RECENT ADVANCES IN COMMUTATIVITY DEGREES and GRAPHS OF GROUPS

For years, pure mathematicians have continually carried out research on the commutativity degrees of groups and applications of groups in graph theory. Containing updates and some related research findings, this book comprises the work of the academic staff, researchers and graduate students associated with Applied Algebraic and Analytical Group (AAAG) in Universiti Teknologi Malaysia. This work includes non-commuting graph of groups, orbit graph and generalized conjugacy class graph. Some properties of these graphs are also presented, including the number of vertices and edges, chromatic number, the clique number, dominating number and the independent number. Two extensions on the theory of commutativity degree are also contained in this book, namely the degree of a product of two subgroups for the abelian groups of order $2n$ and the probability that a group element fixes a set for some finite non-abelian groups where some group actions are used. The results obtained are then applied to find their graphs.
RECENT ADVANCES IN COMMUTATIVITY DEGREES AND GRAPH OF GROUPS

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Preface

Applied Algebra and Analysis Group (AAAG) is one of the research groups under the Frontier Material Research Alliance, Universiti Teknologi Malaysia. The research interests of AAAG are Algebra, Group Theory and Formal Language Theory and Splicing Systems.

This book chapter consists of four chapters that focus on different types of commutativity degrees and their relations with different kinds of graphs. New updates and some research findings by the academic staff and graduate students associated to AAAG are presented in this book chapter.

For a nonabelian finite group, several graphs can be defined including the non-commuting graph. The first chapter deals with one of the problems about non-commuting graph of groups namely, for a nonabelian finite group $G$ and another group $H$, if their non-commuting graphs are isomorphic, then the order of these two groups coincide. In this chapter, the non-abelian finite groups with size $4n$, $6n$ and $8n$ are considered, where $n$ is an integer.

Chapter 2 focuses on graph theory, more precisely on the orbit graph. The orbit graph is found for some finite 2-groups under some group actions. The number of vertices and edges are found. In the second part of this chapter, the conjugacy classes graph is also found. Consequently, some graph properties like the chromatic number, the clique number, the independent number and dominating number are also obtained.

Chapter 3 discusses about the extension of relative commutativity degree which is named as the degree of a product of two subgroups. This is found only for the dihedral groups of order $2n$. The generalization of the degree of a product of two subgroups of a dihedral group are presented through some propositions and
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Theorems. Furthermore, some examples are given to illustrate the results.

Chapter 4 discusses one of the commutativity degrees’ generalizations, which is the probability that a group element fixes a set. This concept is found for some finite non-abelian groups where some group actions are used. As a consequence, the results obtained are applied to graph theory, more specifically to the orbit graph and generalized conjugacy class graph. Some graph properties such as the chromatic number, the clique number, the independent number and dominating number are provided.

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1 Non-commuting Graph of Some Nonabelian Finite Groups
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1.1 INTRODUCTION

Let $G$ be a group and $Z(G)$ be its center. For each group $G$, we will associate a graph which is called the non-commuting graph of $G$, denoted by $\Gamma_G$. The vertex set $V(\Gamma_G)$ is $G - Z(G)$ and the edge set $E(\Gamma_G)$ consists of $\{x, y\}$, where $x$ and $y$ are two distinct vertices of $V(\Gamma_G)$ are joined together if and only if $xy \neq yx$. The non-commuting graph of a group was introduced by Erdos in 1975. The non-commuting graph of a finite group has been studied by many researchers [1].

One of the problems about non-commuting graph of groups is given in the following conjecture:

**Conjecture 1.1.** Let $G$ be a non-abelian finite group and $H$ a group such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.

**Definition 1.1** $T_{4n}$ is a non-abelian finite group with order $4n$. Its structure is defined as

$$T_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle. \quad (1.1)$$

**Definition 1.2** $U_{6n}$ is a non-abelian finite group with order $6n$. Its
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structure is defined as

\[ U_{6n} = \langle a, b | a^{2n} = 1, b^3, a^{-1}ba = b^{-1} \rangle. \quad (1.2) \]

**Definition 1.3** \( V_{8n} \) is a non-abelian finite group with order \( 8n \). Its structure is defined as

\[ V_{8n} = \langle a, b | a^{2n} = 1, b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle. \quad (1.3) \]

The main objective of this chapter is to prove Conjecture 1.1 for three groups \( T_{4n}, U_{6n} \) and \( V_{8n} \). In fact, we show that if \( \Gamma_G \cong \Gamma_{T_{4n}}, \Gamma_G \cong \Gamma_{U_{6n}}, \Gamma_G \cong \Gamma_{V_{8n}}, \) then \( |G| = |T_{4n}|, |G| = |U_{6n}| \) or \( |G| = |V_{8n}| \) respectively. For more details see Conway et al. [2] and Rose [3].

1.2  **NON-COMMUTING GRAPH OF** \( T_{4n} \)

In this section, we show that if \( G \) is a non-abelian finite group such that \( \Gamma_G \cong \Gamma_{T_{4n}}, \) then \( |G| = |T_{4n}| \). In the lemmas, we refer the degree of the vertex \( x \), which is denoted by \( \text{deg}(x) \), as the number of edges through \( x \). We first state some lemmas which will be used throughout this section.

**Lemma 1.1** [4] Let \( G \) be a non-abelian finite group and \( x \) is a vertex of \( \Gamma_G \). Then

\[ \text{deg}(x) = |G| - |C_G(x)|. \quad (1.4) \]

**Lemma 1.2** [4] Let \( G \) be a non-abelian finite group. If \( H \) is a group such that \( \Gamma_G \cong \Gamma_{H}, \) then \( H \) is a non-abelian finite group such that \( |Z(H)| \) divides each of the following:

\[ |G| - |Z(G)|, |G| - |C_G(x)|, |C_G(x)| - |Z(G)|, \text{ for } x \in (G - Z(G)). \]

**Lemma 1.3** Let \( T_{4n} \) be a group. Then

\[ |C_{T_{4n}}(a)| = 2n, |C_{T_{4n}}(b)| = 4 \text{ and } |Z(T_{4n})| = 2. \]
Proof. All elements of $T_{4n}$ are denoted as $a^i b^j$ such that $1 \leq i \leq 2n$, $1 \leq j \leq 4$. The center of $T_{4n}$ is defined by

$$\{a^i b^j | (a^i b^j)a = a(a^i b^j), b(a^i b^j) = (a^i b^j)b, 1 \leq i \leq 2n, 1 \leq j \leq 4\}. $$

Now, we find the elements of $Z(T_{4n})$. If $a^i b^j$ belongs to $Z(T_{4n})$, then $a^i b^j a = a^{i+1} b^j$ and $a^i b^{j+1} = b a^i b^j$. Therefore we have $b^j a = a b^j$ and $b a^i = a^i b$. There exist three cases for $j$ as follows:

(a) If $j = 0$, then $ba^i = a^ib$. According to Definition 1.2, $a^i b = a^{-i} b$ and $i = n$. Hence $a^n \in Z(T_{4n})$.

(b) If $j \neq 2$, then $b^j a = a^{-1} b^j$ and $b^j a = a b^j$. Therefore the order of $a$ is 2, which is a contradiction.

(c) If $j = 2$, then $a^i b^2 = a^{n+i}$ and $a^i b^3 = a^{-i} b^3$. Hence $i = n$ and it shows that $a^n \in Z(T_{4n})$.

So $Z(T_{4n}) = a^n, 1$ and $|Z(T_{4n})| = 2$. We can see easily that $C_{T_{4n}}(a) = \langle a \rangle$ and $C_{T_{4n}}(b) = \langle b \rangle$. Therefore $|C_{T_{4n}}(a)| = 2n, |C_{T_{4n}}(b)| = 4$. \hfill $\square$

**Theorem 1.1** Let $G$ be a finite non-abelian group. If $\Gamma_G \cong \Gamma_{T_{4n}}$, then $|G| = |T_{4n}|$.

**Proof.** We know that $\Gamma_{T_{4n}}$ has two vertices $a$ and $b$ such that $deg(a) = 2n$ and $deg(b) = 4n - 4$. Since $\Gamma_G \cong \Gamma_{T_{4n}}$, we have the following equality:

$$|G| - |Z(G)| = |T_{4n}| - |Z(T_{4n})| = 4n - 2. $$

Therefore $|Z(G)|$ divides $4n - 2$. There exists the corresponding elements $g_1, g_2 \in G - Z(G)$ such that $deg(g_1) = 2n$ and $deg(g_2) = 4n - 4$. By Lemma 1.2, we obtain that $|Z(G)|$ divides 2. Now, we show that $|Z(G)| = 2$. Using the contradiction proof, suppose that $|Z(G)| = 1$ and

$$|G| = 4n - 1, deg(g_1) = |G| - |C_G(g_1)| = 2n. $$

So $|C_G(g_1)| = 2n - 1$. But we know that $2n - 1$ does not divide $4n - 1$. Hence $|Z(G)| = 2$ and $|G| = 4n = |T_{4n}|$. \hfill $\square$
1.3 NON-COMMUTING GRAPH OF $U_{6n}$

According to the definition of $U_{6n}$, we have all of its elements are in the form of $a^ib^j$ such that $0 \leq i \leq 2n - 1$ and $0 \leq j \leq 2$. To obtain our main goal, we start with the following lemma.

Lemma 1.4 Let $U_{6n}$ be a finite group. Then

$$|C_{U_{6n}}(a)| = 2n, \quad |C_{U_{6n}}(b)| = 3 \quad \text{and} \quad |Z(U_{6n})| = 1.$$  

Proof First, we show that $Z(U_{6n}) = 1$. Suppose that there exist $i$ and $j$ such that $a^ib^j \in Z(U_{6n})$ and $i, j \neq 0$. Since $(a^ib^j)a = a(a^ib^j)$, we obtain $a^{i+1}b^j = a^{i+1}b^{-j}$ and $j = 3$. Also we have $b(a^ib^j) = (a^ib^j)b$. Therefore $a^ib = ba^i = a^ib^{-1}$ and the order of $b$ is equal to 2. Hence we conclude that $Z(U_{6n}) = 1$. By the structure of $U_{6n}$, we can easily see that $C_{U_{6n}}(a) = \langle a \rangle$ and $C_{U_{6n}}(b) = \langle b \rangle$. Therefore $|C_{U_{6n}}(a)| = 2n$ and $|C_{U_{6n}}(b)| = 3$. \hfill \Box

Theorem 1.2 Let $G$ be a finite non-abelian group. If $\Gamma_G \cong \Gamma_{U_{6n}}$, then $|G| = |U_{6n}|$.

Proof Since $\Gamma_G \cong \Gamma_{U_{6n}}$, it can concluded that $\Gamma_G$ has two vertices $g_1, g_2$ such that $deg(g_1) = 4n$ and $deg(g_2) = 6n - 3$. Also we have this equality $|G| - |Z(G)| = 6n - 1$.

Since $|Z(G)|$ divides $deg(g_1)$ and $deg(g_2)$, then there exists three cases for $|Z(G)|$ as follows:

(a) If $|Z(G)| = 2$, then $|G| = 6n + 1$ and $|C_G(g_2)| = 4$. This is impossible since $4 \nmid |G|$.

(b) If $|Z(G)| = 3$, then $|G| = 6n + 2$ and $|C_G(g_2)| = 5$. This is impossible since $|Z(G)| \nmid |C_G(g_2)|$.

(c) If $|Z(G)| = 6$, then $|G| = 6n + 5$ and $|C_G(g_2)| = 8$. This is impossible since $|Z(G)| \nmid |C_G(g_2)|$.

Therefore $|Z(G)| = 1$ and $|G| = |U_{6n}| = 6n$. \hfill \Box
1.4 NON-COMMUTING GRAPH OF $V_{8n}$

In this section, we study about $C_{V_{8n}}(a), C_{V_{8n}}(b)$ and $Z(V_{8n})$. We want to show that if $\Gamma_G \cong \Gamma_{V_{8n}}$, then $|G| = |V_{8n}|$. First we start with the following lemma.

**Lemma 1.5** Let $V_{8n}$ be a finite group.

(a) If $n$ is an even number, then $|C_{V_{8n}}(b)| = 8$, $|C_{V_{8n}}(a)| = 4n$ and $|Z(V_{8n})| = 4$.

(b) If $n$ is an odd number, then $|C_{V_{8n}}(b)| = 4$, $|C_{V_{8n}}(a)| = 4n$ and $|Z(V_{8n})| = 2$.

**Proof** Firstly, we show that $|C_{V_{8n}}(a)| = 4n$. It can be shown that

$$|C_{V_{8n}}(a)| = \{a^ib^j|(a^ib^j)a \}
= a(a^ib^j) \ni 0 \leq i \leq 2n - 1, 0 \leq j \leq 3 \}. \quad (1.5)$$

If $j = 0$, then $(a) \leq C_{V_{8n}}(a)$. Assume that $i = 0$, we have $ab^2 = b^2a$. Now suppose that $i \neq 0$.

If $j = 1$, then $a^ib(a) = a^{i-1}b^{-1}$ and $(a)a^ib = a^{i+1}b$. Since the order of $a$ is not equal to the order of $b$, we can conclude that $a(a^ib) \neq (a^ib)a$.

If $j = 2$, then $a^ib^2(a) = a^{i+1}b^2 = (a)a^ib^2$ for all $0 \leq i \leq 2n - 1$.

If $j = 3$, then $a^ib^3(a) = a^{i-1}b^{-3}$ and $(a)a^ib^3 = a^{i+1}b^3$. Since the order of $a$ is not equal to the order of $b$, we can conclude that $a(a^ib^3) \neq (a^ib^3)a$. Therefore, $|C_{V_{8n}}(a)| = 4n$.

Next, we want to obtain $|C_{V_{8n}}(b)|$, where $n$ is an even number. $C_{V_{8n}}(b) = \{a^ib^j|a^ib^j+1 = ba^ib^j\}$ for all $0 \leq i \leq 2n - 1$ and $0 \leq j \leq 3$. We know that $(b) \leq C_{V_{8n}}(b)$. Suppose that $i \neq 0$, now we have four cases for $j$. If $j = 0$, then we recognize $a^i$ such that $a^ib = ba^i$ for all $i$. Thus,

$$a^ib = ba^i \rightarrow a^{i-1}b^{-1}a^{-1} = ba^i \rightarrow a^{i-2}b = ba^{i+2} \rightarrow b^{(i)} = ba^{2i}$$

The preceding equation shows that $i$ cannot be an odd number. Therefore $i$ is an even number and $i = n$. If $j = 1$, then
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For $j = 2$ and $j = 3$, we have

$$a^i b^2(b) \neq (b)a^i b^2$$

and $a^i b^3(b) \neq (b)a^i b^3$ for all $i$. Hence, $|C_{V_{8n}}(b)| = 8$. Also we have:

$$C_{V_{8n}}(a) = \{1, a, a^2, ..., a^{2n-1}, b^2, b^2a, b^2a^2, ..., b^2a^{2n-1}\}$$

and

$$C_{V_{8n}}(b) = \{1, b, b^2, a^n, ba^n, b^2a^n, b^3a^n\}.$$ 

On the other hand, we know that

$$Z(V_{8n}) = \{g \in V_{8n} | gv = vg \text{ for all } v \in V_{8n}\}$$

$$= \{g \in V_{8n} | ga = ag \text{ and } gb = bg\}$$

$$= C_{V_{8n}}(a) \cap C_{V_{8n}}(b) = \{1, b^2, a^n, b^2a^n\}.$$ 

Therefore $|Z(V_{8n})| = 4$. If $n$ is an odd number, according to the above proof we have four cases for $j$. But in any case, we have $a^i b^j(b) \neq (b)a^i b^j$ for all $0 \leq j \leq 3$. Therefore $C_{V_{8n}}(b) = \langle b \rangle$ and

$$Z(V_{8n}) = \{g \in V_{8n} | gv = vg \text{ for all } v \in V_{8n}\}$$

$$= \{g \in V_{8n} | ga = ag \text{ and } gb = bg\}$$

$$= C_{V_{8n}}(a) \cap C_{V_{8n}}(b)$$

$$= \{1, a, a^2, ..., a^{2n-1}, b^2, b^2a, b^2a^2, ..., b^2a^{2n-1}\}$$

$$\cap \{1, b, b^2, b^3\}$$

$$= \{1, b^2\}.$$ 

Hence, $|Z(V_{8n})| = 2$.  

**Theorem 1.3** Let $G$ be a non-abelian finite group. If $\Gamma_G \cong \Gamma_{V_{8n}}$, then $|G| = |V_{8n}|$. 


**Proof** First, we suppose that \( n \) is an even number. In this case \( \text{deg}(a) = 4n \) and \( \text{deg}(b) = 8(n-1) \). Since the \( \Gamma_G \cong \Gamma_{V_{8n}} \), we have
\[
|G| - |Z(G)| = |V_{8n}| - |Z(V_{8n})|.
\]
Hence, \( |Z(G)| \) divides \( 8n - 4 \). Also \( \Gamma_G \) has two vertices \( g_1 \) and \( g_2 \) such that \( \text{deg}(g_1) = 4n \) and \( \text{deg}(g_2) = 8n - 8 \). We know that \( |Z(G)| \) divides \( 8n - 8 \), so \( |Z(G)| \) divides \( 4 \). Therefore \( |Z(G)| \) can be \( 1, 2 \) or \( 4 \).

If \( |Z(G)| = 1 \), then \( |G| = 8n - 3 \) and \( |C_G(g_2)| = 5 \). Since \( |C_G(g_2)| \) must divide \( |G| \), so \( 5 | |G| \). It occurs only when \( n = 1 \) and it is impossible because \( n \) is an even number.

If \( |Z(G)| = 2 \), then \( |G| = 8n - 2 \) and \( |C_G(g_2)| = 6 \). Since \( |C_G(g_2)| \) must divide \( |G| \), so \( 6 | |G| \). It occurs when \( n = 1 \) and it is impossible because \( n \) is an even number. Therefore \( |Z(G)| = 4 \) and \( |G| = |V_{8n}| = 8n \). Now, suppose that \( n \) be an odd number. In this case,
\[
|G| - |Z(G)| = 8n - 2
\]
and \( \text{deg}(g_1) = 4n \) and \( \text{deg}(g_2) = 8n - 4 \). We have that \( |Z(G)| \) divides \( 8n - 2 \) and \( 8n - 4 \). Thus \( |Z(G)| \) divides \( 2 \). There is two cases for \( |Z(G)| \). It can be \( 1 \) or \( 2 \).

If \( |Z(G)| = 1 \), then \( |G| = 8n - 1 \) and \( |C_G(g_2)| = 3 \). However \( 3 | 8n - 1 \) only when \( n = 2 \) which is impossible since \( n \) is an odd number. Hence \( |Z(G)| = 2 \) and \( |G| = |V_{8n}| = 8n \). \( \square \)

### 1.5 CONCLUSION

In this research, we define three groups \( T_{4n}, U_{6n} \) and \( V_{8n} \) and show that if \( G \) is a non-abelian finite group such that
\[
\Gamma_G \cong \Gamma_{T_{4n}}, \quad \Gamma_G \cong \Gamma_{U_{6n}} \quad \text{or} \quad \Gamma_G \cong \Gamma_{V_{8n}},
\]
then
\[
|G| = |T_{4n}| = 4n, \quad |G| = |U_{6n}| = 6n \quad \text{or} \quad |G| = |V_{8n}| = 8n,
\]
respectively.
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REFERENCES


2 Precise Value of the Orbit Graph and Conjugacy Class Graph of Some Finite 2-Groups

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2.1 INTRODUCTION

Orbit graph is considered as one of the generalizations of conjugate graph that was introduced by Erfanian and Tolu [1]. The first generalization of conjugate graph was done by Omer et al. who introduced the so called orbit graph. An orbit graph is a graph whose vertices are non-central orbits under some group actions on a set in which two vertices are adjacent if they are conjugate. In this chapter, we find the orbit graph for some finite 2-groups when a group acts on itself by conjugation.

Throughout this chapter, $\Gamma$ denotes a simple undirected graph, $G$ denotes a 2-group and $\Gamma^c_G$ denotes an orbit graph where the group acts on itself by conjugation.

Basic concepts and definitions of graph theory that are needed are stated in the following:

**Definition 2.1[2]** A graph $\Gamma$ consists of two sets, namely vertices $V(\Gamma)$ and edges $E(\Gamma)$ together with relation of incidence.

The directed graph is a graph whose edges are identified with ordered pair of vertices. Otherwise, $\Gamma$ is called indirected. Two vertices are adjacent if they are joined by an edge.
Definition 2.2 [2] A graph is called a subgraph $\Gamma_{sub}$ of $\Gamma$ if its vertices and edges are subsets of the vertices and edges of $\Gamma$.

Definition 2.3 [3] A complete graph $K_n$ is a graph where each ordered pair of distinct vertices are adjacent.

Definition 2.4 [3] The graph $\Gamma$ is an empty graph, if there is no adjacent (edges) between its vertices. In this chapter, $K_e$ denotes the empty graph.

Definition 2.5 [2] The graph $\Gamma$ is called null if it has no vertices, denoted by $K_0$.

Definition 2.6 [3] A regular graph is a graph whose all vertices have the same sizes.

Definition 2.7 [2] A line graph $L(\Gamma)$ of $\Gamma$ is a graph with the edges of $\Gamma$ as its vertices.

Definition 2.8 [2] A vertex is incident with an edge if it is one of the two vertices of the edge.

Note that two edges of $\Gamma$ are linked in $L(\Gamma)$ if and only if they are incident in $\Gamma$.

Definition 2.9 [3] The valency of a vertex $x$ is the number of neighbors of $x$.

Proposition 2.1 If $\Gamma$ is a simple graph with $v \geq 3$ and $\deg(v) \geq \left(\frac{|V(\Gamma)|}{2}\right)$, then $\Gamma$ is Hamiltonian.

The following proposition is used to determine the degree of vertex in $\Gamma$.

Proposition 2.2 [2] Let $G$ be a finite group and $\Gamma$ be its graph. Let $V(\Gamma)$ be the set of vertices in $\Gamma$. If $v \in V(\Gamma)$, then the degree of $v$ is

$$\deg(v) = \sum_{d(v)} d(v) = 2E,$$

where $E$ is the number of edges in $\Gamma$. 

The **independent set** is a non-empty set $B$ of $V(\Gamma)$, where there is no adjacent between two elements of $B$ in $\Gamma$, while the **independent number** is the number of vertices in maximum independent set and it is denoted by $\alpha(\Gamma)$. However, the maximum number $m$ for which $\Gamma$ is $m$-vertex colored is known as **chromatic number** and denoted by $\chi(\Gamma)$. The maximum distance between any two vertices of $\Gamma$ is called the **diameter** and is denoted by $d(\Gamma)$. In addition, a **clique** in $\Gamma$ is a complete subgraph in $\Gamma$, while the **clique number** is the size of the largest clique in $\Gamma$. The **dominating set** $X \subseteq V(\Gamma)$ is a set where for each $v$ outside $X$, there exists $x \in X$ such that $v$ is adjacent to $x$. The **dominating number** is the minimum size of $X$ and is denoted by $\gamma(\Gamma)$.

**Definition 2.10** [5] Let $G$ be a finite group, and let $g_1, g_2$ be elements in $G$. The elements $g_1, g_2$ are said to be conjugate if there are some $h$ in $G$ such that

$$g_2 = h g_1 h^{-1}.$$ 

The set of all conjugates of $g_1$ is called the conjugacy classes of $g_1$.

In this chapter, we find the orbit graph of some finite non-abelian 2-groups. The presentations of 2-groups are stated in the following.

**Definition 2.11** [6] If $G$ is a finite 2-group of order $2^n$, then $G$ has the following presentation

$$G \cong \langle a, b : a^n = b^2 = 1, (ab)^2 = 1 \rangle.$$ 

**Definition 2.12** [7] Let $n$ be an integer greater than two. If $G$ is a finite non-abelian group of order $2^n+1$, then

$$G \cong \langle G \cong \langle a, b : a^{2^n-1} = 1, b^2 = a^{2^n-2}, bab^{-1} = a^{-1} \rangle. \rangle \rangle.$$ 

**Definition 2.13** [7] Let $G$ be a finite non-abelian 2-group of order $2^n+1$, where $n \geq 3$. Thus, $G$ has a presentation

$$G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^n-1} \rangle.$$
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Definition 2.14[8] If $G$ is a finite non-abelian 2-group group of order $2^\beta+1$, where $\beta \geq 3$, then $G$ has the following presentation

$$G \cong \langle a, b : a^{2^\beta} = b^2 = e, ab = ba^{2^{\beta-1}+1} \rangle.$$ 

Commutativity degree is the term used to determine the abelianness of groups. This concept was firstly introduced by Miller in 1944 [9]. The idea of the commutativity degree was then investigated for symmetric groups by Erdős and Turan [10]. The results obtained encourage many researchers to work on this topic which has initiated various generalizations. One of these generalizations called the probability that a group element fixes a set introduced by Omer et al. [11]. The followings are some results on the probability that a group element fixes a set, needed in this chapter.

Theorem 2.1[11] Let $G$ be a finite 2-group, $G \cong \langle a, b : a^n = b^2 = 1, (ab)^2 = 1 \rangle$, where $n \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $\text{lcm}(|a|, |b|) = 2$. Let $\Omega$ be the set of all subsets of commuting elements of size two. If $G$ acts on $\Omega$ by conjugation, then

$$P_G(\Omega) = \begin{cases} 
\frac{6}{m+1}, & \text{if } n \text{ is even, } \frac{n}{2} \text{ is odd and } m = \frac{5n}{2}, \\
\frac{7}{m+1}, & \text{if } n \text{ is even, } \frac{n}{2} \text{ is even and } m = \frac{5n}{2}, \\
\frac{1}{n}, & \text{if } n \text{ is odd.}
\end{cases}$$

Theorem 2.2[11] Let $G$ be finite 2-group $G \cong \langle a, b : a^{2^n-1} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $\text{lcm}(|a|, |b|) = 2$. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.

Theorem 2.3[12] Let $G$ be a finite non-Abelian group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$, where $n \geq 3$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$
and \(b\) commute and \(\text{lcm}(|a|,|b|) = 2\). Let \(\Omega\) be the set of all subsets of commuting elements of \(G\) of size two and \(G\) acts on \(\Omega\) by conjugation. Then \(P_G(\Omega) = \frac{4}{2^n + 2^{n-2} + 1}\).

**Theorem 2.4**[12] Let \(G\) be a finite non-Abelian 2-group, \(G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^n-1}+1 \rangle\), where \(n \geq 3\). Let \(S\) be a set of elements of \(G\) of size two in the form of \((a, b)\) where \(a\) and \(b\) commute and \(\text{lcm}(|a|,|b|) = 2\). Let \(\Omega\) be the set of all subsets of commuting elements of \(G\) of size two and \(G\) acts on \(\Omega\) by conjugation. Then \(P_G(\Omega) = \frac{2}{3}\).

### 2.2 GRAPH THEORY

This section provides some works that are related to graph theory. Non-commuting graph is a concept that was introduced in 1975 [13], where many recent publications have been done using this graph. The definition of a non-commuting graph is stated in the following.

**Definition 2.15**[13] Let \(G\) be a finite non-abelian group with the center denoted by \(Z(G)\). A non-commuting graph is a graph whose vertices are non-central elements of \(G\) (i.e. \(G - Z(G)\)). Two vertices \(v_1\) and \(v_2\) are adjacent whenever \(v_1 v_2 \in v_2 v_1\).

In [13], it is mentioned that this concept was firstly introduced by Paul Erdos. Erdos posed a question if there is a finite bound on the cardinalities of cliques of \(\Gamma\). The first confirmed form of Erdos’s question was by Neumann [13]. According to Neumann [13] there is a finite complete subgraph in some groups. Furthermore, Abdollahi et al. [14] emphasized the existence of finite bound on the cardinalities of complete subgraph in \(\Gamma\). Abdollahi et al. [14] used the graph theoretical concepts to investigate the algebraic properties of the graph.

In 1990, Bertram et al. [15] introduced a graph which is called the **conjugacy class graph**. The vertices of this graph are non-central conjugacy classes, where two vertices are adjacent if the
cardinalities are not coprime. The following is definition of conjugacy class graph.

**Definition 2.16**[15] Let \( G \) be a finite group. The conjugacy class graph denoted in this chapter by \( \Gamma^c_G \), is a graph whose vertices are non-central conjugacy classes i.e the number of vertices of conjugacy class graph

\[
|V(\Gamma^c_G)| = K(G) - Z(G),
\]

where \( K(G) \) denotes the number of conjugacy classes and \( Z(G) \) denotes the center of \( G \). Two vertices are adjacent if the cardinalities are not coprime.

As a consequence, numerous works have been done on this graph and many results have been achieved [16–18].

In [19], Berge conjectured that the graph is a perfect graph if and only if the chromatic number and clique number are identical and there is no induced subgraph if the graph is an odd cycle of length greater than three. Berge’s conjectured then became known as the strong perfect graph conjecture. Then, various studies have been done on perfect graph [20, 21].

Recently, Bianchi et al. [22] studied the regularity of conjugacy class graph and provided some results.

Later, Erfanian and Tolue [1] introduced a new graph which is called a conjugate graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate.

In 2013, Omer et al. [23] introduced a new graph called the orbit graph whose vertices a non-central orbit under group action on a set. The following is definition of the orbit graph.

**Definition 2.17**[23] Let \( G \) be a finite group and \( \Omega \) be a set of elements of \( G \). Let \( A \) be the set of commuting elements in \( \Omega \), i.e \( A = \{ v \in \Omega : vg = gv, g \in G \} \). The orbit graph \( \Gamma^\Omega_G \) consists of two sets, namely vertices and edges denoted by \( V(\Gamma^\Omega_G) \) and \( E(\Gamma^\Omega_G) \),
respectively. The vertices of $\Gamma^\Omega_G$ are non central elements in $\Omega$ but not in $A$, that is $V(\Gamma^\Omega_G) = \Omega - A$, while the number of edges are 

$$|E(\Gamma^\Omega_G)| = \sum_{i=1}^{V(\Gamma^\Omega_G)} \binom{v}{2},$$

where $v$ is the size of orbit under group action of $G$ on $\Omega$. Two vertices $v_1, v_2$ are adjacent in $\Gamma^\Omega_G$ if one of the following conditions is satisfied.

(a) If there exists $g \in G$ such that $gv_1 = v_2$,
(b) If the vertices of $\Gamma^\Omega_G$ are conjugate that is, $v_1 = g v_2$.

Some graph properties are investigated as follows:

**Proposition 2.3** [24] Let $G$ be a finite non-Abelian group and let $\Omega$ be a set. If $G$ acts on $\Omega$, then the properties of the orbit graph $\Gamma^\Omega_G$ are described as follows.

(a) $\chi(\Gamma^\Omega_G) = \min\{|cl(v_i)|, v_i \in \Omega\}$,
(b) $\omega(\Gamma^\Omega_G) = \max\{|cl(v_i)|, v_i \in \Omega\}$,
(c) $\alpha(\Gamma^\Omega_G) = K(\Omega) - |A|$,
(d) $\gamma(\Gamma^\Omega_G) = K(\Omega) - |A|$,
(e) $d(\Gamma^\Omega_G) = \max\{d(v, u) : \forall v, u \in V(\Gamma^\Omega_G)\}$.

### 2.3 RESULTS AND DISCUSSION

This section consists of two parts. In the first part, we find the orbit graph for some finite 2-groups, while in the second part, we compute the conjugacy class graph for the mentioned groups.

#### 2.3.1 Orbit Graph

In this section, we introduce our results on the orbit graph of some finite 2-groups.

**Theorem 2.5** Let $G$ be a finite 2-group, $G \cong \langle a, b : a^n = b^2 = 1, (ab)^2 = 1 \rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $\text{lcm}(|a|, |b|) = 2$. Let $\Omega$
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be the set of all subsets of commuting elements of \( G \) of size two. If \( G \) acts on \( \Omega \) by conjugation, then

\[
\Gamma^\Omega_G = \begin{cases} 
\bigcup_{i=1}^{\frac{n}{2}} K_{n_i}, & n \text{ is even and } \frac{n}{2} \text{ is odd,} \\
(\bigcup_{i=1}^{\frac{n}{4}} K_{n_i}) \cup (\bigcup_{i=1}^{\frac{n}{2}} K_{n_i}), & n \text{ and } \frac{n}{2} \text{ are even,} \\
K_n, & n \text{ is odd.}
\end{cases}
\]

Proof Based on Theorem 2.1, the number of elements in \( \Omega \) is \( \frac{5n}{2} + 1 \) and using Definition 2.17, the number of vertices of \( \Gamma^\Omega_G \) is \( \frac{5n}{2} \). Two vertices \( \omega_1 \) and \( \omega_2 \) are adjacent if \( \omega_1 = \omega_2^g \). First, when \( n \) is even and \( \frac{n}{2} \) is odd, there are five complete components of \( K_{n_i} \), since there are six orbits five of them are of size \( \frac{n}{2} \). Second, when \( n \) and \( \frac{n}{2} \) are even. The orbit graph \( \Gamma^\Omega_G \) consists of four complete components of \( K_{n_i} \) and two complete components of \( K_{n_i} \), represented to four orbits of size \( \frac{n}{2} \) and two orbits are of size \( \frac{n}{4} \). However, when \( n \) is odd there is only orbit of size \( n \), hence \( \Gamma^\Omega_G \) consists of one complete component of \( K_n \). \( \blacksquare \)

Remark The orbit graph of Theorem 2.5 is a complete graph graph when \( n \) is odd since there is one complete graph of \( n \) connected vertices.

Theorem 2.6 Let \( G \) be a finite 2-group, \( G \cong \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle \). Let \( S \) be a set of elements of \( G \) of size two in the form of \( \langle a, b \rangle \) where \( a \) and \( b \) commute and \( \text{lcm}(|a|, |b|) = 2 \). Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two. If \( G \) acts on \( \Omega \) by conjugation, then \( \Gamma^\Omega_G \) is a null graph.

Proof According to Theorem 2.2, there is only one orbit of size one. By Definition 2.17, the number of vertices is zero since \( |\Omega| = |A| \). Hence, \( \Gamma^\Omega_G \) is a null graph. \( \blacksquare \)

Theorem 2.7 Let \( G \) be a finite group, \( G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle \). Let \( S \) be a set of elements of \( G \) of size two in the form of \( \langle a, b \rangle \) where \( a \) and \( b \) commute and \( \text{lcm}(|a|, |b|) = 2 \). Let \( \Omega \)
be the set of all subsets of commuting elements of \( G \) of size two. If \( G \) acts on \( \Omega \) by conjugation, then \( \Gamma_G^\Omega = K_{2n-1} \cup K_{2n-1} \cup K_{2n-2} \).

**Proof** Suppose that \( G \) acts on \( \Omega \) by conjugation. The number of vertices of \( \Gamma_G^\Omega \) is \( |V(\Gamma_G^\Omega)| = |\Omega| - |A| \). According to Theorem 2.3, the number of elements in \( \Omega \) is \( 2^n + 2^{n-2} + 1 \) and the number of elements in \( A \) is one. Thus, the number of vertices in \( \Gamma_G^\Omega \) is \( 2^n + 2^{n-2} \). Two vertices of \( \Gamma_G^\Omega \) are linked if they are conjugate. Therefore, the adjacent vertices are described as follows: The vertex of the form \((1, a^i b)\) is adjacent to all vertices of the form \((1, a^j b), 0 \leq i, j \leq 2^n \) where \( i \) and \( j \) are even and \( i \neq j \). Thus, there is one complete component of \( K_{2n-1} \). In addition, all the vertices of the form \((a^{2^n-1}, a^i b)\) are adjacent to the vertices \((a^{2^n-1}, a^j b), 0 \leq i \leq 2^n, i \neq j \), where \( i \) and \( j \) are even. Hence, there is one complete component of \( K_{2n-1} \). The vertices in the form \((a^i b, a^{2^n-1+i} b)\), where \( i \) is even, are connected to each others. It follows that, there is only one complete components of \( K_{2n-2} \). Therefore, \( \Gamma_G^\Omega \) contains three complete components, as required. \( \square \)

In the next corollary, the chromatic number, clique number, independent number and dominating number of the graph in Theorem 2.7 are found.

**Corollary 2.1** Let \( G \) be a finite group, \( G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle \). Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute and \( \text{lcm}(|a|, |b|) = 2 \). Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two. If \( G \) acts on \( \Omega \) by conjugation and \( \Gamma_G^\Omega = K_{2n-1} \cup K_{2n-1} \cup K_{2n-2} \), then \( \alpha(\Gamma_G^\Omega) = 3 \) and \( \chi(\Gamma_G^\Omega) = \omega(\Gamma_G^\Omega) = 2^{n-1} \).

**Proof** Based on Theorem 2.7, there are three orbits and according to Proposition 2.3, \( \alpha(\Gamma_G^\Omega) = 3 \). \( \square \)

Next, the orbit graph of the following 2-group is found.

**Theorem 2.8** Let \( G \) be a finite group, \( G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle \). Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute and \( \text{lcm}(|a|, |b|) = 2 \). Let \( \Omega \)
be the set of all subsets of commuting elements of $G$ of size two. If $G$ acts on $\Omega$ by conjugation, then $\Gamma_G^\Omega = K_2 \cup K_2$.

**Proof** Based on Theorem 2.4, the number of vertices, $|V(\Gamma_G^\Omega)| = 5$. Since there are two orbits of size two, thus $\Gamma_G^\Omega$ consists of two complete components of $K_2$. The proof then follows. \hfill $\square$

**Corollary 2.2** Let $G$ be a finite group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $\text{lcm}(|a|, |b|) = 2$. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two. If $G$ acts on $\Omega$ by conjugation and $\Gamma_G^\Omega = K_2 \cup K_2$, then $\chi(\Gamma_G^\Omega) = \omega(\Gamma_G^\Omega) = 2$.

**Proof** Referring to Proposition 2.3, the result follows.

### 2.3.2 Conjugacy Class Graph

In this part, we find the conjugacy class graph for 2-groups mentioned earlier.

**Theorem 2.9** Let $G$ be a finite non-abelian 2-group, where $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$.

If $G$ acts on itself by conjugation, then $|V(\Gamma_G^c)| = 2^{n-1} + 1$ and $|E(\Gamma_G^c)| = 2^{n-3} + 2^{n-2}$.

**Proof** According to Definition 2.16, the number of vertices is $K(G) = 2^{n-1} + 3$ and $|Z(G)| = 2$. It follows that $|V(\Gamma_G^c)| = 2^{n-1} + 1$. In accordance with Proposition 2.2, the degree of any vertex of $\Gamma_G^c$ is $\text{deg}(v) = |V(\Gamma_G^c)| - 1$. Thus, $\text{deg}(v) = 2^{n-1}$ and the degree of $\Gamma_G^c$ is

$$d(\Gamma_G^c) = \sum_{i=1}^{|V(\Gamma_G^c)|} \text{deg}(v_i) = 2|E(\Gamma_G^c)|.$$

It follows that, $d(\Gamma_G^c) = \sum_{i=1}^{2^{n-1}+1} 2^{n-1} = 2|E(\Gamma_G^c)|$. Therefore, $|E(\Gamma_G^c)| = 2^{n-3} + 2^{n-2}$. \hfill $\square$
Example 2.1 Assume $G$ is a finite non-abelian 2-group, where
\[ G \cong \langle a, b : a^2 = b = e, ab = ba^3 \rangle. \]
If $G$ acts on itself by conjugation, then $\Gamma_G^c = K_5$.

Solution In accordance with Theorem 2.9, the number of vertices in $\Gamma_G^c$ is five. Based on Proposition 2.2, the number of edges is
\[ d(\Gamma_G^c) = \sum_{i=1}^{\mid V(\Gamma_G^c) \mid} \deg(v_i) = 2\mid E(\Gamma_G^c) \mid. \]
It follows that, $\mid E(\Gamma_G^c) \mid = 10$. Thus, the graph here is a complete graph of $K_5$.

Corollary 2.3 Let $G$ be a finite non-abelian 2-group, where
\[ G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}} \rangle. \]
If $G$ acts on itself by conjugation, then the graph is Hamiltonian.

Proof The proof follows from Theorem 2.9 and Proposition 2.1.

Theorem 2.10 Let $G$ be a 2-group, where
\[ G \cong \langle a, b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle. \]
If $G$ acts on itself by conjugation, then $\mid V(\Gamma_G^c) \mid = 2^{n-2} + 1$ and $\mid E(\Gamma_G^c) \mid = 22^{n-5} + 2^n - 3$.

Proof According to Definition 2.16, the number of vertices of $\Gamma_G^c$ is $\mid V(\Gamma_G^c) \mid = K(G) - Z(G)$, from which it follows that $\mid V(\Gamma_G^c) \mid = 2^{n-2} + 1$. However, the degree of any vertex $v$ in $\Gamma_G^c$ is $\deg(v) = \mid V(\Gamma_G^c) \mid - 1$, thus $\deg(v) = 2^{n-2}$. The degree of $\Gamma_G^c$ is
\[ 2\mid E(\Gamma_G^c) \mid = \sum_{i=1}^{\mid V(\Gamma_G^c) \mid} \deg(v_i). \]
Therefore, $\mid E(\Gamma_G^c) \mid = 22^{n-5} + 2^n - 3$. □
Example 2.2 Suppose $G$ is a 2-group, where
$$G \cong \langle a,b : a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle.$$ If $G$ acts on itself by conjugation, then $\Gamma^c_G = K_3$.

Solution Based on Theorem 2.10, the number of vertices is three and referring to Proposition 2.2, the number of edges is three. Therefore, the graph is a complete graph of $K_3$.

According to Theorem 2.10, the following remark can be stated.

Remark The graph in Theorem 2.10 is a Hamiltonian graph, since the degree of any vertex is at least half of the number of vertices.

Theorem 2.11 Let $G$ be a finite non-abelian 2-group, where
$$G \cong \langle a,b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle.$$ If $G$ acts on itself by conjugation, then $|V(\Gamma^c_G)| = 32^{n-2}$ and $|E(\Gamma^c_G)| = \frac{3}{2}(32^{n-4} - 2^{n-2})$.

Proof The number of vertices in $\Gamma^c_G$ are $|V(\Gamma^c_G)| = K(G) - Z(G)$, thus $|V(\Gamma^c_G)| = 32^{n-2}$. The degree of any vertex in $\Gamma^c_G$ is
$$deg(v) = |V(\Gamma^c_G)| - 1.$$ Thus $deg(v) = 32^{n-2} - 1$. Meanwhile, the degree of $\Gamma^c_G$ is
$$d(\Gamma^c_G) = \sum_{v=1}^{V(\Gamma^c_G)} deg(v_i) = 2|E(\Gamma^c_G)|.$$ Therefore
$$d(\Gamma^c_G) = \sum_{v=1}^{32^{n-2}} 32^{n-2} - 1 = 2|E(\Gamma^c_G)|.$$ Thus, $|E(\Gamma^c_G)| = \frac{3}{2}(2^{2n-4} - 2^{n-2})$. 

Example 2.3 Suppose $G \cong \langle a,b : a^{2^3} = b^2 = e, ab = ba^5 \rangle$. If $G$ acts on itself by conjugation, then $\Gamma^c_G = K_6$. 


Solution  Based on Theorem 2.11, the number of vertices is six. Thus, the number of edges is \( |E(\Gamma_G^c)| = \frac{3}{2}(12 - 2) \). It follows that \( |E(\Gamma_G^c)| = 15 \). From these, it follows that \( \Gamma_G^c = K_6 \).

Remark  The graph in Theorem 2.11 is a Hamiltonian graph since \( \text{deg}(v) \geq \frac{|V(\Gamma_G^c)|}{2} \).

2.4 CONCLUSION

In this chapter, we found the orbit graph for some finite 2-groups. Besides, the chromatic number, clique number, independent number, and dominating number were all found for the groups. The number of edges of the graph was computed and the graph for 2-groups mentioned in this chapter is Hamiltonian. In addition, we found the conjugacy class graph for the groups mentioned in this chapter.

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REFERENCES


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Precise Value of the Orbit Graph and Conjugacy Class Graph


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3

The Degree of a Product of Two Subgroups of Dihedral Groups

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3.1 INTRODUCTION

Probabilistic group theory is one of the oldest areas in group theory and plays a major role in determining the abelianness of the group. It has been the center of attention to many authors and that lead us to extend their works and study this concept. If $G$ is a finite group, then the probability that two randomly chosen elements of a finite group $G$ commute is known as the commutativity degree of $G$, denoted by $P(G)$.

The first appearance of this concept was in 1994 by Miller [1]. Then, at the end of the 60s both researchers Erdos and Turan [2] introduced the concept of the commutativity degree for the symmetric groups. The commutativity degree can be generalized and modified in many directions. For instance, two subgroups $H$ and $K$ of $G$ permute if $H=K$. Hence, by changing the role of elements to subgroups in a finite group, one can obtain a modification of the commutativity degree of a finite group.

For any finite group $G$, if $H$ is a subgroup of $G$, then the relative commutativity degree of $G$, denoted by $P(H, G)$, is the probability for an element of $H$ commutes with an element of $G$. This concept was first introduced by Erfanian et al. [3]. Similarly, if $K$ is another subgroup of $G$ then the probability for an element of $H$ to commute
to an element of $K$, is denoted by $P(H, K)$. This probability is called the relative commutativity degree of two subgroups of a finite group.

Meanwhile, in 2011, Erfanian et al. [4] defined the relative $n$-th commutativity degree as the probability that the $n$-th power of a random element of $H$ commutes with a random element of $G$. Later, in 2012, Abdul Hamid [5] computed the relative commutativity degree of some dihedral groups. This relative commutativity degree can be extended to the degree of a product of two subgroups of $G$ and denoted as $P_{G,S}(H, K)$. This can be written as,

$$P_{G,S}(H, K) = \frac{|HK \cap KH|}{|HK \cup KH|}.$$

In this paper, the degree of a product of two subgroups is computed and focused only on the dihedral groups which is denoted as $P_{D_{2n},S}(H, K)$. It can easily be seen that $P_{G,S}(H, K) = 1$ if $H$ and $K$ are trivial or center of a group. It will also give the same result if $H = K$.

### 3.2 PRELIMINARIES

In this section, some definitions and theorems used in this chapter are listed below.

**Definition 3.1** [6] *Dihedral Groups of Degree $n$.* For each $n \in \mathbb{Z}$, and $n \geq 2$, $D_{2n}$ is denoted as the set of symmetries of a regular $n$-gon. Furthermore, the order of $D_{2n}$ is $2n$ or equivalently $|D_{2n}| = 2n$. The dihedral groups, $D_{2n}$ can be represented in a form of generators and relations given in the following representation:

$$D_{2n} = \langle x, y | x^n = 1, y^2 = 1, yx = x^{-1}y \rangle.$$

**Definition 3.2** [2] *The Commutativity Degree of a Group $G$.* Let $G$ be a finite group. The commutativity degree of a group $G$ is
denoted by $P(G)$ which is

$$P(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \ni xy = yx}{\text{Total number of ordered pairs } (x, y) \in G \times G} = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$  

**Definition 3.3[3]** The relative commutativity degree of a subgroup $H$ of a group $G$ is defined as:

$$P(H, G) = \frac{|\{h, g \in H \times G | hg = gh\}|}{|H||G|}.$$  

**Definition 3.4[7]** The probability that the commutator of two subgroups elements is equal to an element of a group $G$ is defined as:

$$Pr_{x}(H, K) = \frac{|\{(h, k) \in H \times K : hkh^{-1}k^{-1} = g\}|}{|H||K|}.$$  

Next, the Cayley Table for the dihedral group of order 10, $D_{10}$ and the dihedral group of order 12, $D_{12}$ are given as in Table 3.1 and Table 3.2, respectively will be used in the next section.

### 3.3 RESULTS

In this section, the generalization of $P_{D_{2n}^{\ast}}(H, K)$ are presented through the following propositions and theorems. Furthermore, some examples are given to illustrate our results.

**Proposition 3.1** Let $G$ be a dihedral group of order $2n$ and $n$ is odd. Suppose $H$ and $K$ are subgroups of $D_{2n}$. If $|H| = 2$ and $|K| = 2$ then $P_{D_{2n}^{\ast}}(H, K) = \frac{3}{5}$.

**Proof** Suppose $H$ and $K$ are subgroups of $D_{2n}$ for $n$ is odd and $e, x$ be two elements in $H$ and $e, y$ elements in $K$ that is

$$H = \{e, x\} \text{ and } K = \{e, y\}.$$
Then $HK = \{e, y, x, xy\}$ and $KH = \{e, x, y, yx\}$. Therefore, $HK \cap KH = \{e, y, x\}$ implies $|HK \cap KH| = 3$. Meanwhile, $HK \cup KH = \{e, y, x, xy, yx\}$ implies $|HK \cup KH| = 5$ since $xy \neq yx$. Thus, by definition, $P_{D_{2n}, s}(H, K) = \frac{3}{5}$.

**Example 3.1** Given $G$ is dihedral group of order 10, $D_{10}$. Let $H = \{e, ab\}$ and $K = \{e, a^2b\}$ since $|H| = 2$ and $|K| = 2$. By referring to Table 3.1, $HK = \{e, a^2b, ab, a^4\}$ and $KH = \{e, ab, a^2b, a\}$. Since $ab * a^2b \neq a^2b * ab$, then $|HK \cap KH| = 3$. Clearly $|HK \cup KH| = 5$. Thus, $P_{D_{2n}, s}(H, K) = \frac{3}{2}$.

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**Table 3.1** 0,1-table for $D_{10}$

**Theorem 3.1** If $H$ is a normal subgroup of $D_{2n}$ and $K \leq D_{2n}$ then $P_{D_{2n}, s}(H, K) = 1$.

**Proof** Let $H$ be a normal subgroup and $K$ be another subgroup of dihedral groups. Suppose $h \in H$, $k \in K$ and $H < D_{2n}$. Thus $gh = hg$ for all $g \in D_{2n}$. Since $k \in K$ implies $k \in D_{2n}$. Hence $hk = kh$. This gives $P_{D_{2n}, s}(H, K) = 1$. □
Example 3.2 Given $G$ is dihedral group of order 12, $D_{12}$. Let

$$H = \{e, a, a^2, a^3, a^4, a^5\}$$

which is a normal subgroup of $G$ and $K = \{e, a^3b\}$. By referring to Table 3.2, $HK = \{e, a^3b, a, a^4b, a^2, a^5b, a^3, b, a^4, ab, a^5, a^2b\}$ and $KH = \{e, a, a^2, a^3, a^4, a^5b, a^3b, a^2b, ab, b, a^5b, a^4b\}$. By using definition, $|HK \cap KH| = 12$ and $|HK \cup KH| = 12$ implies

$$\frac{|HK \cup KH|}{|HK \cap KH|} = 1.$$ 

Thus, $P_{D_{2n},S}(H, K) = 1$.

Theorem 3.2 Let $H$ and $K$ be any cyclic subgroups of dihedral group of order $2p$, where $p$ is prime. If $|H| = 1$ or $p$ or $|K| = 1$ or $p$, then $P_{D_{2n},S}(H, K) = 1$. Furthermore, if $|H| = 2$ and $|K| = 2$ then $P_{D_{2n},S}(H, K) = \frac{3}{5}$.

Proof Suppose $|H| = 1$ or $p$. There is only a subgroup of dihedral group of order $2p$ whose element is one i.e identity and clearly $p$ belongs to normal subgroup. Clearly, if $H$ is identity and $K \leq D_{2p}$ then $P_{D_{2n},S}(H, K) = 1$ and if $H$ is a normal subgroup of $D_{2p}$ and $K \leq D_{2p}$, then $P_{D_{2n},S}(H, K) = 1$. Furthermore, by Proposition 3.1, if $H \leq D_{2p}$ and $K \leq D_{2p}$ where $|H| = 2$ and $|K| = 2$ then $P_{D_{2n},S}(H, K) = \frac{3}{5}$. □

Example 3.3 Given $G$ is dihedral group of order 10, $D_{10}$. Let $H = \{e, a, a^2, a^3, a^4\}$ since $|H| = p$ and let $K$ be another subgroup i.e $K = \{e, ab\}$. Then Table 3.1 gives $HK = \{e, ab, a, a^2b, a^2, a^3b, a^3, a^4b, a^4, b\}$ meanwhile

$$KH = \{e, a, a^2, a^3, a^4, ab, b, a^4b, a^3b, a^2b\}.$$ 

By using the definition, $|HK \cap KH| = 10$ and $|HK \cup KH| = 10$, which implies

$$\frac{|HK \cup KH|}{|HK \cap KH|} = 1.$$
Table 3.2: 0-1-table for $D_{12}$

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Thus $P_{D_{2n},S}(H, K) = 1$. Similar result holds if $H = \{e\}$ since the identity commute with every elements. Furthermore, suppose $H = \{e, b\}$ and $K = \{e, ab\}$ since $|H| = 2$ and $|K| = 2$. From Table 3.1, $HK = \{e, ab, b, a^4\}$ and $KH = \{e, b, ab, a\}$. Since the element $b$ and $ab$ do not commute, thus $|HK \cap KH| = 3$ and $|HK \cup KH| = 5$. Therefore, $P_{D_{2n},S}(H, K) = \frac{3}{2}$.

Next, we give some manual calculation for the case of the dihedral groups, $D_{2n}$ where $n$ is even and $H, K \subseteq G$ where $|H| = |K| = 2$.

**Example 3.4** Let $n = 4$, so we have $$D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$ Suppose $H$ is in the form of $\{e, a^i b\}$ and $K$ is in the form of $\{e, a^{2+i} b\}$ where $0 \leq i \leq 2n$. So when

\begin{align*}
i = 0 & \Rightarrow H = \{e, b\} \text{ and } K = \{e, a^2b\}, \\
i = 1 & \Rightarrow H = \{e, ab\} \text{ and } K = \{e, a^3b\}, \\
i = 2 & \Rightarrow H = \{e, a^2b\} \text{ and } K = \{e, b\}, \\
i = 3 & \Rightarrow H = \{e, a^3b\} \text{ and } K = \{e, ab\}, \\
i = 4 & \Rightarrow H = \{e, b\} \text{ and } K = \{e, a^2b\}, \\
i = 5 & \Rightarrow H = \{e, ab\} \text{ and } K = \{e, a^3b\}, \\
i = 6 & \Rightarrow H = \{e, a^2b\} \text{ and } K = \{e, b\}, \\
i = 7 & \Rightarrow H = \{e, a^3b\} \text{ and } K = \{e, ab\}, \\
i = 8 & \Rightarrow H = \{e, b\} \text{ and } K = \{e, a^2b\}.
\end{align*}

From the values of $i$ given, there are two pairs of subgroups which are:

(a) $H = \{e, b\}$ and $K = \{e, a^2b\}$.
(b) $H = \{e, ab\}$ and $K = \{e, a^3b\}$.

If $H = \{e, b\}$ and $K = \{e, a^2b\}$, then $HK = \{e, a^2b, b, a^2\}$ and $KH = \{e, b, a^2b, a^2\}$. Therefore

$$|HK \cap KH| = 4 |HK \cup KH| = 4.$$
Thus, by definition,

\[ P_{D_{2n},S} (H, K) = \frac{|HK \cap KH|}{|HK \cup KH|} = 1. \]

Similarly, if \( H = \{e, ab\} \) and \( K = \{e, a^3b\} \), then

\[ HK = \{e, a^3b, ab, a^2\} \quad \text{and} \quad KH = \{e, ab, a^3b, a^2\} \]

implies \( |HK \cap KH| = 4 \) and \( |HK \cup KH| = 4 \). Thus,

\[ P_{D_{2n},S} (H, K) = \frac{|HK \cap KH|}{|HK \cup KH|} = 1. \]

**Example 3.5** Let \( n = 6 \), so we have

\[ D_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}. \]

So we have,

\[ i = 0 \Rightarrow H = \{e, b\} \quad \text{and} \quad K = \{e, a^3b\}, \]
\[ i = 1 \Rightarrow H = \{e, ab\} \quad \text{and} \quad K = \{e, a^4b\}, \]
\[ i = 2 \Rightarrow H = \{e, a^2b\} \quad \text{and} \quad K = \{e, a^5b\}, \]
\[ i = 3 \Rightarrow H = \{e, a^3b\} \quad \text{and} \quad K = \{e, b\}, \]
\[ i = 4 \Rightarrow H = \{e, a^4b\} \quad \text{and} \quad K = \{e, ab\}, \]
\[ i = 5 \Rightarrow H = \{e, a^5b\} \quad \text{and} \quad K = \{e, a^2b\}, \]
\[ i = 6 \Rightarrow H = \{e, b\} \quad \text{and} \quad K = \{e, a^3b\}, \]
\[ i = 7 \Rightarrow H = \{e, ab\} \quad \text{and} \quad K = \{e, a^4b\}, \]
\[ i = 8 \Rightarrow H = \{e, a^2b\} \quad \text{and} \quad K = \{e, a^5b\}, \]
\[ i = 9 \Rightarrow H = \{e, a^3b\} \quad \text{and} \quad K = \{e, b\}, \]
\[ i = 10 \Rightarrow H = \{e, a^4b\} \quad \text{and} \quad K = \{e, ab\}, \]
\[ i = 11 \Rightarrow H = \{e, a^5b\} \quad \text{and} \quad K = \{e, a^2b\}, \]
\[ i = 12 \Rightarrow H = \{e, b\} \quad \text{and} \quad K = \{e, a^3b\}. \]

In this case, we only have three pairs of subgroups which are:

- (a) \( H = \{e, b\} \) and \( K = \{e, a^3b\} \).
- (b) \( H = \{e, ab\} \) and \( K = \{e, a^4b\} \).
(c) $H = \{ e, a^2 b \}$ and $K = \{ e, a^5 b \}$.

If $H = \{ e, b \}$ and $K = \{ e, a^3 b \}$, then by referring to Table 3.2, $HK = \{ e, a^3 b, b, a^3 \}$ and $KH = \{ e, b, a^3 b, a^3 \}$. Therefore, $|HK \cap KH| = 4$ and $|HK \cup KH| = 4$. Thus, by definition,

$$P_{D_{2n}, S}(H, K) = \frac{|HK \cap KH|}{|HK \cup KH|} = 1.$$

By using the same method, we found that $P_{D_{2n}, S}(H, K) = 1$ for $n = 8, 10, 12$ and 14. Thus, we conclude the manual calculations with the following conjecture:

**Conjecture 1** Let $G$ be the dihedral groups of order $2n$ and $n$ is even. Suppose $H$ and $K$ are subgroups of $D_{2n}$. If $|H| = |K| = 2$ and $H$ is in the form of $\{ e, a^i b \}$ and $K$ is in the form of $\{ e, a^{\frac{n}{2}+i} b \}$ where $0 \leq i \leq 2n$, then $P_{D_{2n}, S}(H, K) = 1$.

### 3.4 CONCLUSION

In this chapter, the degree of a product of two subgroups of dihedral groups has been obtained. From the results obtained, it is shown that $P_{D_{2n}, S}(H, K)$ is equal to either 1 or $3/5$ for dihedral groups of any order.

**Acknowledgments**

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REFERENCES


4
The Probability That an Element of a Group Fixes a Set and Its Application in Graph Theory
Nor Haniza Sarmin and Sanaa Mohamed Saleh Omer

4.1 INTRODUCTION

The probability that two random elements in a group commute is called the commutativity degree. This concept has been generalized by many authors. One of these generalizations is the probability that a group element fixes a set which is our scope in this chapter. In this chapter, the probability that an element of a group fixes a set is found for some finite groups.

Throughout this chapter, $G$ denotes a finite non-abelian group. The determination of the abelianness of a non-abelian group was firstly introduced by Erdos and Turan [1] who worked on symmetric groups. Few years later, Gustafson [2] and MacHale [3] used this concept for finite groups and showed that the probability is less than or equal to 5/8. However, various researches have later been done on this topic and more results have been obtained. The probability that a random element in a group commute with another one in the same group is denoted as the following ratio:

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$ 

This probability has been used by several authors in various aspects.
of group theory. It is clear that this probability is equal to one if and only if the group is abelian. Intensive researches have been done for finding the commutativity degree for various groups. In the following context, we state some basic concepts that are needed in this chapter. These basic concepts can be found in one of the references [4, 5].

**Definition 4.1** [4] The set of all positive integers less than \( m \) and relatively prime to \( m \) is called a group under multiplication modulo \( m \) and is denoted by \( U(m) \).

**Definition 4.2** [4] A group under addition modulo \( n \) is denoted by \( \mathbb{Z}_n \) where \( n \geq 1 \).

**Definition 4.3** [4] The external direct product is a collection of finite groups defined as follows:

\[
G_1 \oplus G_2 \oplus \ldots \oplus G_n = \{(g_1, g_2, \ldots, g_n) : g_i \in G_i\}
\]

**Definition 4.4** [5] Let \( G \) be a finite Rusin group,

\[
G \cong \langle a, b : a^{2^p} = b^m = e, bab^{-1} = a^r \rangle,
\]

where \( m | (p - 1) \) and \( s^j \equiv 1 \mod p \) iff \( m | j \).

In the following, we state the definition of dicyclic group and its generalization, namely generalized quaternion group.

**Definition 4.5** If \( G \) a finite non-abelian dicyclic group, then \( G \) has the following presentation \( G \cong \langle a, b : a^{2^\beta} = b^2 = e, b^{-1}ab = a^{-1}, a^\beta = b^2 \rangle \).

In the case that \( \beta = 2^{q-1} \), the dicyclic group is generalized to quaternion group defined as follows.

**Definition 4.6** Let \( G \) be a generalized quaternion group, \( Q_{2^{q+1}} \). Then \( G \cong \langle a, b : a^{2^q} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{q-1}} = b^2 \rangle \).

In this chapter, we provide some examples, which help the reader to have a fully understanding of the concept that is under discussion.
4.2 PRELIMINARIES

This section is divided into two parts, the first part presents some previous researches related to the commutativity degree; in particular the probability that an element fixes a set or a subgroup element. Meanwhile, the second part focus on the graph theory, where some earlier and recent results are provided.

4.2.1 The Probability That an Element of a Group Fixes a Set

In this part, we state some information related to this chapter.

A new concept introduced by Sherman [6] in 1975, namely the probability of an automorphism of a finite group fixes an arbitrary element in the group is given in the following.

**Definition 4.7** [6] Let $G$ be a group. Let $X$ be a non-empty set of $G$ (i.e., $G$ is a group of permutations of $X$). Then the probability of an automorphism of a group fixes a random element from $X$ is defined as follows:

$$P_G(X) = \frac{|\{(g, x) | g x = x \ \forall \ g \in G, x \in X\}|}{|X||G|}.$$  

In 2011, Moghaddam et al. [7] explored Shermans definition and introduced a new probability which is called the probability of an automorphism fixes a subgroup element of a finite group. This probability is stated as follows:

$$P_{A_G}(H, G) = \frac{|\{(\alpha, h) | h^\alpha, h \in H, \alpha \in A_G\}|}{|H||G|},$$

where $h$ is a fixed element. It is obvious that when $H = G$, then $P_{A_G}(G, G) = P_{A_G}(G)$. Among other results, some upper and lower bounds were obtained (see [7] for more details).

Omer et al. [8] found the probability that an element of a group fixes a set of size two of commuting element in $G$. Their results are listed in the following.
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Definition 4.8[8] Let $G$ be a group. Let $S$ be a set of all subsets of commuting elements of size two in $G$, where $G$ acts on $S$ by conjugation. Then the probability of an element of a group fixes a set is given as follows:

$$P_G(S) = \frac{|\{ (g,s) | gS = S \, \forall \, g \in G, s \in S \}|}{|S||G|}.$$

Theorem 4.1[8] Let $G$ be a finite group and let $X$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $S$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $S$ by conjugation. Then the probability that an element of a group fixes a set is given by:

$$P_G(S) = \frac{K}{|S|},$$

where $K$ is the number of conjugacy classes of $S$ in $G$.

Moreover, they extended their results where they found the above probability for some finite non-abelian 2-groups [9].

4.2.2 Graph Theory

In the subsection a brief information about some fundamental concepts related to graph. Starting with definition of empty graph.

Definition 4.9[10] The graph $\Gamma$ is an empty graph, if there is no adjacent (edges) between its vertices. In this chapter, $K_e$ denotes the empty graph.

Definition 4.10[10] The graph $\Gamma$ is called null if it has no vertices, denoted by $K_0$

Definition 4.11[10] A complete graph is a graph where each ordered pair of distinct vertices are adjacent, and it is denoted by $K_n$, where $n$ is the number of connected vertices.
The following proposition is used to find the degree of vertex in a graph.

**Proposition 4.1**[10] Let $G$ be a finite group and $\Gamma$ be its graph. The degree of $v \in V(\Gamma)$ in $\Gamma$ is $\deg(v) = |V(\Gamma)| - 1$.

Next, some previous works on graph theory that are used in this chapter is provided. In 1990, Bertram et al. [11] introduced a graph which is called **conjugacy class graph**. The vertices of this graph are non-central conjugacy classes, where two vertices are adjacent if the cardinalities are not coprime. Recently, Bianchi et al. [12] studied the regularity of the graph related to conjugacy classes and provided some results. Moreto et al. [13] classified the finite groups that their conjugacy classes lengths are set-wise relatively prime for any five distinct classes.

Recently, Omer et al. [14] extended the work in [11] by defining the generalized conjugacy class graph whose vertices are non-central orbits under groups action on set. The following is the definition of generalized conjugacy class graph.

**Definition 4.12**[14] Let $G$ be a finite group and $\Omega$ a set of $G$. Let $A$ be the set of commuting element in $\Omega$, i.e $\{\omega \in \Omega : \omega g = g \omega, g \in G\}$. Then the generalized conjugacy class graph $\Gamma_G^{\Omega_c}$ is defined as a graph whose vertices are non-central orbits under group action on a set, that is $V(\Gamma_G^{\Omega_c}) = K(\Omega) - A$. Two vertices $\omega_1$ and $\omega_2$ in $\Gamma_G^{\Omega_c}$ are adjacent if their cardinalities are not coprime, i.e $\gcd(\omega_1, \omega_2) \neq 1$.

Later, Erfanian and Tolue [15] introduced a new graph which is called a conjugate graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate.

Furthermore, the conjugate graph has been generalized by Omer et al. [16], where they found the graph and its properties under some group actions on a set. They also introduced the orbit graph in [16]. The definition of the **orbit graph** is stated in the following:
**Definition 4.13**[16] Let $G$ be a finite group and $\Omega$ be a set of elements of $G$. Let $A$ be the set of commuting elements in $\Omega$, i.e. $A = \{v \in \Omega : vg = gv, g \in G\}$. The orbit graph $\Gamma_G^\Omega$ consists of two sets, namely vertices and edges denoted by $V(\Gamma_G^\Omega)$ and $E(\Gamma_G^\Omega)$, respectively. The vertices of $\Gamma_G^\Omega$ are non central elements in $\Omega$ but not in $A$, that is $V(\Gamma_G^\Omega) = \Omega - A$, while the number of edges are

$$|E(\Gamma_G^\Omega)| = \sum_{i=1}^{\left|V(\Gamma_G^\Omega)\right|} \binom{v_i}{2},$$

where $v$ is the size of orbit under group action of $G$ on $\Omega$. Two vertices $v_1, v_2$ are adjacent in $\Gamma_G^\Omega$ if one of the following conditions is satisfied.

(a) If there exists $g \in G$ such that $gv_1 = v_2$.
(b) If the vertices of $\Gamma_G^\Omega$ are conjugate that is, $v_1 = gv_2$.

In 2012, Ilangoon and Sarmin [17], found some graph properties of graph related to conjugacy classes of two-generator two-groups of class two.

Recently, Moradipour et al. [18] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups.

### 4.3 MAIN RESULTS

This section contains two subsections. In the first subsection, we compute the probability that an element of a group fixes a set. While, the orbit graph and graph related to conjugacy classes are found in the second subsection.

#### 4.3.1 The Probability That a Group Element Fixes a Set

In this section, we find the probability that an element of $G$ fixes a set. Some theorems are provided and supported by some examples. First, we start with Rusin group, then followed by dicyclic group and its generalization called generalized quaternion group.
The Probability That an Element of a Group Fixes a Set

**Theorem 4.2** Let $G$ be a finite non-abelian Rusin group,

$$G \cong \langle a, b : a^{2^m} = b^m = e, bab^{-1} = a^3 \rangle,$$

where $m|(p - 1)$ and $s^j \equiv 1 \mod p$ iff $m|j$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then

$$P_G(\Omega) = 1, \text{ if } m \text{ is even.}$$

**Proof** If $m$ is odd, the probability cannot be obtained, since there is no element of size two in $\Omega$. In the case that $m$ is even, the element of $\Omega$ of size two is only the elements in the form $(1, a^{2^m})$. Thus, when $G$ acts on $\Omega$ by conjugation, then there is only one conjugacy class namely $\Omega$. The proof then follows. \hfill \Box

**Example 4.1** Let $G$ be a Rusin group,

$$G \cong \langle a, b : a^{2^{1/3}} = b^3 = e, bab^{-1} = a^5 \rangle,$$

where $m|(p - 1)$ and $s^j \equiv 1 \mod p$ iff $m|j$. If $G$ acts on $\Omega$ by conjugation, then $P_G(\Omega) = 1$.

**Solution** There is only one element in $\Omega$, namely $\Omega$ itself thus when $G$ acts on $\Omega$ by conjugation, $P_G(\Omega) = 1$.

**Theorem 4.3** Let $G$ be a finite non-abelian dicyclic group,

$$G \cong \langle a, b : a^{2^\beta} = b^4 = e, b^{-1}ab = a^{-1}, a^\beta = b^2 \rangle.$$

Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$. 

---
Proof Since \( \Omega \) is the set of all subsets of commuting elements of size two in \( G \), then there is only one element in \( \Omega \) namely \((1, a^{\beta_1})\). If \( G \) acts on \( \Omega \) by conjugation, then we have only one conjugacy class, which is \( \Omega \) itself. The proof then follows. \( \square \)

Example 4.2 Let \( G \) be a finite non-abelian dicyclic group,

\[
G \cong \langle a, b : a^6 = b^4 = e, b^{-1}ab = a^{-1}, a^3 = b^2 \rangle.
\]

If \( G \) acts on \( \Omega \), then \( P_G(\Omega) = 1 \).

Solution According to this presentation, there is only one element in \( \Omega \) which is \((1, a^3)\). In the case that \( G \) acts on \( \Omega \) by conjugation, there is only one element, namely \((1, a^3)\). Therefore, \( P_G(\Omega) = 1 \).

The generalized quaternion group is a dicyclic group with \( \beta = 2^{a-1} \). In the following, the probability that an element of a generalized quaternion group, namely \( Q_{2n+1} \), fixes a set is computed.

Theorem 4.4 Let \( G \) be a generalized quaternion group, \( Q_{2n+1} \),

\[
G \cong \langle a, b : a^{2^\alpha} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{\alpha-1}} = b^2 \rangle.
\]

Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( P_G(\Omega) = 1 \).

Proof The proof is similar with Theorem 4.3. \( \square \)

Theorem 4.5 Let \( G \) be a finite group, \( G \cong U(m), m \in \mathbb{N} \). Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts regularly on \( \Omega \). Then \( P_G(\Omega) = \frac{K(\Omega)}{\lvert \Omega \rvert} \), where \( K(\Omega) \) is the number of conjugacy classes of \( \Omega \).
Proof Since $\Omega$ is the set of all subsets of commuting elements of size two, thus the elements in $\Omega$ are in the form of $(1, a), (1, b)$ and $(a, b)$, where $a, b$ are relatively prime to $m$ and commute. By the regular action of $G$ on $\Omega$, there exists $g \in G, \omega_1, \omega_2 \in \Omega$ such that $g\omega_1 = \omega_2$. Hence, $cl(\omega) = \{ g\omega : g \in G \}$. It follows that the number of conjugacy classes are $K(\Omega)$. According to [8], $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$. \hfill $\square$

Example 4.3 Suppose $G \cong U(8)$ and $\Omega$ be the set of all subsets of commuting elements in $U(8)$. If $G$ acts regularly on $\Omega$, find the probability that $g \in G$ acts on $\Omega$.

Solution The elements of $U(8) = \{1, 3, 5, 7\}$. Thus, the elements of $\Omega$ are stated as follows:

$$\Omega = \{(1, 3), (1, 5), (1, 7), (3, 5), (3, 7), (5, 7)\}.$$ 

If $G$ acts on $\Omega$, the conjugacy classes are described as follows:

$$cl((1, 3)) = \{(1, 3), (5, 7)\},$$
$$cl((1, 5)) = \{(1, 5), (3, 7)\},$$
$$cl((1, 7)) = \{(1, 7), (3, 5)\}.$$ 

It follows that $K(\Omega) = 3$. Therefore, $P_G(\Omega) = \frac{1}{2}$.

Theorem 4.6 Let $G$ be a finite group, $G \cong U(n), n \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.

Proof We know that $\Omega$ is the set of all subsets of commuting elements of size two, thus the elements in $\Omega$ are in the form $(1, a), (1, b)$ and $(a, b)$, where $a, b$ are relatively prime to $n$ and commute. Since $G$ acts on $\Omega$ by conjugation, then exists $g \in G, \omega_1, \omega_2 \in \Omega$ such that $g\omega_1 g^{-1} = \omega_2$. Hence, $cl(\omega) = \{ g\omega g^{-1} : g \in G \}$. Since all elements are relatively prime to $n$ and are all of size two, then $cl(\omega) = \omega \ \forall \omega \in \Omega$. It follows that $P_G(\Omega) = 1$. \hfill $\square$
Example 4.4  Suppose $G \cong U(8)$ and $\Omega$ be the set of all subsets of commuting elements in $U(8)$. If $G$ acts on $\Omega$ by conjugation, find the probability that $g \in G$ acts on $\Omega$.

Solution  The elements of $U(8) = \{1, 3, 5, 7\}$. Thus, the elements of $\Omega$ are stated as follows

$$\Omega = \{(1, 3), (1, 5), (1, 7), (3, 5), (3, 7), (5, 7)\}.$$  

If $G$ acts on $\Omega$ by conjugation, the conjugacy classes described as follows:

$$cl((1, 3)) = \{(1, 3)\},$$
$$cl((1, 5)) = \{(1, 5)\},$$
$$cl((1, 7)) = \{(1, 7)\},$$
$$cl((3, 5)) = \{(3, 5)\},$$
$$cl((3, 7)) = \{(3, 7)\},$$
$$cl((5, 7)) = \{(5, 7)\}.$$

It follows that $K(\Omega) = 6$. Based on Definition 4.8, $P_G(\Omega) = 1$.

Theorem 4.7  Let $G$ be a finite group, $G \cong U(n) \oplus U(m)$, $n, m \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$.

Proof  First, we find the elements of $\Omega$. Since elements of $\Omega$ are of size two, then $|\omega| = lcm(|g_1|, |g_2|) = 2$, where $g_1 \in U(n), g_2 \in U(m)$ thus this case is reduced to the same problem as in Theorem 4.5. In the case that $G$ acts regularly on $\Omega$, the proof then follows Theorem 4.5.  

Example 4.5  Suppose $G \cong U(3) \oplus U(4)$. If $G$ acts regularly on $\Omega$, find the probability that $g \in G$ fixes $\Omega$.  


Solution Since \( U(3) = \{1, 2\} \) and \( U(4) = \{1, 3\} \), thus the elements of \( G = \{(1, 1), (1, 3), (2, 1), (2, 3)\} \), and the elements of \( \Omega = \{(1, 3), (2, 1), (2, 3)\} \). If \( G \) acts regularly on \( \Omega \), there exists \( g \in G \) such that \( gw \in \Omega \). Therefore, the conjugacy classes are

\[
cl((1, 3)) = \{g(1, 3) : g \in G\} = \{(1, 3), (2, 1), (2, 3)\}.
\]

Hence \( cl((1, 2)) = cl((2, 3)) = cl(1, 3) \). Thus \( K(\Omega) = 1 \). It follows that, \( P_G(\Omega) = 1/3 \).

**Theorem 4.8** Let \( G \) be a finite group, \( G \cong U(n) \oplus U(m), n, m \in \mathbb{N} \). Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( P_G(\Omega) = 1 \).

**Proof** The proof is similar to that of Theorem 4.6.

**Example 4.6** Suppose \( G \cong U(3) \oplus U(4) \). If \( G \) acts on \( \Omega \) by conjugation, find the probability that \( g \in G \) fixes \( \Omega \).

Solution The elements of \( G = \{(1, 1), (1, 3), (2, 1), (2, 3)\} \), thus the elements of \( \Omega = \{(1, 3), (2, 1), (2, 3)\} \). If \( G \) acts on \( \Omega \) by conjugation, then \( cl(\omega) = \{g^{-1}\omega g : g \in G\} \). Therefore, the conjugacy classes are \( cl((1, 3)) = \{g^{-1}(1, 3)g : g \in G\} = \{(1, 3)\} \), \( cl((2, 1)) = \{g^{-1}(2, 1)g : g \in G\} = \{(2, 1)\} \) and \( cl((2, 3)) = \{g^{-1}(2, 3)g : g \in G\} = \{(2, 3)\} \). Therefore, \( K(\Omega) = 3 \). Thus, \( P_G(\Omega) = 1 \).

**Theorem 4.9** Let \( G \) be a finite group, \( G \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_{2q} \), where \( p \) and \( q \) prime numbers. Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( P_G(\Omega) = 1 \).
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Proof The order of any \( \omega \in \Omega \) is \( \omega = \text{lcm}(|g_1|,|g_2|) = 2 \), where \( g_1 \in \mathbb{Z}_{2p}, g_2 \in \mathbb{Z}_{2q} \), thus \( |g_1| = \{1, 2, p, 2p\}, |g_2| = \{1, 2, q, 2q\} \) but the order of \( \omega \) is two, thus the elements of \( \Omega \) are \( \{(0, p), (0, q), (p, q)\} \). In case that \( G \) acts on \( \Omega \) by conjugation, then the number of conjugacy classes is equal to the order of \( \Omega \). Hence, \( P_G(\Omega) = 1 \).

Example 4.7 Suppose \( G \cong \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \). Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( P_G(\Omega) = 1 \).

Solution The elements of \( G \) are \( \{(0, 0), (0, 1), (0, 2), \ldots, (5, 9)\} \). Thus, the elements in \( \Omega \) are \( \{(3, 0), (0, 5), (3, 5)\} \). When \( G \) acts on \( \Omega \) by conjugation, then \( cl(3, 0) = \{(0, 3)\}, cl(0, 5) = \{(0, 5)\} \) and \( cl(3, 5) = \{(3, 5)\} \). Based on Definition 4.8, the probability is equal to one.

Theorem 4.10 Let \( G \) be a finite group, \( G \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_{2q} \), where \( p \) and \( q \) are prime numbers. Let \( S \) be a set of elements of \( G \) of size two in the form of \( (a, b) \) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts regularly on \( \Omega \). Then \( P_G(\Omega) = \frac{1}{|\Omega|} \).

Proof The proof follows from Theorem 4.7.

Theorem 4.11 Let \( G \) be a finite group, \( G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q \), where \( p \) and \( q \) are relatively prime. Let \( S \) be a set of elements of \( G \) of size two in the form of \( (a, b) \) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then

\[
P_G(\Omega) = 1, \text{if } p \neq q = 2 \text{ and } p = q = 2.
\]

Proof The elements of \( \Omega \) are \( \{(p, 0), (0, q), (p, q)\} \). Thus in the case that \( p \neq q = 2 \), \( \Omega = \{(p/2, 0)\} \) and when \( G \) acts on \( \Omega \) by conjugation, there is only one conjugacy class, namely \( \Omega \) itself. Hence, \( P_G(\Omega) = 1 \).
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If $p \neq q \neq 2$, then there is no element of size two hence the probability cannot be computed.

**Theorem 4.12** Let $G$ be a finite group, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, where $p$ and $q$ are relatively prime. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then

$$P_G(\Omega) = \begin{cases} \frac{1}{|\Omega|}, & \text{if } p = q = 2, \\ 1, & \text{if } p \neq q = 2. \end{cases}$$

**Proof** The elements of $\Omega$ are $\{(p, 0), (0, q), (p, q)\}$. Thus in the case that $p \neq q = 2$, $\Omega = \{(p/2, 0)\}$. When $G$ acts regularly on $\Omega$ and $p = q = 2$, the proof follows from Theorem 4.7. In the case that $p \neq q = 2$ the proof follows from Theorem 4.9. However, in the case $p \neq q \neq 2$, there is no element of size two since $p$ and $q$ are relatively primes. Thus, there is no possibility to compute $P_G(\Omega)$. \(\square\)

### 4.3.2 The Orbit Graph and Generalized Conjugacy Class Graph

In this section, we find both the orbit graph and generalized conjugacy class graph based on the obtained results in the previous section, starting with the results on orbit graph.

#### 4.3.2.1 Orbit Graph

In this part, we find the orbit graph for all theorems in Section 1 and Section 2, starting with the Rusin group.

**Theorem 4.13** Let $G$ be a finite non-abelian Rusin group,

$$G \cong \langle a, b : a^{2^p} = b^m = e, bab^{-1} = a^s \rangle,$$

where $m | (p - 1) \text{ and } s^j \equiv 1 \mod p \text{ iff } m | j$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$...
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commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( \Gamma^\Omega_G \) is an empty graph.

**Proof** According to Theorem 4.2, the proof is clear since the elements of \( \Omega \) of size two is only the element \((1,a^2)\). Thus, the graph is an empty graph.

**Theorem 4.14** Let \( G \) be a finite non-abelian dicyclic group,

\[
G \cong \langle a, b : a^{2^{\beta}} = b^4 = e, b^{-1}ab = a^{-1}, a^\beta = b^2 \rangle.
\]

Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( \Gamma^\Omega_G \) is an empty graph.

**Proof** The proof is similar to that of Theorem 4.13.

Next, the orbit graph of the generalized quaternion group is found.

**Theorem 4.15** Let \( G \) be a generalized quaternion group, \( Q_{2n+1} \), \( G \cong \langle a, b : a^{2^{\alpha}} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{\alpha-1}} = b^2 \rangle \). Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two and \( G \) acts on \( \Omega \) by conjugation. Then \( \Gamma^\Omega_G \) is an empty graph.

**Proof** The proof is similar to that of Theorem 4.13.

**Theorem 4.16** Let \( G \) be a finite group, \( G \cong U(m), m \in \mathbb{N} \). Let \( S \) be a set of elements of \( G \) of size two in the form of \((a, b)\) where \( a \) and \( b \) commute. Let \( \Omega \) be the set of all subsets of commuting elements of \( G \) of size two. regularly on \( \Omega \). Then the orbit graph is an empty graph.

**Proof** The graph is empty since \( |V(\Gamma^\Omega_G)| = |\Omega| - |A| \), where \( A = \{g\omega_1 = \omega_1 \forall g \in G\} \), and since \( G \) is \( U(m) \), all elements in \( \Omega \) commute with the elements in \( G \).
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Remark There is no orbit graph for Theorems 4.6, 4.7, 4.8, 4.9, 4.10, 4.11 and 4.12 since all elements in $\Omega$ commute with the elements in $G$.

Theorem 4.17 Let $G$ be a finite group, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, where $p$ and $q$ relatively prime. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then

$$
\Gamma^\Omega_G = \begin{cases} 
K_\Omega, & \text{if } p = q = 2, \\
K_e, & \text{if } p \neq q = 2.
\end{cases}
$$

Proof The element of $\Omega$ are $\{(p, 0), (0, q), (p, q)\}$. Thus in case that $p = q = 2$, and two vertices are joined by an edge if there is $g \in G$ such that $g \omega_1 = \omega_2$. According to Theorem 4.12 there is a complete graph, namely $K_\Omega$. However, if $p \neq q = 2$, $\Omega = \{(p/2, 0)\}$ and when $G$ acts on regularly on $\Omega$, then the graph is empty since the element $\{(p/2, 0)\}$ adjacent to itself. In the second case, there is no element of size two, thus it is impossible to find the graph. However, in case three when $p \neq q \neq 2$, there is no element of size two since $p$ and $q$ relatively primes. Thus, there is no possibility to find a graph. \qed

Theorem 4.18 Let $G$ be a finite group, $G \cong U(n) \oplus U(m)$, $n, m \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then

$$
\Gamma^\Omega_G = \begin{cases} 
K_3, & \text{if } G \text{ acts regularly on } \Omega, \\
K_e, & \text{if } G \text{ acts on } \Omega \text{ by conjugation}.
\end{cases}
$$

Proof According to Theorem 4.7, there is no adjacency between elements in $\Omega$ thus the graph is empty. In the case that $G$ acts regularly on $\Omega$, thus $|V(\Gamma^\Omega_G)| = \Omega - A$. Therefore, $|V(\Gamma^\Omega_G)| = 3$. Two vertices are linked by an edge if and only if there exists $g \in G$ such that $g \omega_1 = \omega_2$. From which it follows that there is a complete graph of $K_3$. \qed
4.3.2.2 Generalized Conjugacy Class Graph

In this part, the generalized conjugacy class graph is found for all groups mentioned in the introduction section. We start with the generalized conjugacy class graph of Rusin group.

Theorem 4.19 Let $G$ be a finite non-abelian Rusin group,

$$G \cong \langle a, b : a^{2^m} = b^m = e, bab^{-1} = a^s \rangle,$$

where $m | (p - 1)$ and $s^j \equiv 1 \mod p$ iff $m | j$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $\Gamma_{G}^{\Omega} = K_e$.

Proof According to Theorem 4.2, there is only one conjugacy class. Thus, the graph is empty. \qed

Remark There is no generalized conjugacy class graph in Theorem 4.3 and 4.4, the reason is similar to that in the previous theorem. In the following, the generalized conjugacy class for $U(n)$ namely, group under multiplication modulo $n$.

Theorem 4.20 Let $G$ be a finite group, $G \cong U(m), m \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then $|V(\Gamma_{G}^{\Omega})| = K(\Omega) - |A|.

Proof According to Theorem 4.5, we find that $|V(\Gamma_{G}^{\Omega})| = K(\Omega) - |A|$. According to Proposition 4.1, $\deg(\omega) = |V(\Gamma_{G}^{\Omega})| - 1$, thus $\deg(\omega) = n$ where $n = K(\Omega) - |A|$. \qed

Remark The generalized conjugacy class graph can be obtained only if $\Omega \neq A$, since all elements in $\Omega$ are relatively prime. This restricted condition is true for the rest of the theorems.
4.4 CONCLUSION

In this chapter, the probability that a group element fixes a set is found for some finite groups mentioned in Section 4.1. As consequences of obtained results in Section 4.1, we associated the results in the probability that an element of a group fixes a set to graph theory, more precisely with orbit graph and generalized conjugacy class graph.

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