1

Non-commuting Graph of Some Nonabelian Finite Groups

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1.1 INTRODUCTION

Let G be a group and $Z(G)$ be its center. For each group G, we will associate a graph which is called the non-commuting graph of G, denoted by Γ_G. The vertex set $V(\Gamma_G)$ is $G - Z(G)$ and the edge set $E(\Gamma_G)$ consists of $\{x, y\}$, where x and y are two distinct vertices of $V(\Gamma_G)$ are joined together if and only if $xy \neq yx$. The non-commuting graph of a group was introduced by Erdos in 1975. The non-commuting graph of a finite group has been studied by many researchers [1].

One of the problems about non-commuting graph of groups is given in the following conjecture:

Conjecture 1.1. Let G be a non-abelian finite group and H a group such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.

Definition 1.1 T_{4n} is a non-abelian finite group with order $4n$. Its structure is defined as

$$T_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle. \tag{1.1}$$

Definition 1.2 U_{6n} is a non-abelian finite group with order $6n$. Its structure is defined as

$$U_{6n} = \langle a, b | a^{2n} = 1 = b^3, a^{-1}ba = b^{-1} \rangle. \tag{1.2}$$

Definition 1.3 V_{8n} is a non-abelian finite group with order $8n$. Its structure is defined as

$$V_{8n} = \langle a, b | a^{2n} = 1 = b^4 = 1,\ ab = b^{-1}a^{-1},\ ab^{-1} = ba^{-1} \rangle. \tag{1.3}$$

The main objective of this chapter is to prove Conjecture 1.1 for three groups T_{4n}, U_{6n} and V_{8n}. In fact, we show that if $\Gamma_G \cong \Gamma_{T_{4n}}$, $\Gamma_G \cong \Gamma_{U_{6n}}$, $\Gamma_G \cong \Gamma_{V_{8n}}$, then $|G| = |T_{4n}|$, $|G| = |U_{6n}|$ or $|G| = |V_{8n}|$ respectively. For more details see Conway *et al.* [2] and Rose [3].

1.2 NON-COMMUTING GRAPH OF T_{4n}

In this section, we show that if G is a non-abelian finite group such that $\Gamma_G \cong \Gamma_{T_{4n}}$, then $|G| = |T_{4n}|$. In the lemmas, we refer the degree of the vertex x, which is denoted by $deg(x)$, as the number of edges through x. We first state some lemmas which will be used throughout this section.

Lemma 1.1 [4] *Let G be a non-abelian finite group and x is a vertex of* Γ_G. *Then*

$$deg(x) = |G| - |C_G(x)|. \tag{1.4}$$

Lemma 1.2 [4] *Let G be a non-abelian finite group. If H is a group such that* $\Gamma_G \cong \Gamma_H$, *then H is a non-abelian finite group such that* $|Z(H)|$ *divides each of the following:*

$$|G|-|Z(G)|, |G|-|C_G(x)|, |C_G(x)|-|Z(G)|, \textit{for } x \in (G-Z(G)).$$

Lemma 1.3 *Let* T_{4n} *be a group. Then*

$$|C_{T_{4n}}(a)| = 2n, |C_{T_{4n}}(b)| = 4 \ \textit{ and } \ |Z(T_{4n})| = 2.$$

Proof All elements of T_{4n} are denoted as $a^i b^j$ such that $1 \leq i \leq 2n$, $1 \leq j \leq 4$. The center of T_{4n} is defined by

$$\{a^i b^j | (a^i b^j)a = a(a^i b^j), b(a^i b^j) = (a^i b^j)b, 1 \leq i \leq 2n, 1 \leq j \leq 4\}.$$

Now, we find the elements of $Z(T_{4n})$. If $a^i b^j$ belongs to $Z(T_{4n})$, then $a^i b^j a = a^{i+1} b^j$ and $a^i b^{j+1} = b a^i b^j$. Therefore we have $b^j a = a b^j$ and $b a^i = a^i b$. There exist three cases for j as follows:

(a) If $j = 0$, then $ba^i = a^i b$. According to Definition 1.2, $a^i b = a^{-i} b$ and $i = n$. Hence $a^n \in Z(T_{4n})$.

(b) If $j \neq 2$, then $b^j a = a^{-1} b^j$ and $b^j a = a b^j$. Therefore the order of a is 2, which is a contradiction.

(c) If $j = 2$, then $a^i b^2 = a^{n+i}$ and $a^i b^3 = a^{-i} b^3$. Hence $i = n$ and it shows that $a^n \in Z(T_{4n})$.

So $Z(T_{4n}) = a^n, 1$ and $|Z(T_{4n})| = 2$. We can see easily that $C_{T_{4n}}(a) = \langle a \rangle$ and $C_{T_{4n}}(b) = \langle b \rangle$. Therefore $|C_{T_{4n}}(a)| = 2n, |C_{T_{4n}}(b)| = 4$. □

Theorem 1.1 *Let G be a finite non-abelian group. If $\Gamma_G \cong \Gamma_{T_{4n}}$, then $|G| = |T_{4n}|$.*

Proof We know that $\Gamma_{T_{4n}}$ has two vertices a and b such that $deg(a) = 2n$ and $deg(b) = 4n - 4$. Since $\Gamma_G \cong \Gamma_{T_{4n}}$, we have the following equality:

$$|G| - |Z(G)| = |T_{4n}| - |Z(T_{4n})| = 4n - 2.$$

Therefore $|Z(G)|$ divides $4n - 2$. There exists the corresponding elements $g_1, g_2 \in G - Z(G)$ such that $deg(g_1) = 2n$ and $deg(g_2) = 4n - 4$. By Lemma 1.2, we obtain that $|Z(G)|$ divides 2. Now, we show that $|Z(G)| = 2$. Using the contradiction proof, suppose that $|Z(G)| = 1$ and

$$|G| = 4n - 1, \; deg(g_1) = |G| - |C_G(g_1)| = 2n.$$

So $|C_G(g_1)| = 2n - 1$. But we know that $2n - 1$ does not divide $4n - 1$. Hence $|Z(G)| = 2$ and $|G| = 4n = |T_{4n}|$. □

1.3 NON-COMMUTING GRAPH OF U_{6n}

According to the definition of U_{6n}, we have all of its elements are in the form of $a^i b^j$ such that $0 \leq i \leq 2n-1$ and $0 \leq j \leq 2$. To obtain our main goal, we start with the following lemma.

Lemma 1.4 *Let U_{6n} be a finite group. Then*

$$|C_{U_{6n}}(a)| = 2n,\ |C_{U_{6n}}(b)| = 3 \ \text{and}\ |Z(U_{6n})| = 1.$$

Proof First, we show that $Z(U_{6n}) = 1$. Suppose that there exist i and j such that $a^i b^j \in Z(U_{6n})$ and $i, j \neq 0$. Since $(a^i b^j)a = a(a^i b^j)$, we obtain $a^{i+1}b^j = a^{i+1}b^{-j}$ and $j = 3$. Also we have $b(a^i b^j) = (a^i b^j)b$. Therefore $a^i b = ba^i = a^i b^{-1}$ and the order of b is equal to 2. Hence we conclude that $Z(U_{6n}) = 1$. By the structure of U_{6n}, we can easily see that $C_{U_{6n}}(a) = \langle a \rangle$ and $C_{U_{6n}}(b) = \langle b \rangle$. Therefore $|C_{U_{6n}}(a)| = 2n$ and $|C_{U_{6n}}(b)| = 3$. □

Theorem 1.2 *Let G be a finite non-abelian group. If $\Gamma_G \cong \Gamma_{U_{6n}}$, then $|G| = |U_{6n}|$.*

Proof Since $\Gamma_G \cong \Gamma_{U_{6n}}$, it can concluded that Γ_G has two vertices g_1, g_2 such that $deg(g_1) = 4n$ and $deg(g_2) = 6n-3$. Also we have this equality $|G| - |Z(G)| = 6n - 1$.

Since $|Z(G)|$ divides $deg(g_1)$ and $deg(g_2)$, then there exists three cases for $|Z(G)|$ as follows:

(a) If $|Z(G)| = 2$, then $|G| = 6n + 1$ and $|C_G(g_2)| = 4$. This is impossible since $4 \nmid |G|$.

(b) If $|Z(G)| = 3$, then $|G| = 6n + 2$ and $|C_G(g_2)| = 5$. This is impossible since $|Z(G)| \nmid |C_G(g_2)|$.

(c) If $|Z(G)| = 6$, then $|G| = 6n + 5$ and $|C_G(g_2)| = 8$. This is impossible since $|Z(G)| \nmid |C_G(g_2)|$.

Therefore $|Z(G)| = 1$ and $|G| = |U_{6n}| = 6n$. □

1.4 NON-COMMUTING GRAPH OF V_{8n}

In this section, we study about $C_{V_{8n}}(a)$, $C_{V_{8n}}(b)$ and $Z(V_{8n})$. We want to show that if $\Gamma_G \cong \Gamma_{V_{8n}}$, then $|G| = |V_{8n}|$. First we start with the following lemma.

Lemma 1.5 *Let V_{8n} be a finite group.*

(a) *If n is an even number, then $|C_{V_{8n}}(b)| = 8$, $|C_{V_{8n}}(a)| = 4n$ and $|Z(V_{8n})| = 4$.*

(b) *If n is an odd number, then $|C_{V_{8n}}(b)| = 4$, $|C_{V_{8n}}(a)| = 4n$ and $|Z(V_{8n})| = 2$.*

Proof Firstly, we show that $|C_{V_{8n}}(a)| = 4n$. It can be shown that

$$|C_{V_{8n}}(a)| = \{a^i b^j | (a^i b^j)a = a(a^i b^j) \ni 0 \le i \le 2n-1, 0 \le j \le 3\}. \quad (1.5)$$

If $j = 0$, then $\langle a \rangle \le C_{V_{8n}}(a)$. Assume that $i = 0$, we have $ab^2 = b^2a$. Now suppose that $i \neq 0$.

If $j = 1$, then $a^i b(a) = a^{i-1}b^{-1}$ and $(a)a^i b = a^{i+1}b$. Since the order of a is not equal to the order of b, we can conclude that $a(a^i b) \neq (a^i b)a$.

If $j = 2$, then $a^i b^2(a) = a^{i+1}b^2 = (a)a^i b^2$ for all $0 \le i \le 2n-1$.

If $j = 3$, then $a^i b^3(a) = a^{i-1}b^{-3}$ and $(a)a^i b^3 = a^{i+1}b^3$. Since the order of a is not equal to the order of b, we can conclude that $a(a^i b^3) \neq (a^i b^3)a$. Therefore, $|C_{V_{8n}}(a)| = 4n$.

Next, we want to obtain $|C_{V_{8n}}(b)|$, where n is an even number. $C_{V_{8n}}(b) = \{a^i b^j | a^i b^{j+1} = ba^i b^j\}$ for all $0 \le i \le 2n-1$ and $0 \le j \le 3$. We know that $\langle b \rangle \le C_{V_{8n}}(b)$. Suppose that $i \neq 0$, now we have four cases for j. If $j = 0$, then we recognize a^i such that $a^i b = ba^i$ for all i. Thus,

$$a^i b = ba^i \to a^{i-1}b^{-1}a^{-1} = ba^i \to a^{i-2}b = ba^{i+2} \to b^{(-1)^i} = ba^{2i}$$

The preceeding equation shows that i cannot be an odd number. Therefore i is an even number and $i = n$. If $j = 1$, then

$a^i b(b) \neq (b)a^i b$. For $j = 2$ and $j = 3$, we have

$$a^i b^2(b) \neq (b)a^i b^2$$

and $a^i b^3(b) \neq (b)a^i b^3$ for all i. Hence, $|C_{V_{8n}}(b)| = 8$.
Also we have:

$$C_{V_{8n}}(a) = \{1, a, a^2, ..., a^{2n-1}, b^2, b^2a, b^2a^2, ..., b^2a^{2n-1}\}$$

and

$$C_{V_{8n}}(b) = \{1, b, b^2, b^3, a^n, ba^n, b^2a^n, b^3a^n\}.$$

On the other hand, we know that

$$\begin{aligned} Z(V_{8n}) &= \{g \in V_{8n} | gv = vg \text{ for all } v \in V_{8n}\} \\ &= \{g \in V_{8n} | ga = ag \text{ and } gb = bg\} \\ &= C_{V_{8n}}(a) \cap C_{V_{8n}}(b) = \{1, b^2, a^n, b^2a^n\}. \end{aligned}$$

Therefore $|Z(V_{8n})| = 4$.

If n is an odd number, according to the above proof we have four cases for j. But in any case, we have $a^i b^j(b) \neq (b)a^i b^j$ for all $0 \leq j \leq 3$. Therefore $C_{V_{8n}}(b) = \langle b \rangle$ and

$$\begin{aligned} Z(V_{8n}) &= \{g \in V_{8n} | gv = vg \text{ for all } \in V_{8n}\} \\ &= \{g \in V_{8n} | ga = ag \text{ and } gb = bg\} \\ &= C_{V_{8n}}(a) \cap C_{V_{8n}}(b) \\ &= \{1, a, a^2, \ldots, a^{2n-1}, b^2, b^2a, b^2a^2, \ldots, b^2a^{2n-1}\} \\ &\qquad \cap \{1, b, b^2, b^3\} \\ &= \{1, b^2\}. \end{aligned}$$

Hence, $|Z(V_{8n})| = 2$. □

Theorem 1.3 *Let G be a non-abelian finite group. If $\Gamma_G \cong \Gamma_{V_{8n}}$, then $|G| = |V_{8n}|$.*

Proof First, we suppose that n is an even number. In this case $deg(a) = 4n$ and $deg(b) = 8(n-1)$. Since the $\Gamma_G \cong \Gamma_{V_{8n}}$, we have

$$|G| - |Z(G)| = |V_{8n}| - |Z(V_{8n}|.$$

Hence, $|Z(G)|$ divides $8n-4$. Also Γ_G has two vertices g_1 and g_2 such that $deg(g_1) = 4n$ and $deg(g_2) = 8n-8$. We know that $|Z(G)|$ divides $8n-8$, so $|Z(G)|$ divides 4. Therefore $|Z(G)|$ can be 1, 2 or 4.

If $|Z(G)| = 1$, then $|G| = 8n-3$ and $|C_G(g_2)| = 5$. Since $|C_G(g_2)|$ must divide $|G|$, so $5 \mid |G|$. It occurs only when $n = 1$ and it is impossible because n is an even number.

If $|Z(G)| = 2$, then $|G| = 8n-2$ and $|C_G(g_2)| = 6$. Since $|C_G(g_2)|$ must divide $|G|$, so $6 \mid |G|$. It occurs when $n = 1$ and it is impossible because n is an even number. Therefore $|Z(G)| = 4$ and $|G| = |V_{8n}| = 8n$. Now, suppose that n be an odd number. In this case,

$$|G| - |Z(G)| = 8n - 2$$

and $deg(g_1) = 4n$ and $deg(g_2) = 8n-4$. We have that $|Z(G)|$ divides $8n-2$ and $8n-4$. Thus $|Z(G)|$ divides 2. There is two cases for $|Z(G)|$. It can be 1 or 2.

If $|Z(G)| = 1$, then $|G| = 8n-1$ and $|C_G(g_2)| = 3$. However $3 \mid 8n-1$ only when $n = 2$ which is impossible since n is an odd number. Hence $|Z(G)| = 2$ and $|G| = |V_{8n}| = 8n$. □

1.5 CONCLUSION

In this research, we define three groups T_{4n}, U_{6n} and V_{8n} and show that if G is a non-abelian finite group such that

$$\Gamma_G \cong \Gamma_{T_{4n}}, \Gamma_G \cong \Gamma_{U_{6n}} \text{ or } \Gamma_G \cong \Gamma_{V_{8n}},$$

then

$$|G| = |T_{4n}| = 4n, |G| = |U_{6n}| = 6n \text{ or } |G| = |V_{8n}| = 8n,$$

respectively.

Acknowledgments

The authors would like to thank Universiti Teknologi Malaysia (UTM) for financial funding through the Research University Grant (RUG) Vote No. 08H07.

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