# 4

# The Probability That an Element of a Group Fixes a Set and Its Application in Graph Theory

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## 4.1 INTRODUCTION

The probability that two random elements in a group commute is called the commutativity degree. This concept has been generalized by many authors. One of these generalizations is the probability that a group element fixes a set which is our scope in this chapter. In this chapter, the probability that an element of a group fixes a set is found for some finite groups.

Throughout this chapter, $G$ denotes a finite non-abelian group. The determination of the abelianness of a non-abelian group was firstly introduced by Erdos and Turan [1] who worked on symmetric groups. Few years later, Gustafson [2] and MacHale [3] used this concept for finite groups and showed that the probability is less than or equal to 5/8. However, various researches have later been done on this topic and more results have been obtained. The probability that a random element in a group commute with another one in the same group is denoted as the following ratio:

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

This probability has been used by several authors in various aspects of group theory. It is clear that this probability is equal to one if and only if the group is abelian. Intensive researches have been done for finding the commutativity degree for various groups. In the following context, we state some basic concepts that are needed in this chapter. These basic concepts can be found in one of the references [4, 5].

**Definition 4.1 [4]** The set of all positive integers less than $m$ and relatively prime to $m$ is called a group under multiplication modulo $m$ and is denoted by $U(m)$.

**Definition 4.2 [4]** A group under addition modulo $n$ is denoted by $\mathbb{Z}_n$ where $n \geq 1$.

**Definition 4.3 [4]** The external direct product is a collection of finite groups defined as follows:

$$G_1 \oplus G_2 \oplus ... \oplus G_n = \{(g_1, g_2, ..., g_n) : g_i \in G_i\}$$

**Definition 4.4 [5]** Let $G$ be a finite Rusin group,

$$G \cong \langle a, b : a^{2^p} = b^m = e, bab^{-1} = a^s \rangle,$$

where $m|(p-1)$ and $s^j \equiv 1 \bmod p$ iff $m|j$.

In the following, we state the definition of dicyclic group and its generalization, namely generalized quaternion group.

**Definition 4.5** If $G$ a finite non-abelian dicyclic group, then $G$ has the following presentation $G \cong \langle a, b : a^{2\beta} = b^2 = e, b^{-1}ab = a^{-1}, a^{\beta} = b^2 \rangle$.

In the case that $\beta = 2^{\alpha-1}$, the dicyclic group is generalized to quaternion group defined as follows.

**Definition 4.6** Let $G$ be a generalized quaternion group, $Q_{2^{n+1}}$. Then $G \cong \langle a, b : a^{2\alpha} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{\alpha-1}} = b^2 \rangle$.

In this chapter, we provide some examples, which help the reader to have a fully understanding of the concept that is under discussion.

## 4.2 PRELIMINARIES

This section is divided into two parts, the first part presents some previous researches related to the commutativity degree; in particular the probability that an element fixes a set or a subgroup element. Meanwhile, the second part focus on the graph theory, where some earlier and recent results are provided.

### 4.2.1 The Probability That an Element of a Group Fixes a Set

In this part, we state some information related to this chapter.

A new concept introduced by Sherman [6] in 1975, namely the probability of an automorphism of a finite group fixes an arbitrary element in the group is given in the following.

**Definition 4.7 [6]** Let $G$ be a group. Let $X$ be a non-empty set of $G$ (i.e., $G$ is a group of permutations of $X$). Then the probability of an automorphism of a group fixes a random element from $X$ is defined as follows:

$$P_G(X) = \frac{|\{(g, x)|gx = x \ \ \forall \ g \in G, x \in X\}|}{|X||G|}.$$

In 2011, Moghaddam *et al.* [7] explored Shermans definition and introduced a new probability which is called the probability of an automorphism fixes a subgroup element of a finite group. This probability is stated as follows:

$$P_{A_G}(H, G) = \frac{|\{(\alpha, h)|h^{\alpha}, h \in H, \alpha \in A_G\}|}{|H||G|},$$

where $h$ is a fixed element. It is obvious that when $H$=$G$ , then $P_{A_G}(G, G) = P_{A_G}(G)$. Among other results, some upper and lower bounds were obtained (see [7] for more details).

Omer *et al.* [8] found the probability that an element of a group fixes a set of size two of commuting element in $G$. Their results are listed in the following.

**Definition 4.8 [8]** Let $G$ be a group. Let $S$ be a set of all subsets of commuting elements of size two in $G$, where $G$ acts on $S$ by conjugation. Then the probability of an element of a group fixes a set is given as follows:

$$P_G(S) = \frac{|\{(g,s)|gS = S \ \ \forall\, g \in G, s \in S\}|}{|S||G|}.$$

**Theorem 4.1 [8]** *Let $G$ be a finite group and let $X$ be a set of elements of $G$ of size two in the form of $(a,b)$ where $a$ and $b$ commute. Let $S$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $S$ by conjugation. Then the probability that an element of a group fixes a set is given by:*

$$P_G(S) = \frac{K}{|S|},$$

*where $K$ is the number of conjugacy classes of $S$ in $G$.*

Moreover, they extended their results where they found the above probability for some finite non-abelian 2-groups [9].

### 4.2.2 Graph Theory

In the subsection a brief information about some fundamental concepts related to graph. Starting with definition of empty graph.

**Definition 4.9 [10]** The graph $\Gamma$ is an empty graph, if there is no adjacent (edges) between its vertices. In this chapter, $K_e$ denotes the empty graph.

**Definition 4.10 [10]** The graph $\Gamma$ is called null if it has no vertices, denoted by $K_0$

**Definition 4.11 [10]** A complete graph is a graph where each ordered pair of distinct vertices are adjacent, and it is denoted by $K_n$, where $n$ is the number of connected vertices.

The following proposition is used to find the degree of vertex in a graph.

**Proposition 4.1 [10]** *Let $G$ be a finite group and $\Gamma$ be its graph. The degree of $v \in V(\Gamma)$ in $\Gamma$ is deg($v$)=$|V(\Gamma)| - 1$.*

Next, some previous works on graph theory that are used in this chapter is provided. In 1990, Bertram *et al.* [11] introduced a graph which is called **conjugacy class graph** . The vertices of this graph are non-central conjugacy classes, where two vertices are adjacent if the cardinalities are not coprime. Recently, Bianchi *et al.* [12] studied the regularity of the graph related to conjugacy classes and provided some results. Moreto *et al.* [13] classified the finite groups that their conjugacy classes lengths are set-wise relatively prime for any five distinct classes.

Recently, Omer *et al.* [14] extended the work in [11] by defining the generalized conjugacy class graph whose vertices are non-central orbits under groups action on set. The following is the definition of generalized conjugacy class graph .

**Definition 4.12 [14]** Let $G$ be a finite group and $\Omega$ a set of $G$. Let $A$ be the set of commuting element in $\Omega$, i.e $\{\omega \in \Omega : \omega g = g\omega, g \in G\}$. Then the generalized conjugacy class graph $\Gamma_G^{\Omega_c}$ is defined as a graph whose vertices are non-central orbits under group action on a set, that is $V(\Gamma_G^{\Omega_c}) = K(\Omega) - A$. Two vertices $\omega_1$ and $\omega_2$ in $\Gamma_G^{\Omega_c}$ are adjacent if their cardinalities are not coprime, i.e $\gcd(\omega_1, \omega_2) \neq 1$.

Later, Erfanian and Tolue [15] introduced a new graph which is called a conjugate graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate.

Furthermore, the conjugate graph has been generalized by Omer *et al.* [16], where they found the graph and its properties under some group actions on a set. They also introduced the orbit graph in [16]. The definition of the **orbit graph** is stated in the following:

**Definition 4.13 [16]** Let $G$ be a finite group and $\Omega$ be a set of elements of $G$. Let $A$ be the set of commuting elements in $\Omega$, i.e $A = \{v \in \Omega : vg = gv, g \in G\}$. The orbit graph $\Gamma_G^{\Omega}$ consists of two sets, namely vertices and edges denoted by $V(\Gamma_G^{\Omega})$ and $E(\Gamma_G^{\Omega})$, respectively. The vertices of $\Gamma_G^{\Omega}$ are non central elements in $\Omega$ but not in $A$, that is $V(\Gamma_G^{\Omega}) = \Omega - A$, while the number of edges are

$$|E(\Gamma_G^{\Omega})| = \sum_{i=1}^{|V(\Gamma_G^{\Omega})|} \binom{v_i}{2},$$

where $v$ is the size of orbit under group action of $G$ on $\Omega$. Two vertices $v_1, v_2$ are adjacent in $\Gamma_G^{\Omega}$ if one of the following conditions is satisfied.

(a) If there exists $g \in G$ such that $gv_1 = v_2$,
(b) I If the vertices of $\Gamma_G^{\Omega}$ are conjugate that is, $v_1 = g^{v_2}$.

In 2012, Ilangovan and Sarmin [17], found some graph properties of graph related to conjugacy classes of two-generator two-groups of class two.

Recently, Moradipour *et al.* [18] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups.

## 4.3 MAIN RESULTS

This section contains two subsections. In the first subsection, we compute the probability that an element of a group fixes a set. While, the orbit graph and graph related to conjugacy classes are found in the second subsection.

### 4.3.1 The Probability That a Group Element Fixes a Set

In this section, we find the probability that an element of $G$ fixes a set. Some theorems are provided and supported by some examples. First, we start with Rusin group, then followed by dicyclic group and its generalization called generalized quaternion group.

**Theorem 4.2** *Let $G$ be a finite non-abelian Rusin group,*

$$G \cong \langle a, b : a^{2^p} = b^m = e, bab^{-1} = a^s \rangle,$$

*where $m|(p-1)$ and $s^j \equiv 1$ mod $p$ iff $m|j$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then*

$$P_G(\Omega) = 1, \textit{if } m \textit{ is even}.$$

**Proof** If $m$ is odd, the probability cannot be obtained, since there is no element of size two in $\Omega$. In the case that $m$ is even, the element of $\Omega$ of size two is only the elements in the form $(1, a^{\frac{m}{2}})$. Thus, when $G$ acts on $\Omega$ by conjugation, then there is only one conjugacy class namely $\Omega$. The proof then follows. □

**Example 4.1** Let $G$ be a Rusin group,

$$G \cong \langle a, b : a^{2^1 3} = b^3 = e, bab^{-1} = a^s \rangle,$$

where $m|(p-1)$ and $s^j \equiv 1$ mod $p$ iff $m|j$. If $G$ acts on $\Omega$ by conjugation, then $P_G(\Omega) = 1$.

**Solution** There is only one element in $\Omega$, namely $\Omega$ itself thus when $G$ acts on $\Omega$ by conjugation, $P_G(\Omega) = 1$.

**Theorem 4.3** *Let $G$ be a finite non-abelian dicyclic group,*

$$G \cong \langle a, b : a^{2\beta} = b^4 = e, b^{-1}ab = a^{-1}, a^\beta = b^2 \rangle.$$

*Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.*

**Proof** Since $\Omega$ is the set of all subsets of commuting elements of size two in $G$, then there is only one element in $\Omega$ namely $(1, a^{\frac{2\beta}{2}})$. If $G$ acts on $\Omega$ by conjugation, then we have only one conjugacy class, which is $\Omega$ itself. The proof then follows. $\square$

**Example 4.2** Let $G$ be a finite non-abelian dicyclic group,

$$G \cong \langle a, b : a^6 = b^4 = e, b^{-1}ab = a^{-1}, a^3 = b^2 \rangle.$$

If $G$ acts on $\Omega$, then $P_G(\Omega) = 1$.

**Solution** According to this presentation, there is only one element in $\Omega$ which is $(1, a^3)$. In the case that $G$ acts on $\Omega$ by conjugation, there is only one element, namely $(1, a^3)$. Therefore, $P_G(\Omega) = 1$.

The generalized quaternion group is a dicyclic group with $\beta = 2^{\alpha-1}$. In the following, the probability that an element of a generalized quaternion group, namely $Q_{2^{n+1}}$, fixes a set is computed.

**Theorem 4.4** *Let G be a generalized quaternion group, $Q_{2^{n+1}}$,*

$$G \cong \langle a, b : a^{2\alpha} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{\alpha-1}} = b^2 \rangle.$$

*Let S be a set of elements of G of size two in the form of $(a, b)$ where a and b commute. Let $\Omega$ be the set of all subsets of commuting elements of G of size two and G acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.*

**Proof** The proof is similar with Theorem 4.3. $\square$

**Theorem 4.5** *Let G be a finite group, $G \cong U(m), m \in \mathbb{N}$. Let S be a set of elements of G of size two in the form of $(a, b)$ where a and b commute. Let $\Omega$ be the set of all subsets of commuting elements of G of size two and G acts regularly on $\Omega$. Then $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$, where $K(\Omega)$ is the number of conjugacy classes of $\Omega$.*

**Proof** Since $\Omega$ is the set of all subsets of commuting elements of size two, thus the elements in $\Omega$ are in the form of $(1, a)$, $(1, b)$ and $(a, b)$, where $a$, $b$ are relatively prime to $m$ and commute. By the regular action of $G$ on $\Omega$, there exists $g \in G, \omega_1, \omega_2 \in \Omega$ such that $g\omega_1 = \omega_2$. Hence, $cl(\omega) = \{g\omega : g \in G\}$. It follows that the number of conjugacy classes are $K(\Omega)$. According to [8], $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$. □

**Example 4.3** Suppose $G \cong U(8)$ and $\Omega$ be the set of all subsets of commuting elements in $U(8)$. If $G$ acts regularly on $\Omega$, find the probability that $g \in G$ acts on $\Omega$.

**Solution** The elements of $U(8) = \{1, 3, 5, 7\}$. Thus, the elements of $\Omega$ are stated as follows:

$$\Omega = \{(1, 3), (1, 5), (1, 7), (3, 5), (3, 7), (5, 7)\}.$$

If $G$ acts on $\Omega$, the conjugacy classes are described as follows:

$$cl((1, 3)) = \{(1, 3), (5, 7)\},$$
$$cl((1, 5)) = \{(1, 5), (3, 7)\},$$
$$cl((1, 7)) = \{(1, 7), (3, 5)\}.$$

It follows that $K(\Omega) = 3$. Therefore, $P_G(\Omega) = \frac{1}{2}$.

**Theorem 4.6** *Let $G$ be a finite group, $G \cong U(n), n \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.*

**Proof** We know that $\Omega$ is the set of all subsets of commuting elements of size two, thus the elements in $\Omega$ are in the form $(1, a)$, $(1, b)$ and $(a, b)$, where $a$, $b$ are relatively prime to $n$ and commute. Since $G$ acts on $\Omega$ by conjugation, then exists $g \in G, \omega_1, \omega_2 \in \Omega$ such that $g\omega_1 g^{-1} = \omega_2$. Hence, $cl(\omega) = \{g\omega g^{-1} : g \in G\}$. Since all elements are relatively prime to $n$ and are all of size two, then $cl(\omega) = \omega \; \forall \omega \in \Omega$. It follows that $P_G(\Omega) = 1$. □

**Example 4.4** Suppose $G \cong U(8)$ and $\Omega$ be the set of all subsets of commuting elements in $U(8)$. If $G$ acts on $\Omega$ by conjugation, find the probability that $g \in G$ acts on $\Omega$.

**Solution** The elements of $U(8) = \{1, 3, 5, 7\}$. Thus, the elements of $\Omega$ are stated as follows

$$\Omega = \{(1, 3), (1, 5), (1, 7), (3, 5), (3, 7), (5, 7)\}.$$

If $G$ acts on $\Omega$ by conjugation, the conjugacy classes described as follows:

$$\begin{aligned} cl((1,3)) &= \{(1,3)\}, \\ cl((1,5)) &= \{(1,5)\}, \\ cl((1,7)) &= \{(1,7)\}, \\ cl((3,5)) &= \{(3,5)\}, \\ cl((3,7)) &= \{(3,7)\}, \\ cl((5,7)) &= \{(5,7)\}. \end{aligned}$$

It follows that $K(\Omega) = 6$. Based on Definition 4.8, $P_G(\Omega) = 1$.

**Theorem 4.7** *Let G be a finite group, $G \cong U(n) \oplus U(m), n, m \in \mathbb{N}$. Let S be a set of elements of G of size two in the form of $(a, b)$ where a and b commute. Let $\Omega$ be the set of all subsets of commuting elements of G of size two and G acts regularly on $\Omega$. Then $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$.*

**Proof** First, we find the elements of $\Omega$. Since elements of $\Omega$ are of size two, then $|\omega| = lcm(|g_1|, |g_2|) = 2$, where $g_1 \in U(n), g_2 \in U(m)$ thus this case is reduced to the same problem as in Theorem 4.5. In the case that $G$ acts regularly on $\Omega$, the proof then follows Theorem 4.5. $\square$

**Example 4.5** Suppose $G \cong U(3) \oplus U(4)$. If $G$ acts regularly on $\Omega$, find the probability that $g \in G$ fixes $\Omega$.

**Solution** Since $U(3) = \{1, 2\}$ and $U(4) = \{1, 3\}$, thus the elements of $G = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$, and the elements of $\Omega = \{(1, 3), (2, 1), (2, 3)\}$. If $G$ acts regularly on $\Omega$, there exists $g \in G$ such that $gw \in \Omega$. Therefore, the conjugacy classes are

$$cl((1, 3)) = \{g(1, 3) : g \in G\} = \{(1, 3), (2, 1), (2, 3)\}.$$

Hence $cl((1, 2)) = cl((2, 3)) = cl(1, 3)$. Thus $K(\Omega) = 1$. It follows that, $P_G(\Omega) = 1/3$.

**Theorem 4.8** *Let G be a finite group, $G \cong U(n) \oplus U(m), n, m \in \mathbb{N}$. Let S be a set of elements of G of size two in the form of $(a, b)$ where a and b commute. Let $\Omega$ be the set of all subsets of commuting elements of G of size two and G acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.*

**Proof** The proof is similar to that of Theorem 4.6. □

**Example 4.6** Suppose $G \cong U(3) \oplus U(4)$. If $G$ acts on $\Omega$ by conjugation, find the probability that $g \in G$ fixes $\Omega$.

**Solution** The elements of $G = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$, thus the elements of $\Omega = \{(1, 3), (2, 1), (2, 3)\}$. If $G$ acts on $\Omega$ by conjugation, then $cl(\omega) = \{g^{-1}\omega g : g \in G\}$. Therefore, the conjugacy classes are $cl((1, 3)) = \{g^{-1}(1, 3)g : g \in G\} = \{(1, 3)\}$, $cl((2, 1)) = \{g^{-1}(2, 1)g : g \in G\} = \{(2, 1)\}$ and $cl((2, 3)) = \{g^{-1}(2, 3)g : g \in G\} = \{(2, 3)\}$. Therefore, $K(\Omega) = 3$. Thus, $P_G(\Omega) = 1$.

**Theorem 4.9** *Let G be a finite group, $G \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_{2q}$,where p and q prime numbers. Let S be a set of elements of G of size two in the form of $(a, b)$ where a and b commute. Let $\Omega$ be the set of all subsets of commuting elements of G of size two and G acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.*

**Proof** The order of any $\omega \in \Omega$ is $\omega = lcm(|g_1|, |g_2|) = 2$, where $g_1 \in \mathbb{Z}_{2p}, g_2 \in \mathbb{Z}_{2q}$, thus $|g_1| = \{1, 2, p, 2p\}, |g_2| = \{1, 2, q, 2q\}$ but the order of $\omega$ is two, thus the elements of $\Omega$ are $\{(0, p), (0, q), (p, q)\}$. In case that $G$ acts on $\Omega$ by conjugation, then the number of conjugacy classes is equal to the order of $\Omega$. Hence, $P_G(\Omega) = 1$. □

**Example 4.7** Suppose $G \cong \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $P_G(\Omega) = 1$.

**Solution** The elements of $G$ are $\{(0, 0), (0, 1), (0, 2), ..., (5, 9)\}$. Thus, the elements in $\Omega$ are $\{(3, 0), (0, 5), (3, 5)\}$. When $G$ acts on $\Omega$ by conjugation, then $cl(3, 0) = \{(0, 3)\}, cl(0, 5) = \{(0, 5)\}$ and $cl(3, 5) = \{(3, 5)\}$. Based on Definition 4.8, the probability is equal to one.

**Theorem 4.10** *Let $G$ be a finite group, $G \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_{2q}$,where $p$ and $q$ are prime numbers. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then $P_G(\Omega) = \frac{1}{|\Omega|}$.*

**Proof** The proof follows from Theorem 4.7. □

**Theorem 4.11** *Let $G$ be a finite group, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$,where $p$ and $q$ are relatively prime. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then*

$$P_G(\Omega) = 1, \textit{if } p \neq q = 2 \textit{ and } p = q = 2.$$

**Proof** The elements of $\Omega$ are $\{(p, 0), (0, q), (p, q)\}$. Thus in the case that $p \neq q = 2$, $\Omega = \{(p/2, 0)\}$ and when $G$ acts on $\Omega$ by conjugation, there is only one conjugacy class, namely $\Omega$ itself. Hence, $P_G(\Omega) = 1$.

If $p \neq q \neq 2$, then there is no element of size two hence the probability cannot be computed. □

**Theorem 4.12** *Let $G$ be a finite group, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, where $p$ and $q$ are relatively prime. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then*

$$P_G(\Omega) = \begin{cases} \frac{1}{|\Omega|}, \text{if } p = q = 2, \\ 1, \text{if } p \neq q = 2. \end{cases}$$

**Proof** The elements of $\Omega$ are $\{(p, 0), (0, q), (p, q)\}$. Thus in the case that $p \neq q = 2$, $\Omega = \{(p/2, 0)\}$. When $G$ acts regularly on $\Omega$ and $p = q = 2$, the proof follows from Theorem 4.7. In the case that $p \neq q = 2$ the proof follows from Theorem 4.9. However, in the case $p \neq q \neq 2$, there is no element of size two since $p$ and $q$ are relatively primes. Thus, there is no possibility to compute $P_G(\Omega)$. □

### 4.3.2 The Orbit Graph and Generalized Conjugacy Class Graph

In this section, we find both the orbit graph and generalized conjugacy class graph based on the obtained results in the previous section, starting with the results on orbit graph.

#### *4.3.2.1 Orbit Graph*

In this part, we find the orbit graph for all theorems in Section 1 and Section 2, starting with the Rusin group.

**Theorem 4.13** *Let $G$ be a finite non-abelian Rusin group,*

$$G \cong \langle a, b : a^{2^p} = b^m = e, bab^{-1} = a^s \rangle,$$

*where $m|(p-1)$ and $s^j \equiv 1 \bmod p$ iff $m|j$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$*

*commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $\Gamma_G^{\Omega}$ is an empty graph.*

**Proof** According to Theorem 4.2, the proof is clear since the elements of $\Omega$ of size two is only the element $(1, a^2)$. Thus, the graph is an empty graph. □

**Theorem 4.14** *Let $G$ be a finite non-abelian dicyclic group,*

$$G \cong \langle a, b : a^{2\beta} = b^4 = e, b^{-1}ab = a^{-1}, a^{\beta} = b^2\rangle.$$

*Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $\Gamma_G^{\Omega}$ is an empty graph.*

**Proof** The proof is similar to that of Theorem 4.13. □

Next, the orbit graph of the generalized quaternion group is found.

**Theorem 4.15** *Let $G$ be a generalized quaternion group, $Q_{2^{n+1}}$, $G \cong \langle a, b : a^{2\alpha} = b^4 = e, b^{-1}ab = a^{-1}, a^{2^{\alpha-1}} = b^2\rangle$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$ by conjugation. Then $\Gamma_G^{\Omega}$ is an empty graph.*

**Proof** The proof is similar to that of Theorem 4.13. □

**Theorem 4.16** *Let $G$ be a finite group, $G \cong U(m), m \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two. regularly on $\Omega$. Then the orbit graph is an empty graph.*

**Proof** The graph is empty since $|V(\Gamma_G^{\Omega})| = |\Omega| - |A|$, where $A = \{g\omega_1 = \omega_1 \forall g \in G\}$, and since $G$ is $U(m)$, all elements in $\Omega$ commute with the elements in $G$. □

**Remark** There is no orbit graph for Theorems 4.6, 4.7, 4.8, 4.9, 4.10, 4.11 and 4.12 since all elements in $\Omega$ commute with the elements in $G$.

**Theorem 4.17** *Let $G$ be a finite group, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, where $p$ and $q$ relatively prime. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then*

$$\Gamma_G^\Omega = \begin{cases} K_\Omega, \text{if } p = q = 2, \\ K_e, \text{if } p \neq q = 2. \end{cases}$$

**Proof** The element of $\Omega$ are $\{(p, 0), (0, q), (p, q)\}$. Thus in case that $p = q = 2$, and two vertices are joined by an edge if there is $g \in G$ such that $g\omega_1 = \omega_2$. According to Theorem 4.12 there is a complete graph, namely $K_\Omega$. However, if $p \neq q = 2$, $\Omega = \{(p/2, 0)\}$ and when $G$ acts on regularly on $\Omega$, then the graph is empty since the element $\{(p/2, 0)\}$ adjacent to itself. In the second case, there is no element of size two, thus it is impossible to find the graph. However, in case three when $p \neq q \neq 2$, there is no element of size two since $p$ and $q$ relatively primes. Thus, there is no possibility to find a graph. □

**Theorem 4.18** *Let $G$ be a finite group, $G \cong U(n) \oplus U(m), n, m \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two and $G$ acts regularly on $\Omega$. Then*

$$\Gamma_G^\Omega = \begin{cases} K_3, \text{if } G \text{ acts regularly on } \Omega, \\ K_e, \text{if } G \text{ acts on } \Omega \text{ by conjugation.} \end{cases}$$

**Proof** According to Theorem 4.7, there is no adjacency between elements in $\Omega$ thus the graph is empty. In the case that $G$ acts regularly on $\Omega$, thus $|V(\Gamma_G^\Omega)| = \Omega - A$. Therefore, $|V(\Gamma_G^\Omega)| = 3$. two vertices are linked by an edge if and only if there exists $g \in G$ such that $g\omega_1 = \omega_2$. From which it follows that there is a complete graph of $K_3$. □

#### 4.3.2.2 Generalized Conjugacy Class Graph

In this part, the generalized conjugacy class graph is found for all groups mentioned in the introduction section. We start with the generalized conjugacy class graph of Rusin group.

**Theorem 4.19** *Let G be a finite non-abelian Rusin group,*

$$G \cong \langle a, b : a^{2^p} = b^m = e, bab^{-1} = a^s \rangle,$$

*where* $m|(p-1)$ *and* $s^j \equiv 1 \mod p$ *iff* $m|j$. *Let S be a set of elements of G of size two in the form of* $(a, b)$ *where a and b commute. Let* $\Omega$ *be the set of all subsets of commuting elements of G of size two and G acts on* $\Omega$ *by conjugation. Then* $\Gamma_G^{\Omega_c} = K_e$.

**Proof** According to Theorem 4.2, there is only one conjugacy class. Thus, the graph is empty. □

**Remark** There is no generalized conjugacy class graph in Theorem 4.3 and 4.4, the reason is similar to that in the previous theorem. In the following, the generalized conjugacy class for $U(n)$ namely, group under multiplication modulo $n$.

**Theorem 4.20** *Let G be a finite group,* $G \cong U(m), m \in \mathbb{N}$. *Let S be a set of elements of G of size two in the form of* $(a, b)$ *where a and b commute. Let* $\Omega$ *be the set of all subsets of commuting elements of G of size two and G acts regularly on* $\Omega$. *Then* $|V(\Gamma_G^{\Omega_c})| = K(\Omega) - |A|$.

**Proof** According to Theorem 4.5, we find that $|V(\Gamma_G^{\Omega_c})| = K(\Omega) - |A|$. According to Proposition 4.1, $deg(\omega) = |V(\Gamma_G^{\Omega_c})| - 1$, thus $deg(\omega) = n$ where $n = K(\Omega) - |A|$. □

**Remark** The generalized conjugacy class graph can be obtained only if $\Omega \neq A$, since all elements in $\Omega$ are relatively prime.This restricted condition is true for the rest of the theorems.

## 4.4 CONCLUSION

In this chapter, the probability that a group element fixes a set is found for some finite groups mentioned in Section 4.1. As consequences of obtained results in Section 4.1, we associated the results in the probability that an element of a group fixes a set to graph theory, more precisely with orbit graph and generalized conjugacy class graph.

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