2

On the Automorphism of Two-Generator p-Groups of Nilpotency Class Two

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2.1 INTRODUCTION

An automorphism on a group G is a homomorphism of G, which is one to one and onto. Recall that a homomorphism of G is a function f from G into itself that preserves the operation on G, that is $f(gh) = f(g)f(h)$ for every $g, h \in G$. The set of all automorphisms on G together with composition forms a group that is called the automorphism group of G, and denoted by $Aut(G)$.

Now, let G be a finite nonabelian 2-generated p-group of class two. In Chapter 1, some classifications of G have been stated. In this chapter, the latest version of the classifications given in Theorem 1.4 is applied to give two techniques to find the automorphisms on G. Some examples are provided to show how these methods work. Additionally, some properties of $Aut(G)$ are provided to help characterize it.

2.2 METHODS TO RECOGNIZE AUTOMORPHISMS ON FINITE 2-GENERATED p-GROUPS OF CLASS TWO

The group of exactly class two is usually considered nonabelian, more precisely, the derived subgroup is assumed nontrivial. If G is a finite nonabelian 2-generated p-group of class two, then according to Theorem 1.4, G is presented as follows:

$$\Big\langle a, b \mid [a,b]^{p^{\gamma}} = [a,b,a] = [a,b,b] = 1,\\ a^{p^{\alpha}} = [a,b]^{p^{\rho}}, b^{p^{\beta}} = [a,b]^{p^{\sigma}} \Big\rangle. \tag{2.1}$$

This presentation can be used to find the automorphisms on G. Recall that an automorphism is onto, so its image should span G. This fact is used to give the first technique for finding the automorphisms on G. This technique is stated in Theorem 2.1. To elaborate our results, the following proposition is applied which shows the multiplication of commutators in nilpotent groups of class two.

Proposition 2.1 [1] *Let H be a group of nilpotency class two. For any $x, y, z \in H$ and $n \in \mathbb{Z}$, the following equations hold:*
(a) $[x, yz] = [x, y][x, z]$;
(b) $[xy, z] = [x, z][y, z]$;
(c) $[x^n, y] = [x, y]^n = [x, y^n]$;
(d) $(xy)^n = x^n y^n [y, x]^{(n(n-1))/2}$.

Theorem 2.1 [2] First Method to Find Automorphisms on G
Let G be a finite 2-generated p-group of class two and f be a map of G to itself. Then f extends to an automorphism on G if and only if it satisfies in the following conditions:
(a) $G = \langle f(a), f(b)\rangle$;
(b) $[f(a), f(b)]^{p^{\gamma}} = [f(a), f(b), f(a)] = [f(a), f(b), f(b)] = 1$;
(c) $[f(a)]^{p^{\alpha}} = [f(a), f(b)]^{p^{\rho}}$;
(d) $[f(b)]^{p^{\beta}} = [f(a), f(b)]^{p^{\sigma}}$.

Sketch of Proof. If f is an automorphism on G, then it is onto and its image group is G. In other words, $f(a)$ and $f(b)$ are generators of G that satisfy in (2.1), considering the fact that (2.1) is a presentation of G. In converse, let f be a mapping on G into itself that satisfies in conditions (a) - (d). Since these conditions are written based on properties of G that are given in (2.1), it can be concluded that f extends to an automorphism on G.

The next example gives an application of the first method.

Example 2.1 Let G be a nonabelian group of order p^3. Then the following map

$$f : \begin{cases} a \longmapsto a^{p+1}, \\ b \longmapsto b \end{cases} \tag{2.2}$$

extends to an automorphism on G.

Solution According to the discussion given in Section 1.3, if $p = 2$, then $G \cong D_4$ or Q_8. However for both groups, $[a, b]^2 = 1$ and $a^2 = [a, b]$. Hence,

$$a^4 = 1, \quad [a, b]^{-1} = [a, b], \quad f(a) = a^{p+1} = a^3 = a^{-1}.$$

Thus,

$$\left\langle [f(a)]^{-1}, f(b) \right\rangle = \langle a, b \rangle = G.$$

Moreover,

$$[f(a), f(b)] = [a^{-1}, b] = [a, b]^{-1} = [a, b] \in G' = Z(G).$$

Therefore,

$$|[f(a), f(b)]| = |[a, b]|,$$
$$[f(a), f(b), f(a)] = [f(a), f(b), f(b)] = 1.$$

Finally,

$$|[f(b)]| = |b|,$$
$$[f(a)]^2 = (a^3)^2 = a^4a^2 = a^2 = [a, b] = [f(a), f(b)].$$

Hence, f satisfies conditions (a) – (d) of Theorem 2.1.

Now, let $p > 2$. If G is of the form (1.3), then

$$a^p = b^p = [a, b]^p = 1.$$

Hence, $f(a) = a^{p+1} = a$ or f is the identity map, which obviously extends to identity automorphism on G. On the other hand, if G is presented by (1.4), then $(\alpha, \beta, \gamma; \rho, \sigma) = (1, 1, 1; 0, 1)$, $a^{p^2} = b^p = [a, b]^p = 1$ and $a^p = [a, b]$. Hence, we find

$$a = a^{(p^3+1)} = [a^{p+1}]^{(p^2-p+1)} = [f(a)]^{(p^2-p+1)}.$$

This leads to $G = \left\langle [f(a)]^{(p^2-p+1)}, f(b) \right\rangle$. In addition, we have

$$[f(a), f(b)] = [a^{p+1}, b] = [a, b]^{p+1} = [a, b] \in G' \leq Z(G).$$

This implies that f satisfies all relations that are given in condition (b) of Theorem 2.1. The next statement completes the solution:

$$\begin{aligned}[f(a)]^{p^\alpha} &= [f(a)]^p = [a^{p+1}]^p \\ &= a^{p^2+p} = a^p = [a, b] \\ &= [f(a), f(b)] = [f(a), f(b)]^{p^\rho}.\end{aligned}$$

Next, the Frattini subgroup and some of its properties are applied to find the second method, as provided in the following:

Definition 2.1 [3] Let H be an arbitrary group. A non-generator element of H is an element that could be removed from any generating set. The set of all non-generators of H forms a normal subgroup that is called Frattini subgroup, and denoted by $\Phi(H)$. Indeed,

$$H = \langle \Phi(H), x_1, x_2, \ldots, x_n \rangle$$

if and only if

$$H = \langle x_1, x_2, \ldots, x_n \rangle .$$

Proposition 2.2 [3] *Let H be a p-group. Then*

(a) $H' \le \Phi(H)$;
(b) $h^p \in \Phi(H)$ *for all* $h \in H$.

In our study, G is considered a finite 2-generated p-group of nilpotency class two. Let $\{a, b\}$ spans G. Then by Proposition 2.2, $a^p, b^p \in \Phi(G)$. Hence, $|a\Phi(G)| = |b\Phi(G)| = p$, and so then

$$G/\Phi(G) = \langle a\Phi(G), b\Phi(G)\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

is of order p^2. The automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$ is isomorphic to $GL(2, p)$ the general linear group of degree two. Moreover, $|GL(2, p)| = (p^2 - 1)(p^2 - p)$ [1]. It is known that $GL(2, p)$ consists of all nonsingular 2×2 matrices. i.e.

$$GL(2, p) \cong \left\{ \begin{pmatrix} k & l \\ m & n \end{pmatrix} : k, l, m, n \in \mathbb{Z}_p, kn - lm \not\equiv 0 \pmod{p} \right\}.$$

This fact and the following proposition are used in the proof of method 2, which is given in [4].

Proposition 2.3 [4] *Let $G = \langle a, b\rangle$ be a finite 2-generated group of class two. Then every element $g \in G$ is of the form $g = a^{x_1} b^{x_2} [a, b]^{x_3}$ where x_i's are positive integers such that $x_1 \le |a|$, $x_2 \le |b|$ and $x_3 < |[a, b]|$. Moreover, we have $ba^k = a^k b[a, b]^{-k}$, for any integer k.*

Theorem 2.2 [4] Second Method to Find Automorphisms on G
Let G be a 2-generated group of class two and order p^n that corresponded to $(\alpha, \beta, \gamma; \rho, \sigma)$. Let x_i and y_i be nonnegative integers for $i = 1, 2, 3$ such that $0 \le x_1, y_1 < |a|$, $0 \le x_2, y_2 < |b|$ and $0 \le x_3, y_3 < |[a, b]|$. Then the map defined as:

$$f : \begin{cases} a \longmapsto a^{x_1} b^{x_2} [a, b]^{x_3}, \\ b \longmapsto a^{y_1} b^{y_2} [a, b]^{y_3} \end{cases}$$

can be extended to a unique automorphism on G if and only if the following conditions hold:

(a) $d(f) := det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \not\equiv 0 \pmod{p}$;
(b) $[f(a), f(b)]^{p^\gamma} = [f(a), f(b), f(a)]$
$= [f(a), f(b), f(b)] = 1$;
(c) $[f(a)]^{p^\alpha} = [f(a), f(b)]^{p^\rho}$;
(d) $[f(b)]^{p^\beta} = [f(a), f(b)]^{p^\sigma}$.

An example of the usage of the second method can be found in [5], where all automorphisms on D_4 were found by applying Method 2. As another example of the application of Method 2, one can consider the map f given in (2.2). Clearly, conditions (b) - (d) in both methods are similar. Thus, it is enough to show that the map f given in (2.2) satisfies in condition (a) of Method 2. However,

$$d(f) = det \begin{pmatrix} p+1 & 0 \\ 0 & 1 \end{pmatrix} = p + 1 \equiv 1 \not\equiv 0 \pmod{p}.$$

The second method enables us to find an upper bound for the order of $Aut(G)$ in our case. To achieve this goal, the following proposition that reveals the orders of a and b in G is used.

Proposition 2.4 [4] *Let $G = \langle a, b\rangle$ be a nilpotent group of class two and order p^n that is corresponded to $(\alpha, \beta, \gamma; \rho, \sigma)$. Then*

$$|a| = p^{\alpha+\gamma-\rho}, \quad |b| = p^{\beta+\gamma-\sigma}, \quad |[a, b]| = p^\gamma.$$

Lemma 2.1 *Let $G = \langle a, b\rangle$ be a nilpotent group of class two and order p^n that is corresponded to $(\alpha, \beta, \gamma; \rho, \sigma)$. Then*

$$|Aut(G)| \leq p^{2[(n+(\gamma-\rho)+(\gamma-\sigma)]}.$$

Proof According to Theorem 2.2, we have

$$|Aut(G)| \leq |a|^2|b|^2|[a, b]|^2.$$

Proposition 2.4 and the fact that $\alpha + \beta + \gamma = n$, which is excerpted from Theorem 1.4, imply that:

$$\begin{aligned} |Aut(G)| &\leq p^{2(\alpha+\gamma-\rho)}p^{2(\beta+\gamma-\sigma)}p^{2\gamma} \\ &= p^{[2(\alpha+\beta+\gamma)+2(2\gamma-\rho-\sigma)]} \\ &= p^{2[(n+(\gamma-\rho)+(\gamma-\sigma)]}. \end{aligned}$$

In the next section, an idea on the characterization of $Aut(G)$, where G is a finite 2-generated p-group of nilpotency class two is discussed.

2.3 CHARACTERIZATION OF AUTOMORPHISMS ON FINITE TWO-GENERATED *p*-GROUPS OF CLASS TWO

To proceed our discussion, another concept is needed. In [4], $A_\Phi(G)$ is introduced as the following:

Definition 2.2 [4] Let $G = \langle a, b\rangle$ be a nilpotent p-group of class two. Then $A_\Phi(G)$ is defined to be the set consisting of all those elements f in $Aut(G)$ that induce the identity automorphism on $G/\Phi(G)$. i.e. $\bar{f} \in A_\Phi(G)$ if and only if for each $g\Phi(G) \in G/\Phi(G)$, we have:

$$\bar{f}(g\Phi(G)) = f(g)\Phi(G) = \Phi(G) = \mathbf{i}_{G/\Phi(G)}.$$

Proposition 2.5 [4] *Let G be a finite 2-generated p-group of nilpotency class two. Then $A_\Phi(G)$ is a normal subgroup of $Aut(G)$.*

The following lemma explains the reason of our interest in $A_\Phi(G)$. In fact, it shows that how $A_\Phi(G)$ can be used to study and characterize $Aut(G)$.

Theorem 2.3 *Let G be a finite 2-generated p-group of nilpotency class two. Then $Aut(G)/A_\Phi(G)$ is isomorphic to a subgroup of $GL(2, p)$.*

Proof Consider the following map:

$$\begin{cases} \tau : Aut(G) \to Aut(G/\Phi(G)), \\ \tau(f) = \bar{f}, \end{cases}$$

where, for every $\bar{f} \in Aut(G/\Phi(G))$ we have $\bar{f}(g\Phi(G)) = f(g)\Phi(G)$. We need to prove that τ is a homomorphism and $A_\Phi(G)$ is its kernel. In other words, $A_\Phi(G) = \tau^{-1}(\{\mathbf{i}_{G/\Phi(G)}\})$.

To show that τ is well-defined, consider $f_1 = f_2$ in $Aut(G)$. Then, $f_1(g) = f_2(g)$ for every $g \in G$. Hence, $f_1(g)\Phi(G) = f_2(g)\Phi(G)$, or $\tau(f_1) = \bar{f_1} = \bar{f_2} = \tau(f_2)$, which prove that τ is well-defined. Furthermore, τ is a homomorphism since the following relations hold:

$$\begin{aligned}\tau(f_1 f_2)(g\Phi(G)) &= \overline{f_1 f_2}(g\Phi(G)) \\ &= [f_1 f_2(g)]\Phi(G) \\ &= [f_1(f_2(g))]\Phi(G) \\ &= \bar{f_1}(f_2(g)\Phi(G)) \\ &= \bar{f_1}(\bar{f_2}(g\Phi(G))) \\ &= [\bar{f_1}\bar{f_2}](g\Phi(G)) \\ &= \tau(f_1)\tau(f_2)(g\Phi(G)).\end{aligned}$$

Therefore, according to the first theorem of isomorphism $Aut(G)/Ker(\tau) \cong Im(\tau)$. Recall that

$$Im(\tau) \leq Aut(G/\Phi(G)) \cong GL(2, p).$$

In other words, $Im(\tau)$ is isomorphic to a subgroup of $GL(2, p)$. Hence, $Im(\tau)$ is almost known, and it remains to characterize $Ker(\tau)$. The following relations show that $Ker(\tau) = A_\Phi(G)$.

$$\begin{aligned}Ker(\tau) &= \{f \in Aut(G) : \tau(f) = \mathbf{i}_{G/\Phi(G)}\} \\ &= \{f \in Aut(G) : \bar{f} = \mathbf{i}_{G/\Phi(G)}\} \\ &= A_\Phi(G).\end{aligned}$$

This shows the importance of $A_\Phi(G)$. □

2.4 CONCLUSION

Let G be a 2-generated group of nilpotency class two and order p^n. In this chapter, firstly two methods with examples have been stated to recognize the automorphisms on G among the maps that can be defined from G to itself. Next, a technique that can be applied to characterize and study $Aut(G)$ is provided. This technique is based on introducing a specific normal subgroup of $Aut(G)$ called $A_\Phi(G)$ and its quotient group.

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