# 2

# Precise Value of the Orbit Graph and Conjugacy Class Graph of Some Finite 2-Groups

Nor Haniza Sarmin, Sanaa Mohamed Saleh Omer, and Ahmad Erfanian

## 2.1 INTRODUCTION

Orbit graph is considered as one of the generalizations of conjugate graph that was introduced by Erfanian and Tolue [1]. The first generalization of conjugate graph was done by Omer *et al.* who introduced the so called orbit graph. An orbit graph is a graph whose vertices are non-central orbits under some group actions on a set in which two vertices are adjacent if they are conjugate. In this chapter, we find the orbit graph for some finite 2-groups when a group acts on itself by conjugation.

Throughout this chapter, $\Gamma$ denotes a simple undirected graph, $G$ denotes a 2-group and $\Gamma_G^c$ denotes an orbit graph where the group acts on itself by conjugation.

Basic concepts and definitions of graph theory that are needed are stated in the following :

**Definition 2.1 [2]** A graph $\Gamma$ consists of two sets, namely vertices $V(\Gamma)$ and edges $E(\Gamma)$ together with relation of incidence.

The **directed** graph is a graph whose edges are identified with ordered pair of vertices. Otherwise, $\Gamma$ is called **indirected**. Two vertices are **adjacent** if they are joined by an edge.

**Definition 2.2 [2]** A graph is called a **subgraph** $\Gamma_{sub}$ of $\Gamma$ if its vertices and edges are subsets of the vertices and edges of $\Gamma$.

**Definition 2.3 [3]** A **complete** graph $K_n$ is a graph where each ordered pair of distinct vertices are adjacent.

**Definition 2.4 [3]** The graph $\Gamma$ is an **empty graph**, if there is no adjacent (edges) between its vertices. In this chapter, $K_e$ denotes the empty graph.

**Definition 2.5 [2]** The graph $\Gamma$ is called **null** if it has no vertices, denoted by $K_0$

**Definition 2.6 [3]** A **regular** graph is a graph whose all vertices have the same sizes.

**Definition 2.7 [2]** A **line** graph $L(\Gamma)$ of $\Gamma$ is a graph with the edges of $\Gamma$ as its vertices.

**Definition 2.8 [2]** A vertex is **incident** with an edge if it is one of the two vertices of the edge.

Note that two edges of $\Gamma$ are linked in $L(\Gamma)$ if and only if they are incident in $\Gamma$.

**Definition 2.9 [3]** The **valency** of a vertex $x$ is the number of neighbors of $x$.

**Proposition 2.1** *If $\Gamma$ is a simple graph with $v \geq 3$ and $deg(v) \geq (|V(\Gamma)|)/2$, then $\Gamma$ is Hamiltonian.*

The following proposition is used to determine the degree of vertex in $\Gamma$.

**Proposition 2.2 [2]** *Let $G$ be a finite group and $\Gamma$ be its graph. Let $V(\Gamma)$ be the set of vertices in $\Gamma$. If $v \in V(\Gamma)$, then the degree of $v$ is deg($v$)=$\sum_{v \in V)\Gamma)} d(v) = 2E$, where $E$ is the number of edges in $\Gamma$.*

The **independent set** is a non-empty set $B$ of V($\Gamma$), where there is no adjacent between two elements of $B$ in $\Gamma$, while the **independent number** is the number of vertices in maximum independent set and it is denoted by $\alpha(\Gamma)$. However, the maximum number $m$ for which $\Gamma$ is $m$-vertex colored is known as **chromatic number** and denoted by $\chi(\Gamma)$. The maximum distance between any two vertices of $\Gamma$ is called the **diameter** and is denoted by $d(\Gamma)$. In addition, a **clique** $\omega(\Gamma)$ is a complete subgraph in $\Gamma$, while the **clique number** is the size of the largest clique in $\Gamma$. The **dominating set** $X \subseteq V(\Gamma)$ is a set where for each $v$ outside $X$, there exists $x \in X$ such that $v$ is adjacent to $x$. The **dominating number** is the minimum size of $X$ and is denoted by $\gamma(\Gamma)$.

**Definition 2.10 [5]** Let $G$ be a finite group, and let $g_1, g_2$ be elements in $G$. The elements $g_1, g_2$ are said to be conjugate if there are some $h$ in $G$ such that

$$g_2 = hg_1h^{-1}.$$

The set of all conjugates of $g_1$ is called the conjugacy classes of $g_1$.

In this chapter, we find the orbit graph of some finite non-abelian 2-groups. The presentations of 2-groups are stated in the following.

**Definition 2.11 [6]** If $G$ is a finite 2-group of order $2n$, then $G$ has the following presentation

$$G \cong \langle a, b : a^n = b^2 = 1, (ab)^2 = 1\rangle.$$

**Definition 2.12 [7]** Let $n$ be an integer greater than two. If $G$ is a finite non-abelian group of order $2^{n+1}$, then

$$G \cong \langle G \cong \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1}\rangle.$$

**Definition 2.13 [7]** Let $G$ be a finite non-abelian 2-group of order $2^{n+1}$, where $n \geq 3$. Thus, $G$ has a presentation

$$G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1}\rangle.$$

**Definition 2.14 [8]** If $G$ is a finite non-abelian 2-group group of order $2^{\beta+1}$, where $\beta \geq 3$, then $G$ has the following presentation

$$G \cong \langle a, b : a^{2^{\beta}} = b^2 = e, ab = ba^{2^{\beta-1}+1} \rangle.$$

Commutativity degree is the term used to determine the abelianness of groups. This concept was firstly introduced by Miller in 1944 [9]. The idea of the commutativity degree was then investigated for symmetric groups by Erdös and Turan [10]. The results obtained encourage many researchers to work on this topic which has initiated various generalizations. One of these generalizations called the probability that a group element fixes a set introduced by Omer *et al.* [11]. The followings are some results on the probability that a group element fixes a set, needed in this chapter.

**Theorem 2.1 [11]** *Let $G$ be a finite 2-group, $G \cong \langle a, b : a^n = b^2 = 1, (ab)^2 = 1 \rangle$, where $n \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) = 2$. Let $\Omega$ be the set of all subsets of commuting elements of size two. If $G$ acts on $\Omega$ by conjugation, then*

$$P_G(\Omega) = \begin{cases} \frac{6}{m+1}, \text{ if } n \text{ is even, } \frac{n}{2} \text{ is odd and } m = \frac{5n}{2}, \\ \frac{7}{m+1}, \text{ if } n \text{ is even, } \frac{n}{2} \text{ is even and } m = \frac{5n}{2}, \\ \frac{1}{n}, \quad \text{ if } n \text{ is odd.} \end{cases}$$

**Theorem 2.2 [11]** *Let $G$ be finite 2-group $G \cong \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) = 2$. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.*

**Theorem 2.3 [12]** *Let $G$ be a finite non-Abelian group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$, where $n \geq 3$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$*

*and b commute and lcm*$(|a|, |b|) = 2$*. Let* $\Omega$ *be the set of all subsets of commuting elements of G of size two and G acts on* $\Omega$ *by conjugation. Then* $P_G(\Omega) = \frac{4}{2^n+2^{n-2}+1}$.

**Theorem 2.4 [12]** *Let G be a finite non-Abelian 2-group,* $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1}\rangle$*, where* $n \geq 3$*. Let S be a set of elements of G of size two in the form of* $(a, b)$ *where a and b commute and lcm*$(|a|, |b|) = 2$*. Let* $\Omega$ *be the set of all subsets of commuting elements of G of size two and G acts on* $\Omega$ *by conjugation. Then* $P_G(\Omega) = \frac{2}{3}$.

## 2.2 GRAPH THEORY

This section provides some works that are related to graph theory. Non-commuting graph is a concept that was introduced in 1975 [13], where many recent publication have been done using this graph. The definition of a non-commuting graph is stated in the following.

**Definition 2.15 [13]** Let $G$ be a finite non-abelian group with the center denoted by $Z(G)$. A non-commuting graph is a graph whose vertices are non central elements of $G$ (i.e $G - Z(G)$). Two vertices $v_1$ and $v_2$ are adjacent whenever $v_1v_2 \neq v_2v_1$.

In [13], it is mentioned that this concept was firstly introduced by Paul Erdos. Erdos posed a question if there is a finite bound on the cardinalities of cliques of $\Gamma$. The first conformed of Erdos's question was by Neumann [13]. According to Neumann [13] there is a finite complete subgraph in some groups. Furthermore, Abdollahi *et al.* [14] emphasized the existence of finite bound on the cardinalities of complete subgraph in $\Gamma$. Abdollahi *et al.* [14] used the graph theoretical concepts to investigate the algebraic properties of the graph.

In 1990, Bertram *et al.* [15] introduced a graph which is called the **conjugacy class graph**. The vertices of this graph are non-central conjugacy classes, where two vertices are adjacent if the cardinalities are not coprime. The following is definition definition of conjugacy class graph.

**Definition 2.16 [15]** Let $G$ be a finite group. The conjugacy class graph denoted in this chapter by $\Gamma_G^c$, is a graph whose vertices are non-central conjugacy classes i.e the number of vertices of conjugacy class graph

$$|V(\Gamma_G^c)| = K(G) - Z(G),$$

where $K(G)$ denotes the number of conjugacy classes and $Z(G)$ denotes the center of $G$. Two vertices are adjacent if the cardinalities are not coprime.

As a consequence, numerous works have been done on this graph and many results have been achieved [16–18].

In [19], Berge conjectured that the graph is a perfect graph if and only if the chromatic number and clique number are identical and there is no induced subgraph if the graph is an odd cycle of length greater than three. Berge's conjectured then became known as the strong perfect graph conjecture. Then, various studies have been done on perfect graph [20, 21].

Recently, Bianchi *et al.* [22] studied the regularity of conjugacy class graph and provided some results.

Later, Erfanian and Tolue [1] introduced a new graph which is called a **conjugate** graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate.

In 2013, Omer *et al.* [23] introduced a new graph called the orbit graph whose vertices a non-central orbit under group action on a set. The following is definition of the orbit graph.

**Definition 2.17 [23]** Let $G$ be a finite group and $\Omega$ be a set of elements of $G$. Let $A$ be the set of commuting elements in $\Omega$, i.e $A = \{v \in \Omega : vg = gv, g \in G\}$. The orbit graph $\Gamma_G^\Omega$ consists of two sets, namely vertices and edges denoted by $V(\Gamma_G^\Omega)$ and $E(\Gamma_G^\Omega)$, respectively. The vertices of $\Gamma_G^{\Omega}$ are non central elements in $\Omega$ but not in $A$, that is $V(\Gamma_G^{\Omega}) = \Omega - A$, while the number of edges are $|E(\Gamma_G^{\Omega})| = \sum_{i=1}^{|V(\Gamma_G^{\Omega})|} \binom{v_i}{2}$, where $v$ is the size of orbit under group action of $G$ on $\Omega$. Two vertices $v_1, v_2$ are adjacent in $\Gamma_G^{\Omega}$ if one of the following conditions is satisfied.

(a) If there exists $g \in G$ such that $gv_1 = v_2$,
(b) If the vertices of $\Gamma_G^{\Omega}$ are conjugate that is, $v_1 = g^{v_2}$.

Some graph properties are investigated as follows:

**Proposition 2.3** *[24] Let G be a finite non-Abelian group and let $\Omega$ be a set. If G acts on $\Omega$, then the properties of the orbit graph $\Gamma_G^{\Omega}$ are described as follows.*

(a) $\chi(\Gamma_G^{\Omega}) = min\{|cl(v_i)|, v_i \in \Omega\}$,
(b) $\omega(\Gamma_G^{\Omega}) = max\{|cl(v_i)|, v_i \in \Omega\}$,
(c) $\alpha(\Gamma_G^{\Omega}) = K(\Omega) - |A|$,
(d) $\gamma(\Gamma_G^{\Omega}) = K(\Omega) - |A|$,
(e) $d(\Gamma_G^{\Omega}) = max\{d(v,u) : \forall\ v, u \in V(\Gamma_G^{\Omega})\}$.

## 2.3 RESULTS AND DISCUSSION

This section consists of two parts. In the first part, we find the orbit graph for some finite 2-groups, while in the second part, we compute the conjugacy class graph for the mentioned groups.

### 2.3.1 Orbit Graph

In this section, we introduce our results on the orbit graph of some finite 2-groups.

**Theorem 2.5** *Let G be a finite 2-group, $G \cong \langle a, b : a^n = b^2 = 1, (ab)^2 = 1\rangle$. Let S be a set of elements of G of size two in the form of $(a, b)$ where a and b commute and $lcm(|a|, |b|) = 2$. Let $\Omega$*

*be the set of all subsets of commuting elements of G of size two. If G acts on* $\Omega$ *by conjugation, then*

$$\Gamma_G^{\Omega} = \begin{cases} \cup_{i=1}^{5} K_{\frac{n}{2}i}, & n \text{ is even and } \frac{n}{2} \text{ is odd,} \\ (\cup_{i=1}^{4} K_{\frac{n}{2}i}) \cup (\cup_{i=1}^{2} K_{\frac{n}{4}i}), & n \text{ and } \frac{n}{2} \text{ are even,} \\ K_n, & n \text{ is odd.} \end{cases}$$

**Proof** Based on Theorem 2.1, the number of elements in $\Omega$ is $\frac{5n}{2}+1$ and using Definition 2.17, the number of vertices of $\Gamma_G^{\Omega}$ is $\frac{5n}{2}$. Two vertices $\omega_1$ and $\omega_2$ are adjacent if $\omega_1 = \omega_2{}^g$. First, when $n$ is even and $\frac{n}{2}$ is odd. there are five complete components of $K_{\frac{n}{2}}$, since there are six orbits five of them are of size $\frac{n}{2}$. Second, when $n$ and $\frac{n}{2}$ are even. The orbit graph $\Gamma_G^{\Omega}$ consists of four complete components of $K_{\frac{n}{2}}$ and two complete components of $K_{\frac{n}{4}}$, represented to four orbits of size $\frac{n}{2}$ and two orbits are of size $\frac{n}{4}$. However, when $n$ is odd there is only orbit of size $n$, hence $\Gamma_G^{\Omega}$ consists of one complete component of $K_n$. □

**Remark** The orbit graph of Theorem 2.5 is a complete graph graph when $n$ is odd since there is one complete graph of $n$ connected vertices.

**Theorem 2.6** *Let G be a finite 2-group,* $G \cong \langle a,b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1}$. *Let S be a set of elements of G of size two in the form of* $(a,b)$ *where a and b commute and* $lcm(|a|,|b|) = 2$. *Let* $\Omega$ *be the set of all subsets of commuting elements of G of size two. If G acts on* $\Omega$ *by conjugation, then* $\Gamma_G^{\Omega}$ *is a null graph.*

**Proof** According to Theorem 2.2, there is only one orbit of size one. By Definition 2.17, the number of vertices is zero since $|\Omega| = |A|$. Hence, $\Gamma_G^{\Omega}$ is a null graph. □

**Theorem 2.7** *Let G be a finite group,* $G \cong \langle a,b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1}\rangle$. *Let S be a set of elements of G of size two in the form of* $(a,b)$ *where a and b commute and* $lcm(|a|,|b|) = 2$. *Let* $\Omega$

*be the set of all subsets of commuting elements of $G$ of size two. If $G$ acts on $\Omega$ by conjugation, then $\Gamma_G^{\Omega} = K_{2^{n-1}} \cup K_{2^{n-1}} \cup K_{2^{n-2}}$.*

**Proof** Suppose that $G$ acts on $\Omega$ by conjugation. The number of vertices of $\Gamma_G^{\Omega}$ is $|V(\Gamma_G^{\Omega})| = |\Omega| - |A|$. According to Theorem 2.3, the number of elements in $\Omega$ is $2^n + 2^{n-2} + 1$ and the number of elements in $A$ is one. Thus, the number of vertices in $\Gamma_G^{\Omega}$ is $2^n + 2^{n-2}$. Two vertices of $\Gamma_G^{\Omega}$ are linked if they are conjugate. Therefore, the adjacent vertices are described as follows: The vertex of the form $(1, a^i b)$ is adjacent to all vertices of the form $(1, a^j b), 0 \leq i, j \leq 2^n$ where $i$ and $j$ are even and $i \neq j$. Thus, there is one complete component of $K_{2^{n-1}}$. In addition, all the vertices of the form $(a^{2^{n-1}}, a^i b)$ are adjacent to the vertices $(a^{2^{n-1}}, a^j b), 0 \leq i \leq 2^n, i \neq j$, where $i$ and $j$ are even. Hence, there is one complete component of $K_{2^{n-1}}$. The vertices in the form $(a^i b, a^{2^{n-1}+i} b)$, where $i$ is even, are connected to each others. It follows that, there is only one complete components of $K_{2^{n-2}}$. Therefore, $\Gamma_G^{\Omega}$ contains three complete components, as required. □

In the next corollary, the chromatic number, clique number, independent number and dominating number of the graph in Theorem 2.7 are found.

**Corollary 2.1** *Let $G$ be a finite group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) = 2$. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two. If $G$ acts on $\Omega$ by conjugation and $\Gamma_G^{\Omega} = K_{2^{n-1}} \cup K_{2^{n-1}} \cup K_{2^{n-2}}$, then $\alpha(\Gamma_G^{\Omega})| = \gamma(\Gamma_G^{\Omega}) = 3$ and $\chi(\Gamma_G^{\Omega}) = \omega(\Gamma_G^{\Omega}) = 2^{n-1}$.*

**Proof** Based on Theorem 2.7, there are three orbits and according to Proposition 2.3, $\alpha(\Gamma_G^{\Omega})| = \gamma(\Gamma_G^{\Omega}) = 3$. □

Next, the orbit graph of the following 2-group is found.

**Theorem 2.8** *Let $G$ be a finite group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) = 2$. Let $\Omega$*

*be the set of all subsets of commuting elements of G of size two. If G acts on* $\Omega$ *by conjugation, then* $\Gamma_G^{\Omega} = K_2 \cup K_2$.

**Proof** Based on Theorem 2.4, the number of vertices, $|V(\Gamma_G^{\Omega})| = 5$. Since there are two orbits of size two, thus $\Gamma_G^{\Omega}$ consists of two complete components of $K_2$. The proof then follows. □

**Corollary 2.2** *Let G be a finite group,* $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1}\rangle$. *Let S be a set of elements of G of size two in the form of* $(a, b)$ *where a and b commute and* $lcm(|a|, |b|) = 2$. *Let* $\Omega$ *be the set of all subsets of commuting elements of G of size two. If G acts on* $\Omega$ *by conjugation and* $\Gamma_G^{\Omega} = K_2 \cup K_2$, *then* $\chi(\Gamma_G^{\Omega}) = \omega(\Gamma_G^{\Omega}) = 2$.

**Proof** Referring to Proposition 2.3, the result follows.

### 2.3.2 Conjugacy Class Graph

In this part, we find the conjugacy class graph for 2-groups mentioned earlier.

**Theorem 2.9** *Let G be a finite non-abelian 2-group, where*

$$G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1}\rangle.$$

*If G acts on itself by conjugation, then* $|V(\Gamma_G^c)| = 2^{n-1} + 1$ *and* $|E(\Gamma_G^c)| = 2^{2n-3} + 2^{n-2}$.

**Proof** According to Definition 2.16, the number of vertices is Hence, $K(G) = 2^{n-1} + 3$ and $|Z(G)| = 2$. It follows that $|V(\Gamma_G^c)| = 2^{n-1} + 1$. In accordance with Proposition 2.2, the degree of any vertex of $\Gamma_G^c$ is $deg(v) = |V(\Gamma_G^c)| - 1$. Thus, $deg(v) = 2^{n-1}$ and the degree of $\Gamma_G^c$ is

$$d(\Gamma_G^c) = \sum_{i=1}^{|V(\Gamma_G^c)|} deg(v_i) = 2|E(\Gamma_G^c)|.$$

It follows that, $d(\Gamma_G^c) = \sum_{i=1}^{2^{n-1}+1} 2^{n-1} = 2|E(\Gamma_G^c)|$. Therefore, $|E(\Gamma_G^c)| = 2^{2n-3} + 2^{n-2}$. □

**Example 2.1** Assume $G$ is a finite non-abelian 2-group, where

$$G \cong \langle a, b : a^{2^3} = b^2 = e, ab = ba^3 \rangle.$$

If $G$ acts on itself by conjugation, then $\Gamma_G^c = K_5$.

**Solution** In accordance with Theorem 2.9, the number of vertices in $\Gamma_G^c$ is five. Based on Proposition 2.2, the number of edges is

$$d(\Gamma_G^c) = \sum_{i=1}^{|V(\Gamma_G^c)|} deg(v_i) = 2|E(\Gamma_G^c)|.$$

It follows that, $|E(\Gamma_G^c)| = 10$. Thus, the graph here is a complete graph of $K_5$.

**Corollary 2.3** *Let $G$ be a finite non-abelian 2-group, where*

$$G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle.$$

*If $G$ acts on itself by conjugation, then the graph is Hamiltonian.*

**Proof** The proof follows from Theorem 2.9 and Proposition 2.1.□

**Theorem 2.10** *Let $G$ be a 2-group, where*

$$G \cong \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle.$$

*If $G$ acts on itself by conjugation, then $|V(\Gamma_G^c)| = 2^{n-2} + 1$ and $|E(\Gamma_G^c)| = 22^{n-5} + 2^{n-3}$.*

**Proof** According to Definition 2.16, the number of vertices of $\Gamma_G^c$ is $|V(\Gamma_G^c)| = K(G) - Z(G)$, from which it follows that $|V(\Gamma_G^c)| = 2^{n-2} + 1$. However, the degree of any vertex $v$ in $\Gamma_G^c$ is $deg(v) = |V(\Gamma_G^c)| - 1$, thus $deg(v) = 2^{n-2}$. The degree of $\Gamma_G^v$ is

$$2|E(\Gamma_G^c)| = \sum_{i=1}^{|V(\Gamma_G^c)|} deg(v_i).$$

Therefore, $|E(\Gamma_G^c)| = 2^{2n-5} + 2^{n-3}$. □

**Example 2.2** Suppose $G$ is a 2-group, where

$$G \cong \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle.$$

If $G$ acts on itself by conjugation, then $\Gamma_G^c = K_3$.

**Solution** Based on Theorem 2.10, the number of vertices is three and referring to Proposition 2.2, the number of edges is three. Therefore, the graph is a complete graph of $K_3$.

According to Theorem 2.10, the following remark can be stated.

**Remark** The graph in Theorem 2.10 is a Hamiltonian graph, since the degree of any vertex is at least half of the number of vertices.

**Theorem 2.11** *Let G be a finite non-abelian 2-group, where*

$$G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle.$$

*If G acts on itself by conjugation, then* $|V(\Gamma_G^c)| = 32^{n-2}$ *and*

$$|E(\Gamma_G^c)| = \frac{3}{2}(32^{n-4} - 2^{n-2}).$$

**Proof** The number of vertices in $\Gamma_G^c$ are $|V(\Gamma_G^c)| = K(G) - Z(G)$, thus $|V(\Gamma_G^c)| = 32^{n-2}$. The degree of any vertex in $\Gamma_G^c$ is

$$deg(v) = |V(\Gamma_G^c)| - 1.$$

Thus $deg(v) = 32^{n-2} - 1$. Meanwhile, the degree of $\Gamma_G^c$ is

$$d(\Gamma_G^c) = \sum_{v=1}^{|V(\Gamma_G^c)|} deg(v_i) = 2|E(\Gamma_G^c)|.$$

Therefore

$$d(\Gamma_G^c) = \sum_{v=1}^{32^{n-2}} 32^{n-2} - 1 = 2|E(\Gamma_G^c)|.$$

Thus, $|E(\Gamma_G^c)| = \frac{3}{2}(2^{2n-4} - 2^{n-2})$. □

**Example 2.3** Suppose $G \cong \langle a, b : a^{2^3} = b^2 = e, ab = ba^5 \rangle$. If $G$ acts on itself by conjugation, then $\Gamma_G^c = K_6$.

**Solution** Based on Theorem 2.11, the number of vertices is six. Thus, the number of edges is $|E(\Gamma_G^c)| = \frac{3}{2}(12-2)$. It follows that $|E(\Gamma_G^c)| = 15$. From these, it follows that $\Gamma_G^c = K_6$.

**Remark** The graph in Theorem 2.11 is a Hamiltonian graph since $deg(v) \geq \frac{|V(\Gamma_G^c)|}{2}$.

## 2.4 CONCLUSION

In this chapter, we found the orbit graph for some finite 2-groups. Besides, the chromatic number, clique number, independent number, and dominating number were all found for the groups. The number of edges of the graph was computed and the graph for 2-groups mentioned in this chapter is Hamiltonian. In addition, we found the conjugacy class graph for the groups mentioned in this chapter.

### Acknowledgments

The authors would like to thank Universiti Teknologi Malaysia (UTM) for financial funding through the Research University Grant (RUG) Vote No. 08H07. The second author would also like to acknowledge UTM Johor Bahru, Malaysia for her International Doctoral Fellowship (IDF).

## REFERENCES

[1] Erfanian, A. and B. Tolue. 2012. "Conjugate Graphs of Finite Groups." *Discrete Mathematics, Algorithms and Applications*, 4(2): 35–43.

[2] Bondy, J. and G. Murty. 1982. *Graph Theory with Application*. 5th Edition. Boston New York: North Holand.

[3] Godsil, C. and G. Royle. 2001. *Algebraic Graph Theory*. 5th Edition. Boston New York: Springer.

[4] Xu, J. 2003. *Theory and Application of Graphs*. 5th Edition. Boston New York: Kluwer Academic Publishers.

[5] Beardon, A. F. 2005. *Algebra and Geometry*. Cambridge: Cambridge University Press.

[6] Bogopolski, O. 2008. *Introduction to Group Theory*. London: European Mathematical Society.

[7] Humphreys, J. 2004. *A Course in Group Theory*. London: Oxford University Press.

[8] Leedham-Green, C. R. and S. R. Mckay. 2002. *The Structure of Groups of Prime Power Order*. 1st Edition. London: Oxford University Press.

[9] Miller, G. A. 1944. "Relative Number of Non-invariant Operators in a Group." *Proceeding National Academy of Science USA*, 30(2): 25–28.

[10] Erdos, P. and P. Turan. 1968. "On Some Problems of a Statistical Group Theory." *IV, Acta Mathematica Hungaricae*, 19: 413–435.

[11] Omer, S. M. S., N. H. Sarmin, K. Moradipour, and A. Erfanian. 2013. "The Probability That an Element of a Group Fixes a Set and the Group Acts on Set by Conjugation." *International Journal of Applied Mathematics and Statistics*, 32(2): 111–117.

[12] Omer, S. M. S., N. H. Sarmin, K. Moradipour, and A. Erfanian. 2013. "The Probability That an Element of a Group Fixes a Set of Size Two for Some Finite 2-groups." *International Graduate Conference on Engineering Science and Humanity*, 30(2): 971–973.

[13] Neumann, B. H. 1976. "A Problem of Paul Erdos on Groups." *Journal of Australian Mathematical Society*, 21: 467–472.

[14] Abdollahi, A., S. Akbari, and , H. R. Maimani. 2006. "Non-commuting Graph of a Group." *Journal of Algebra*, 89(98): 468–492.

[15] Bertram, E. A., M. Herzog, and A. Mann. 1990. "On a Graph Related to Conjugacy Classes of Groups." *Bulletin of London Mathematical Society*, 22: 569–575.

[16] Bianchi, M., D. Chillag, A. Mauri, M. Herzog, and C. Scoppola. 1992. "Applications of a Graph Related to Conjugacy Classes in Finite Groups." *Archiv der Mathematik*, 58: 126–132.

[17] Chillag, D., M. Herzog, and A. Mann. 1993. "On the Diameter of a Graph Related to Conjugacy Classes of Groups." *Bulletin of London Mathematical Society*, 25: 255–262.

[18] You, X., G. Qian, and W. Shi. 2005. "A New Graph Related to Conjugacy Classes of Finite Groups." *arXiv:math/0510015*.

[19] Moreto, A., G. Qian, and W. Shi. 2005. "Finite Groups Whose Conjugacy Class Graphs Have Few Vertices." *Archiv der Mathematik*, 85: 101–107.

[20] Zverovicha, I. E. and E. V. Zverovichb. 2005. "Basic Perfect Graphs and Their Extensions." *Discrete Mathematics*, 293: 291–311.

[21] Hougardy, S. 2006. "Classes of Perfect Graphs." *Discrete Mathematics*, 306: 2529–2571.

[22] Bianchi, M., M. Herzog, E. Pacifici, and G. Saffirio. 2012. "On the Regularity of a Graph Related to Conjugacy Classes of Groups." *European Journal of Combinatorics*. 33: 1402–1407.

[23] Omer, S. M. S., N. H. Sarmin, and A. Erfanian. 2014. "The Orbit Graph for Some Finite Solvable Groups." *AIP Conference Proceeding 1602*, 863–869.

[24] Omer, S. M. S. 2014. "Extension of the Commutativity Degrees of Some Finite Groups and Their Related Graphs." Ph.D. Thesis. Universiti Teknologi Malaysia.