1

Non-commuting Graph of Some Nonabelian Finite Groups

Nor Haniza Sarmin, Maryam Jahandideh, and Mohammad Reza Darafsheh

1.1 INTRODUCTION

Let $G$ be a group and $Z(G)$ be its center. For each group $G$, we will associate a graph which is called the non-commuting graph of $G$, denoted by $\Gamma_G$. The vertex set $V(\Gamma_G)$ is $G - Z(G)$ and the edge set $E(\Gamma_G)$ consists of $\{x, y\}$, where $x$ and $y$ are two distinct vertices of $V(\Gamma_G)$ are joined together if and only if $xy \neq yx$. The non-commuting graph of a group was introduced by Erdos in 1975. The non-commuting graph of a finite group has been studied by many researchers [1].

One of the problems about non-commuting graph of groups is given in the following conjecture:

**Conjecture 1.1.** Let $G$ be a non-abelian finite group and $H$ a group such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.

**Definition 1.1** $T_{4n}$ is a non-abelian finite group with order $4n$. Its structure is defined as

$$T_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$ (1.1)

**Definition 1.2** $U_{6n}$ is a non-abelian finite group with order $6n$. Its
structure is defined as
\[
U_{6n} = \langle a, b | a^{2n} = 1 = b^3, a^{-1}ba = b^{-1} \rangle.
\] (1.2)

**Definition 1.3** \( V_{8n} \) is a non-abelian finite group with order \( 8n \). Its structure is defined as
\[
V_{8n} = \langle a, b | a^{2n} = 1 = b^4 = 1, ab = b^{-1}a^{-1}, \]
\[
ab^{-1} = ba^{-1} \).
\] (1.3)

The main objective of this chapter is to prove Conjecture 1.1 for three groups \( T_{4n} \), \( U_{6n} \) and \( V_{8n} \). In fact, we show that if \( \langle G \rangle \cong \Gamma_{T_{4n}}, \Gamma_{G} \cong \Gamma_{U_{6n}}, \Gamma_{G} \cong \Gamma_{V_{8n}} \), then \( |G| = |T_{4n}|, |G| = |U_{6n}| \) or \( |G| = |V_{8n}| \) respectively. For more details see Conway *et al.* [2] and Rose [3].

### 1.2 Non-Commuting Graph of \( T_{4n} \)

In this section, we show that if \( G \) is a non-abelian finite group such that \( \langle G \rangle \cong \Gamma_{T_{4n}} \), then \( |G| = |T_{4n}| \). In the lemmas, we refer the degree of the vertex \( x \), which is denoted by \( \text{deg}(x) \), as the number of edges through \( x \). We first state some lemmas which will be used throughout this section.

**Lemma 1.1** [4] *Let* \( G \) *be a non-abelian finite group and* \( x \) *is a vertex of* \( \Gamma_{G} \). *Then*
\[
\text{deg}(x) = |G| - |C_{G}(x)|.
\] (1.4)

**Lemma 1.2** [4] *Let* \( G \) *be a non-abelian finite group. If* \( H \) *is a group such that* \( \langle G \rangle \cong \Gamma_{H} \), *then* \( H \) *is a non-abelian finite group such that* \( |Z(H)| \) *divides each of the following:
\[
|G| - |Z(G)|, |G| - |C_{G}(x)|, |C_{G}(x)| - |Z(G)|, \text{ for } x \in (G - Z(G)).
\]

**Lemma 1.3** *Let* \( T_{4n} \) *be a group. Then
\[
|C_{T_{4n}}(a)| = 2n, |C_{T_{4n}}(b)| = 4 \text{ and } |Z(T_{4n})| = 2.
\]
Proof. All elements of $T_{4n}$ are denoted as $a^i b^j$ such that $1 \leq i \leq 2n, 1 \leq j \leq 4$. The center of $T_{4n}$ is defined by
\[
\{a^i b^j | (a^i b^j) a = a (a^i b^j), b (a^i b^j) = (a^i b^j) b, 1 \leq i \leq 2n, 1 \leq j \leq 4 \}.
\]

Now, we find the elements of $Z(T_{4n})$. If $a^i b^j$ belongs to $Z(T_{4n})$, then $a^i b^j a = a^{i+1} b^j$ and $a^i b^j + 1 = ba^j b^j$. Therefore we have $b^j a = ab^j$ and $ba^j = a^j b$. There exist three cases for $j$ as follows:
(a) If $j = 0$, then $ba^j = a^j b$. According to Definition 1.2, $a^j b = a^{-j} b$ and $i = n$. Hence $a^n \in Z(T_{4n})$.
(b) If $j \neq 0$, then $b^j a = a^{-j} b^j$ and $b^j a = ab^j$. Therefore the order of $a$ is 2, which is a contradiction.
(c) If $j = 2$, then $a^i b^2 = a^{i+2}$ and $a^i b^3 = a^{-i} b^3$. Hence $i = n$ and it shows that $a^n \in Z(T_{4n})$.

So $Z(T_{4n}) = a^n, 1$ and $|Z(T_{4n})| = 2$. We can see easily that $C_{T_{4n}}(a) = \{a\}$ and $C_{T_{4n}}(b) = \{b\}$. Therefore $|C_{T_{4n}}(a)| = 2n, |C_{T_{4n}}(b)| = 4$.

Theorem 1.1 Let $G$ be a finite non-abelian group. If $\Gamma_G \cong \Gamma_{T_{4n}}$, then $|G| = |T_{4n}|$.

Proof. We know that $\Gamma_{T_{4n}}$ has two vertices $a$ and $b$ such that $\text{deg}(a) = 2n$ and $\text{deg}(b) = 4n - 4$. Since $\Gamma_G \cong \Gamma_{T_{4n}}$, we have the following equality:
\[
|G| - |Z(G)| = |T_{4n}| - |Z(T_{4n})| = 4n - 2.
\]

Therefore $|Z(G)|$ divides $4n - 2$. There exists the corresponding elements $g_1, g_2 \in G - Z(G)$ such that $\text{deg}(g_1) = 2n$ and $\text{deg}(g_2) = 4n - 4$. By Lemma 1.2, we obtain that $|Z(G)|$ divides 2. Now, we show that $|Z(G)| = 2$. Using the contradiction proof, suppose that $|Z(G)| = 1$ and
\[
|G| = 4n - 1. \text{deg}(g_1) = |G| - |C_G(g_1)| = 2n.
\]

So $|C_G(g_1)| = 2n - 1$. But we know that $2n - 1$ does not divide $4n - 1$. Hence $|Z(G)| = 2$ and $|G| = 4n = |T_{4n}|$. 

\[ \square \]
1.3 NON-COMMUTING GRAPH OF $U_{6n}$

According to the definition of $U_{6n}$, we have all of its elements are in the form of $a^i b^j$ such that $0 \leq i \leq 2n - 1$ and $0 \leq j \leq 2$. To obtain our main goal, we start with the following lemma.

**Lemma 1.4** Let $U_{6n}$ be a finite group. Then

$$|C(a)| = 2n, \quad |C(b)| = 3 \quad \text{and} \quad |Z(U_{6n})| = 1.$$  

**Proof** First, we show that $Z(U_{6n}) = 1$. Suppose that there exist $i$ and $j$ such that $a^i b^j \in Z(U_{6n})$ and $i, j \neq 0$. Since $(a^i b^j)a = a(a^i b^j)$, we obtain $a^{i+1} b^{-j}$ and $j = 3$. Also we have $b(a^i b^j) = (a^i b^j)b$. Therefore $a^i b = ba^i = a^{i+1} b^{-1}$ and the order of $b$ is equal to 2. Hence we conclude that $Z(U_{6n}) = 1$. By the structure of $U_{6n}$, we can easily see that $C(a) = \langle a \rangle$ and $C(b) = \langle b \rangle$. Therefore $|C(a)| = 2n$ and $|C(b)| = 3$. □

**Theorem 1.2** Let $G$ be a finite non-abelian group. If $\Gamma_G \cong \Gamma_{U_{6n}}$, then $|G| = |U_{6n}|$.

**Proof** Since $\Gamma_G \cong \Gamma_{U_{6n}}$, it can concluded that $\Gamma_G$ has two vertices $g_1, g_2$ such that $\deg(g_1) = 4n$ and $\deg(g_2) = 6n - 3$. Also we have this equality $|G| - |Z(G)| = 6n - 1$.

Since $|Z(G)|$ divides $\deg(g_1)$ and $\deg(g_2)$, then there exists three cases for $|Z(G)|$ as follows:

(a) If $|Z(G)| = 2$, then $|G| = 6n + 1$ and $|C_G(g_2)| = 4$. This is impossible since $4 \nmid |G|$.

(b) If $|Z(G)| = 3$, then $|G| = 6n + 2$ and $|C_G(g_2)| = 5$. This is impossible since $|Z(G)| \nmid |C_G(g_2)|$.

(c) If $|Z(G)| = 6$, then $|G| = 6n + 5$ and $|C_G(g_2)| = 8$. This is impossible since $|Z(G)| \nmid |C_G(g_2)|$.

Therefore $|Z(G)| = 1$ and $|G| = |U_{6n}| = 6n$. □
1.4 NON-COMMUTING GRAPH OF $V_{8n}$

In this section, we study about $C_{V_{8n}}(a)$, $C_{V_{8n}}(b)$ and $Z(V_{8n})$. We want to show that if $\Gamma_G \cong \Gamma_{V_{8n}}$, then $|G| = |V_{8n}|$. First we start with the following lemma.

**Lemma 1.5** Let $V_{8n}$ be a finite group.
(a) If $n$ is an even number, then $|C_{V_{8n}}(b)| = 8$, $|C_{V_{8n}}(a)| = 4n$ and $|Z(V_{8n})| = 4$.
(b) If $n$ is an odd number, then $|C_{V_{8n}}(b)| = 4$, $|C_{V_{8n}}(a)| = 4n$ and $|Z(V_{8n})| = 2$.

**Proof** Firstly, we show that $|C_{V_{8n}}(a)| = 4n$. It can be shown that

$$|C_{V_{8n}}(a)| = \{a^ib^j \mid (a^ib^j)a \}$$

$$= a(a^ib^j) \equiv 0 \leq i \leq 2n-1, 0 \leq j \leq 3.$$

(1.5)

If $j = 0$, then $\langle a \rangle \leq C_{V_{8n}}(a)$. Assume that $i = 0$, we have $ab^2 = b^2a$. Now suppose that $i \neq 0$.

If $j = 1$, then $a^ib(a) = a^{-1}b^{-1}$ and $(a)a^ib = a^{i+1}b$. Since the order of $a$ is not equal to the order of $b$, we can conclude that $a(a^ib) \neq (a^ib)a$.

If $j = 2$, then $a^ib^2(a) = a^{i+1}b^2 = (a)a^ib^2$ for all $0 \leq i \leq 2n-1$.

If $j = 3$, then $a^ib^3(a) = a^{-1}b^{-3}$ and $(a)a^ib^3 = a^{i+1}b^3$. Since the order of $a$ is not equal to the order of $b$, we can conclude that $a(a^ib^3) \neq (a^ib^3)a$. Therefore, $|C_{V_{8n}}(a)| = 4n$.

Next, we want to obtain $|C_{V_{8n}}(b)|$, where $n$ is an even number. $C_{V_{8n}}(b) = \{a^ib^j \mid a^ib^{j+1} = ba^ib^j \}$ for all $0 \leq i \leq 2n-1$ and $0 \leq j \leq 3$. We know that $\langle b \rangle \leq C_{V_{8n}}(b)$. Suppose that $i \neq 0$, now we have four cases for $j$. If $j = 0$, then we recognize $a^i$ such that $a^ib = ba^i$ for all cases. Thus,

$$a^ib = ba^i \rightarrow a^{i-1}b^{-1}a^{-1} = ba^i \rightarrow a^{i-2}b = ba^{i+2} \rightarrow b^{(-1)^i} = ba^{2i}$$

The preceding equation shows that $i$ cannot be an odd number. Therefore $i$ is an even number and $i = n$. If $j = 1$, then
6 Recent Advances in Commutativity Degrees and Graphs of Groups

\(a^j b(b) \neq (b)a^j b\). For \(j = 2\) and \(j = 3\), we have

\[a^j b^2(b) \neq (b)a^j b^2\]

and \(a^j b^3(b) \neq (b)a^j b^3\) for all \(i\). Hence, \(|C_{V_{8n}}(b)| = 8\).

Also we have:

\[C_{V_{8n}}(a) = \{1, a, a^2, \ldots, a^{2n-1}, b^2, b^2a, b^2a^2, \ldots, b^2a^{2n-1}\}\]

and

\[C_{V_{8n}}(b) = \{1, b, b^2, b^3, a^n, ba^n, b^2a^n, b^3a^n\}\].

On the other hand, we know that

\[Z(V_{8n}) = \{g \in V_{8n}|gv = vg \text{ for all } v \in V_{8n}\} = \{g \in V_{8n}|ga = ag \text{ and } gb = bg\}\]

\[= C_{V_{8n}}(a) \cap C_{V_{8n}}(b) = \{1, b, a^n, b^2a^n\}\].

Therefore \(|Z(V_{8n})| = 4\).

If \(n\) is an odd number, according to the above proof we have four cases for \(j\). But in any case, we have \(a^j b^i(b) \neq (b)a^j b^i\) for all \(0 \leq j \leq 3\). Therefore \(C_{V_{8n}}(b) = \langle b \rangle\) and

\[Z(V_{8n}) = \{g \in V_{8n}|gv = vg \text{ for all } v \in V_{8n}\} = \{g \in V_{8n}|ga = ag \text{ and } gb = bg\}\]

\[= C_{V_{8n}}(a) \cap C_{V_{8n}}(b) = \{1, a, a^2, \ldots, a^{2n-1}, b^2, b^2a, b^2a^2, \ldots, b^2a^{2n-1}\}\]

\[\cap \{1, b, b^2, b^3\} = \{1, b^2\}.
\]

Hence, \(|Z(V_{8n})| = 2\).

\[\Box\]

**Theorem 1.3** Let \(G\) be a non-abelian finite group. If \(\Gamma_G \cong \Gamma_{V_{8n}}\), then \(|G| = |V_{8n}|\).
Proof  First, we suppose that $n$ is an even number. In this case $\deg(a) = 4n$ and $\deg(b) = 8(n-1)$. Since the $\Gamma_G \cong \Gamma_{V_{8n}}$, we have

$$|G| - |Z(G)| = |V_{8n}| - |Z(V_{8n})|.$$  

Hence, $|Z(G)|$ divides $8n-4$. Also $\Gamma_G$ has two vertices $g_1$ and $g_2$ such that $\deg(g_1) = 4n$ and $\deg(g_2) = 8n - 8$. We know that $|Z(G)|$ divides $8n-8$, so $|Z(G)|$ divides 4. Therefore $|Z(G)|$ can be 1, 2 or 4.

If $|Z(G)| = 1$, then $|G| = 8n - 3$ and $|C_{\Gamma_G}(g_2)| = 5$. Since $|C_{\Gamma_G}(g_2)|$ must divide $|G|$, so 5 $|G|$. It occurs only when $n = 1$ and it is impossible because $n$ is an even number.

If $|Z(G)| = 2$, then $|G| = 8n - 2$ and $|C_{\Gamma_G}(g_2)| = 6$. Since $|C_{\Gamma_G}(g_2)|$ must divide $|G|$, so 6 $|G|$. It occurs when $n = 1$ and it is impossible because $n$ is an even number. Therefore $|Z(G)| = 4$ and $|G| = |V_{8n}| = 8n$. Now, suppose that $n$ be an odd number. In this case,

$$|G| - |Z(G)| = 8n - 2$$

and $\deg(g_1) = 4n$ and $\deg(g_2) = 8n - 4$. We have that $|Z(G)|$ divides $8n-2$ and $8n-4$. Thus $|Z(G)|$ divides 2. There are two cases for $|Z(G)|$. It can be 1 or 2.

If $|Z(G)| = 1$, then $|G| = 8n - 1$ and $|C_{\Gamma_G}(g_2)| = 3$. However $3 | 8n - 1$ only when $n = 2$ which is impossible since $n$ is an odd number. Hence $|Z(G)| = 2$ and $|G| = |V_{8n}| = 8n$. \qed

1.5 CONCLUSION

In this research, we define three groups $T_{4n}$, $U_{6n}$ and $V_{8n}$ and show that if $G$ is a non-abelian finite group such that

$$\Gamma_G \cong \Gamma_{T_{4n}}, \Gamma_G \cong \Gamma_{U_{6n}} \text{ or } \Gamma_G \cong \Gamma_{V_{8n}},$$

then

$$|G| = |T_{4n}| = 4n, |G| = |U_{6n}| = 6n \text{ or } |G| = |V_{8n}| = 8n,$$

respectively.
Acknowledgments

The authors would like to thank Universiti Teknologi Malaysia (UTM) for financial funding through the Research University Grant (RUG) Vote No. 08H07.

REFERENCES


