconnected otherwise. Subgraphs $\Gamma(V_1), \Gamma(V_2), \ldots, \Gamma(V_n)$ are all components of Γ. The graph is connected if it has precisely one component. On the other hand, a graph is said to be complete if each ordered pair of vertices are adjacent to each other, denoted by K_n, where n is the number of adjacent vertices. Also, the graph is empty if there is no adjacent edge between its vertices. Additionally, a graph is called null if it has no vertices, which is denoted by K_0, the null graph.

Moreover, a non-empty set S of $V(\Gamma)$ is called an independent set of Γ if there is no adjacent edge between two elements of S in Γ. Thus, the independent number is the number of vertices in maximum independent set and it is denoted as $\alpha(\Gamma)$. In addition, the chromatic number is defined as the maximum number c in which Γ is c-vertex colorable, denoted by $\chi(\Gamma)$. Meanwhile, $d(\Gamma)$ is used to denote the diameter of Γ, which is the maximum distance between any two vertices in Γ. A complete subgraph in Γ is called a clique, while the clique number is the size of the largest clique in Γ, denoted by $\omega(\Gamma)$. Further, the dominating set $X \subseteq V(\Gamma)$ is a set where for each v outside X, $\exists x \in X$ such that v is adjacent to x. The dominating number, denoted by $\gamma(\Gamma)$ is the minimum size of X.

This paper is divided into three parts. The first part browses on some backgrounds on commutativity degree and probability, and also some basic concepts in graph theory. Simultaneously, the second part discusses on previous researches and recent publications that have been done related to probability and graphs. In the third part, our main result on the probability that a group element fixes a set and its application which is the orbit graph are presented.

PRELIMINARIES

In this part, several previous works that have been done related to the commutativity degree and the probability that a group element fixes a set are discussed. Moreover, recent publications on graph theory related to probability are also considered.

In 1975, a new concept was introduced by Sherman [13], namely the probability of an automorphism of a finite group fixes an arbitrary element in the group. The definition of this probability is given in the following.

Definition 6 [13] Let G be a group. Let X be a non-empty set of G (G is a group of permutations of X). Then the probability of an automorphism of a group fixes a random element from X is defined as follows:

$$P_G(X) = \frac{\left|\left\{(g,x) \middle| gx = x \ \forall g \in G, x \in X\right\}\right|}{|X||G|}.$$

In 2011, Moghaddam [14] explored Sherman's definition and introduced a new probability, which is called the probability of an automorphism fixes a subgroup element of a finite group, the probability is stated as follows:

$$P_{A_G}(H,G) = \frac{\left|\left\{(\alpha,h) \mid h^{\alpha} : h \in H, \alpha \in A_G\right\}\right|}{|H||G|},$$

where A_G is the group of automorphisms of a group G. It is clear that if $H = G$, then $P_{A_G}(G,G) = P_{A_G}$.

Later on, Omer *et al.* [5] generalized the commutativity degree by defining the probability that an element of a group fixes a set of size two. Their results are given in the following.

Definition 7 [5] Let G be a group. Let S be a set of all subsets of commuting elements of size two in G, G acts on S by conjugation. Then the probability of an element of a group fixes a set is defined as follows:

$$P_G(S) = \frac{\left|\left\{(g,s) \middle| gs = s \ \forall g \in G, s \in S\right\}\right|}{|S||G|}.$$

Theorem 2 [5] Let G be a finite group and let X be a set of elements of G of size two in the form of (a,b) where a and b commute. Let S be the set of all subsets of commuting elements of G of size two and G acts on S by conjugation. Then the probability that an element of a group fixes a set is given by $P_G(S)=\frac{K(S)}{|S|}$, where $K(S)$ is the number of conjugacy classes of S in G.

Moreover, Omer *et al.* [15] found the probability that a group element fixes a set where their results are then applied to graph theory. Recently, Mustafa *et al.* [16] extended the work in [4] by restricting the order of Ω. The following theorem illustrates their results.

Theorem 3 [16] Let G be a finite group and let S be a set of elements of G of size two in the form of (a,b), where a, b commute and $|a|=|b|=2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω .Then the probability that an element of a group fixes a set is given by $P_G(\Omega)=\frac{K(\Omega)}{|\Omega|}$, where $K(\Omega)$ is the number of conjugacy classes of Ω in G.

Proposition 1 [16] If G is an abelian group, the probability that an element of a group fixes a set $P_G(\Omega)$ is equal to one.

Next, we state some works related to graph theory, particularly conjugacy class graph and orbit graph. The study on conjugacy class graph was found by Bianchi *et al*. [17] on the regularity of a graph and Moreto *et al.* [18] on the sizes of conjugacy with distinct classes. Later on, Moradipour *et al.* [19] used the graph related to conjugacy classes to find the graph properties on some finite metacyclic 2-groups.

Moreover, Erfanian and Tolue [20] introduced the conjugate graph in which two vertices of this graph are connected if they are conjugate. Later on, Omer *et al.* [21] generalized this graph by introducing a new graph called the orbit graph, denoted by Γ_G^Ω. The vertices of this graph are non-central orbits under group action on a set. The following is the definition of an orbit graph.

Definition 8 [21] Let G be a finite group and Ω be a set of elements of G. Let A be the set of commuting elements in Ω, i.e $A=\{v\in\Omega : vg=gv, g\in G\}$. The orbit graph Γ_G^Ω consists of two sets, namely vertices and edges, denoted by $V(\Gamma_G^\Omega)$ and $E(\Gamma_G^\Omega)$, respectively. The vertices of $V(\Gamma_G^\Omega)$ are non-central elements in Ω but not in A, that is $V(\Gamma_G^\Omega)=\Omega-A$, while the number of edges are $\left|E(\Gamma_G^\Omega)\right|=\sum_{i=1}^{|V(\Gamma_G^\Omega)|}\binom{v_i}{2}$, where v is the size of orbit under group action G on Ω. Two vertices v_1, v_2 are adjacent in Γ_G^Ω if one the following conditions are satisfied:

i. If there exists $g\in G$ such that $gv_1=v_2$,
ii. If the vertices of Γ_G^Ω are conjugate, that is $v_1=g^{v_2}$.

The orbit graph is also found for solvable groups [22], symmetric groups and alternating groups [23].

Later on, Omer *et al.* [24] generalized the conjugacy class graph, where the vertices are orbits under the group action on a set. The definition of the generalized conjugacy class graph is stated in the following.

Definition 9 [24] Let G be a finite non-abelian group and Ω is a set of G. If G acts on Ω, the the number of vertices of generalized conjugacy classes graph is $\left|V\left(\Gamma_G^{\Omega_c}\right)\right|=K(\Omega)-|A|$, where A is $\{g\omega=\omega g : \omega\in\Omega\}$. Two vertices ω_1 and ω_2 in $\Gamma_G^{\Omega_c}$ are adjacent if their cardinalities are not coprime.

MAIN RESULTS

This part provides our main result which is the computation of the probability that an element of a metacyclic 3-group of negative type fixes a set and also its application which is the orbit graph. In this paper, we focus only on the condition when $\alpha=\beta=\delta=1$ as given in the following proposition.

The Probability That an Element of a Group Fixes a Set

We start this part with the first result on the metacyclic 3-groups of negative type under the condition of $\alpha = \beta = \delta = 1$ as given in the following proposition.

Proposition 1 Let G be a group such that $G \cong \left\langle a,b \middle| a^{3^\alpha} = b^{3^\beta} = 1, [b,a] = a^{3^{\alpha-\delta}} \right\rangle$, where $\alpha, \beta, \delta \in \mathbb{N}, \delta - 1 \le \alpha < 2\delta, \delta \le \beta$ and $\alpha = \beta = \delta = 1$. Let S be a set of elements of G of size three in the form of (a,b) where a and b commute and $\operatorname{lcm}(|a|,|b|) = 3$. Let Ω be the set of all subsets of commuting elements of G of size three. If G acts on Ω by conjugation, then $P_G(\Omega) = \frac{5}{|\Omega|}$.

Proof The elements of G of order one and three are $\{1, a, a^2, b, b^2\}$. Therefore, there are 10 possible pairs of elements of size three. However, only eight of the possible pairs commute with each other. Hence, the elements of Ω of size three are $\{(1,a),(1,a^2),(1,b),(1,b^2),(a,a^2),(a,b^2),(a^2,b^2),(b,b^2)\}$. Therefore, $|\Omega| = 8$. Since the action here is by conjugation, thus there are five orbits divided as follows: There are two orbits in the form of $\{(1,a),(1,a^2)\}$ of size two, two orbits in the form of $\{(1,b),(1,b^2)\}$ of size two, one orbit in the form of $\{(a,a^2)\}$ of size one, two orbits in the form of $\{(a,b^2),(a^2,b^2)\}$ and lastly one orbit of $\{(b,b^2)\}$ of size one. Using Theorem 2, $P_G(\Omega) = \frac{5}{|\Omega|}$, as claimed.

The Orbit Graph

In this part, the results obtained from the probability are applied to orbit graph. According to Omer *et al*. [21], the number of vertices of orbit graph Γ_G^Ω, is equal to $\left|V\left(\Gamma_G^\Omega\right)\right| = |\Omega| - |A|$, where $A = \{g\omega = \omega g : \omega \in \Omega\}$. In our case, $|A| = 0$, therefore the $\left|V\left(\Gamma_G^\Omega\right)\right| = |\Omega| - 0$, which is 8. Our results on orbit graph is given in the following proposition.

Proposition 3 Let G be a finite non-abelian group and let Ω be the set of all subsets of commuting elements of G of size three. If G acts on Ω by conjugation and $P_G(\Omega) = \frac{5}{|\Omega|}$, then $\Gamma_G^\Omega = K_2 \cup K_2 \cup K_2$.

Proof Suppose $P_G(\Omega) = \frac{5}{|\Omega|}$. According to Omer *et al*., two vertices ω_1 and ω_2 of Γ_G^Ω are adjacent if $\omega_1 = \omega_2{}^g$. By the proof of Proposition 1, there are three complete components of size two, K_2, while the other two vertices are isolated. Then, $\Gamma_G^\Omega = K_2 \cup K_2 \cup K_2$.

The orbit graph is illustrated in Figure 1.

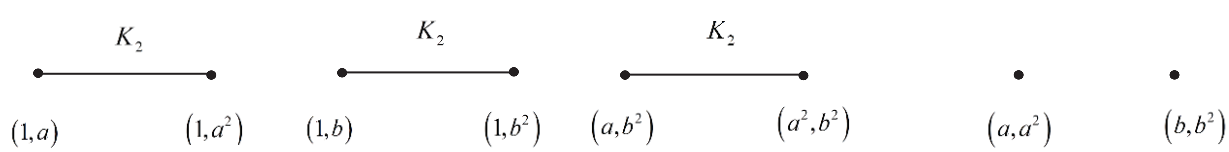

FIGURE 1. The orbit graphs of the probability that a group element of a metacyclic 3-group fixes a set

CONCLUSION

In this paper, the probability that an element of a group fixes a set has been computed for a metacyclic 3-group of negative type of nilpotency class at least three. Moreover, the results obtained from the probability are then applied to graph theory, specifically on the orbit graph.

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