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Two-Generator p-Groups of Nilpotency Class Two: A Review

Nor Haniza Sarmin and Yasamin Barakat

1.1 INTRODUCTION

Let g and h be two elements in an arbitrary group G. Then, the commutator of g and h is defined to be $[g,h] = g^{-1}h^{-1}gh$. The group generated by all commutators is called the derived subgroup and is denoted by G'. Indeed, $G' = \langle [g,h] : g,h \in G \rangle$. The group G is of nilpotency class two if and only if $G' \leq Z(G)$, where $Z(G)$ is the centre of G. Moreover, a group G is referred to be a p-group, in case the order of every element in G is p^k for some nonnegative integer k. Hence, every finite subgroup of a p-group is of order p^n for a nonnegative integer n [1].

Now, let G be a finite 2-generated p-group of nilpotency class two. If G is generated by a and b, then its derived subgroup, G' is generated by $[a,b]$ [2]. Moreover, if G' is of order p^m, then $Z(G) \cap \langle a \rangle = \langle a^{p^m} \rangle$ and $Z(G) \cap \langle b \rangle = \langle b^{p^m} \rangle$ [2]. Consequently, we find $Z(G) = \langle a^{p^m}, b^{p^m}, [a,b] \rangle$. Usually, 2-generated groups of exactly class two are considered nonabelian, or in other words $[a,b] \neq 1$.

There are several classifications of finite 2-generated p-groups of nilpotency class two. Some of these classifications are stated in the next section. These classifications will be applied in the next chapters to present some recent findings concerning these kind of groups.

1.2 CLASSIFICATIONS OF 2-GENERATED p-GROUPS OF CLASS TWO

In 1993, Bacon and Kappe [3] published their findings and gave a classification for finite 2-generated p-groups of nilpotency exactly two, where p is an odd prime. This classification is stated in Theorem 1.1.

Theorem 1.1 [3] *Let G be a finite 2-generated p-group of class two, where p is an odd prime. Then G is isomorphic to exactly one group of the following three types:*

(a) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b] = c, [a,c] = [b,c] = 1, |a| = p^{\alpha}, |b| = p^{\beta}, |c| = p^{\gamma}$; α, β, γ *are integers, and* $\alpha \geq \beta \geq \gamma$;

(b) $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a,b] = a^{p^{\alpha-\gamma}}, |a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}$; α, β, γ *are integers,* $\alpha \geq 2\gamma$ *and* $\beta \geq \gamma$;

(c) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b] = a^{p^{\alpha-\gamma}}c, [c,b] = a^{-p^{2\alpha-\gamma}}c^{-p^{\alpha-\gamma}}, |a| = p^{\alpha}, |b| = p^{\beta}, |c| = p^{\sigma}, |[a,b]| = p^{\gamma}$; α, β, γ *are integers,* $\beta \geq \gamma \geq \sigma \geq 1$ *and* $\alpha + \sigma \geq 2\gamma$.

In 1999, Kappe *et al.* [2] extended the previous classification to include the case $p = 2$. Their result is given in Theorem 1.2 as follows:

Theorem 1.2 [2] *Let G be a finite 2-generated 2-group of class two. Then G is isomorphic to exactly one group of the following four types:*

(a) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b] = c, [a,c] = [b,c] = 1, |a| = 2^{\alpha}, |b| = 2^{\beta}, |c| = 2^{\gamma}$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$;

(b) $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a,b] = a^{2^{\alpha-\gamma}}, |a| = 2^{\alpha}, |b| = 2^{\beta}, |[a,b]| = 2^{\gamma}$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma, \beta \geq \gamma, \alpha + \beta > 3$;

(c) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b] = a^{2^{\alpha-\gamma}}c, [c,b] = a^{-2^{2\alpha-\gamma}}c^{-2^{\alpha-\gamma}}, |a| = 2^{\alpha}, |b| = 2^{\beta}, |c| = 2^{\sigma}, |[a,b]| = 2^{\gamma}$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\beta \geq \gamma > \sigma, \alpha + \sigma \geq 2\gamma$;

(d) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b] = a^{2}c, [c,b] = a^{-4}c^{-2}, |a| = |b| = 2^{\gamma+1}, |c| = 2^{\gamma-1}, |[a,b]| = 2^{\gamma}$,

$a^{2\gamma} = b^{2\gamma}$, $\gamma \in \mathbb{N}$.

Clearly, Theorem 1.2 is applicable for finite 2-generated 2-groups of class two and hence covers the case $p = 2$. Several years later, Magidin [4] gives the classification in Theorem 1.2 using presentation notations. This version is stated in Theorem 1.3.

Theorem 1.3 [4] *Let G be a finite* 2*-generated* 2*-group of class two. Then G is isomorphic to exactly one group of the following three types:*

(a) $G \cong \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = [a,b]^{2^\gamma} = [a,b,a] = [a,b,b] = 1\rangle$, *where* α, β *and* γ *are positive integers satisfying* $\alpha \geq \beta \geq \gamma$*;*

(b) $G \cong \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = [a,b,a] = [a,b,b] = 1, a^{2^{\alpha+\sigma-\gamma}} = [a,b]^{2^\sigma}\rangle$, *with* $\alpha, \beta, \gamma, \sigma$ *integers satisfying* $\beta \geq \gamma > \sigma \geq 0$*,* $\alpha + \sigma \geq 2\gamma$ *and* $\alpha + \beta + \sigma > 3$*;*

(c) $G \cong \langle a, b \mid a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a,b]^{2^\gamma} = [a,b,a] = [a,b,b] = 1, a^{2^\gamma} = b^{2^\gamma} = [a,b]^{2^{\gamma-1}}\rangle$, *where* $\gamma \in \mathbb{N}$.

Finally, Ahmad *et al.* [5] modified this version to include all finite 2-generated p-groups of class two. Their result, which is presented in Theorem 1.4 is the most updated version that is published recently in 2012.

Theorem 1.4 [5] *Let* p *be a prime and* $n > 2$ *an integer. Every* $2-$*generated* $p-$*group of class exactly two and order* p^n*, corresponds to an ordered* $5-$*tuple of integers,* $(\alpha, \beta, \gamma; \rho, \sigma)$ *such that:*

(a) $\alpha \geq \beta \geq \gamma \geq 1$*;*

(b) $\alpha + \beta + \gamma = n$*;*

(c) $0 \leq \rho \leq \gamma$ *and* $0 \leq \sigma \leq \gamma$*;*

where $(\alpha, \beta, \gamma; \rho, \sigma)$ *corresponds to the group presented by*

$$G = \Big\langle a, b \mid [a,b]^{p^\gamma} = [a,b,a] = [a,b,b] = 1, \quad a^{p^\alpha} = [a,b]^{p^\rho}, b^{p^\beta} = [a,b]^{p^\sigma} \Big\rangle.$$

Moreover,

(a) *if* $\alpha > \beta$*, then* G *is isomorphic to:*
   (i) $(\alpha, \beta, \gamma; \rho, \gamma)$ *when* $\rho \leq \sigma$*;*
   (ii) $(\alpha, \beta, \gamma; \gamma, \sigma)$ *when* $0 \leq \sigma \leq \sigma + \alpha - \beta \leq \rho$ *or* $\sigma < \rho = \gamma$*;*
   (iii) $(\alpha, \beta, \gamma; \rho, \sigma)$ *when* $0 \leq \sigma \leq \rho < \min(\gamma, \sigma + \alpha - \beta)$*;*
(b) *if* $\alpha = \beta > \gamma$*, or* $\alpha = \beta = \gamma$ *and* $p > 2$*, then* G *is isomorphic to* $(\beta, \beta, \gamma; \min(\rho, \sigma), \gamma)$*;*
(c) *if* $\alpha = \beta = \gamma$ *and* $p = 2$*, then* G *is isomorphic to:*
   (i) $(\gamma, \gamma, \gamma; \min(\rho, \sigma), \gamma)$ *when* $0 \leq \min(\rho, \sigma) < \gamma - 1$*;*
   (ii) $(\gamma, \gamma, \gamma; \gamma - 1, \gamma - 1)$ *when* $\rho = \sigma = \gamma - 1$*;*
   (iii) $(\gamma, \gamma, \gamma; \gamma, \gamma)$ *when* $\min(\rho, \sigma) \geq \gamma - 1$ *and* $\max(\rho, \sigma) = \gamma$.

The groups listed in 1(a)-3(c) are pairwise non-isomorphic.

According to Ahmad *et al.* [5], those groups classified in part 1(c) of Theorem 1.4 were missing in the previous classifications.

1.3 AN EXAMPLE

Let p be a prime positive integer. The class of nonabelian groups of order p^3 is an example of finite nonabelian 2-generated p-groups of nilpotency class two. It is known that $G' = Z(G)$ if G is a nonabelian group of order p^3. Hence, G is of nilpotency class two. Moreover, G is 2-generated, since it is isomorphic to one of the following groups, in case $p = 2$:

$$D_4 = \left\langle a, b | a^4 = b^2 = 1, a^b = a^{-1} \right\rangle, \tag{1.1}$$

or,

$$Q_8 = \left\langle a, b | a^4 = 1, a^2 = b^2, a^b = a^{-1} \right\rangle; \tag{1.2}$$

where D_4 is the dihedral group of order eight, and Q_8 is the quaternion group. Moreover, in case $p > 2$, the group G is isomorphic to:

$$\left\langle a, b \mid a^p = 1 = b^p, [a, b]^a = [a, b] = [a, b]^b \right\rangle, \tag{1.3}$$

when the exponent of G is p, and it is isomorphic to:

$$\left\langle a, b \mid a^{p^2} = 1 = b^p, a^b = a^{p+1} \right\rangle, \tag{1.4}$$

if the exponent of G is p^2. Recall that the exponent of a group H is the least positive integer m such that $h^m = 1$ for any $h \in H$. Clearly, the exponent of every p-group is a power of p.

A presentation of G in the form given in Theorem 1.4 was provided in [6], where G is a nonabelian group of order p^3. For instance, it was shown that the groups given in (1.3) and (1.4) are corresponded to $(1, 1, 1; 1, 1)$ and $(1, 1, 1; 0, 1)$, respectively. Additionally, applying Theorem 1.1 implies that groups of the form (1.3) are isomorphic to $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, and groups of the form (1.4) are isomorphic to $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$.

1.4 CONCLUSION

In this chapter some properties of finite 2-generated p-groups of nilpotency class two have been provided. For instance, the structure of the derived subgroup and also the centre. Moreover, some classifications for this kind of groups are presented, which are applied in the next chapters.

REFERENCES

[1] Rotman, J. J. 1994. *The Theory of Groups: An Introduction*. 4th Edition. New York: Springer-Verlag.

[2] Kappe, L. C., N. H. Sarmin, and M. P. Visscher. 1999. “Two-generator Two-groups of Class Two and Their Nonabelian Tensor Squares.” *Glasgow Mathematical Journal*, 41: 417–430.

[3] Bacon, M. R. and L. C. Kappe. 1993. “The Nonabelian Tensor Square of a 2-generator p-group of Class 2.” *Archiv der Mathematik*, 61: 508–516.

[4] Magidin, A. 2006. "Capable Two-generator Two-groups of Class Two." *Communications in Algebra*, 34 (6): 2183–2193.

[5] Ahmad, A., A. Magidin, and R. F. Morse. 2012. "Two Generator p-groups of Nilpotency Class 2 and Their Conjugacy Classes." *Publications Mathematicae Debrecen*, 81: 1–2(10).

[6] Barakat, Y. and N. H. Sarmin. 2012. "Verification of an Old Conjecture on Nonabelian Two Generated Groups of Order p^3." *Jurnal Teknologi*, 1557 (41): 41–45.