Jurnal
Teknologi

Full Paper

The Central Subgroups of the Nonabelian Tensor Squares of Some Bieberbach Groups with Elementary Abelian 2-Group Point Group

Nor Fadzilah Abdul Ladi[a], Rohaidah Masri[a]*, Nor'ashiqin Mohd Idrus[a], Nor Haniza Sarmin[b], Tan Yee Ting[a]

[a]Department of Mathematics, Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris, 35900, Tanjung Malim, Perak, Malaysia
[b]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310, UTM Johor Bahru, Johor, Malaysia

Article history
Received
30 March 2017
Received in revised form
20 June 2017
Accepted
10 October 2017

*Corresponding author
rohaidah@fsmt.upsi.edu.my

Graphical abstract

$$S_1(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \;\middle|\; \begin{array}{l} a_0^{\,2} = l_1^{-1}, a_1^{\,2} = l_2^{-1}, {}^{a_0}a_1 = a_1 l_1^{-1} l_2 l_3^{-1} \\ {}^{a_0}l_1 = l_1, {}^{a_0}l_2 = l_2^{-1}, {}^{a_0}l_3 = l_3^{-1}, \\ {}^{a_1}l_1 = l_1^{-1}, {}^{a_1}l_2 = l_2, {}^{a_1}l_3 = l_3^{-1}, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right\rangle$$

$$S_2(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \;\middle|\; \begin{array}{l} a_0^{\,2} = l_2^{-1} l_3^{-1}, a_1^{\,2} = l_1^{-1}, {}^{a_0}a_1 = a_1 l_1 l_2^{-1} \\ {}^{a_0}l_1 = l_1^{-1}, {}^{a_0}l_2 = l_2, {}^{a_0}l_3 = l_3, \\ {}^{a_1}l_1 = l_1, {}^{a_1}l_2 = l_2^{-1}, {}^{a_1}l_3 = l_3, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right\rangle$$

Abstract

Bieberbach groups are torsion free crystallographic groups. In this paper, our focus is on the Bieberbach groups with elementary abelian 2-group point group, $C_2 \times C_2$. The central subgroup of the nonabelian tensor square of a group G is generated by $g \otimes g$ for all g in G. The purpose of this paper is to compute the central subgroups of the nonabelian tensor squares of two Bieberbach groups with elementary abelian 2-point group of dimension three.

Keywords: Group theory, Bieberbach group, central subgroup, nonabelian tensor square, elementary abelian group

Abstrak

Kumpulan Bieberbach dikenali sebagai kumpulan kristalografi yang bebas kilasan. Dalam makalah ini, fokus kami adalah kepada kumpulan Bieberbach dengan kumpulan abelan asas dua sebagai kumpulan titik, $C_2 \times C_2$. Subkumpulan pusat bagi kuasa dua tensor tak abelan bagi kumpulan G dihasilkan oleh $g \otimes g$ bagi semua g dalam G. Tujuan makalah ini adalah untuk mengira subkumpulan pusat bagi kuasa dua tensor tak abelan bagi dua kumpulan Bieberbach dengan kumpulan abelan asas dua sebagai kumpulan titik yang berdimensi tiga.

Kata kunci: Teori Kumpulan, kumpulan Bieberbach, Subkumpulan pusat, kuasa dua tensor tak abelan, kumpulan abelan permulaan

1.0 INTRODUCTION

Bieberbach groups are torsion free crystallographic group. This group is an extension of a finite point group P and free abelian group L of finite rank is given by the short exact sequence

$$1 \longrightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \longrightarrow 1$$

such that $G/\varphi(L) \cong P.$

The central subgroup of the nonabelian tensor square, $\nabla(G)$ is one of the homological functors that can reveal the properties of a group. The nonabelian tensor square of the group, $G \otimes G$ is generated by the symbols, $g \otimes h$, for all $g, h \in G$ subject to the relations,

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and}$$
$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h'). \; [1]$$

The subgroup $\nabla(G)$ is normal which is generated by $g \otimes g$ for all $g \in G$. The abelianization of the group, G^{ab} must be first determined before $\nabla(G)$ can be computed. Researches on the computation of $\nabla(G)$ of Bieberbach groups with certain point group have been started since 2009. Masri [2] focused on the central subgroups of the nonabelian tensor squares for some families of Bieberbach groups with cyclic point group of order two. Masri [2] also provided a method to create a family of Bieberbach groups with same point group. Mohd Idrus [3] and Wan Mohd Fauzi *et al.* [4] computed the central subgroups of the nonabelian tensor squares of some centerless Bieberbach groups with dihedral point group. Tan *et al.* [5] computed the central subgroup of the nonabelian tensor square of Bieberbach group of dimension six with symmetric point group of order six and Mohammad *et al.* [6] computed the central subgroup of the nonabelian tensor square of a torsion free space group.

In this paper, two Bieberbach groups with elementary abelian 2-group point group of dimension 3, denoted by $S_1(3)$ and $S_2(3)$, are considered. These groups have consistent polycyclic presentations as given in the following [7].

$$S_1(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \;\middle|\; \begin{array}{l} a_0^{\,2} = l_1^{-1}, a_1^{\,2} = l_2^{-1}, {}^{a_0}a_1 = a_1 l_1^{-1} l_2 l_3^{-1} \\ {}^{a_0}l_1 = l_1, {}^{a_0}l_2 = l_2^{-1}, {}^{a_0}l_3 = l_3^{-1}, \\ {}^{a_1}l_1 = l_1^{-1}, {}^{a_1}l_2 = l_2, {}^{a_1}l_3 = l_3^{-1}, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right\rangle \quad (1.1)$$

$$S_2(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \;\middle|\; \begin{array}{l} a_0^{\,2} = l_2^{-1} l_3^{-1}, a_1^{\,2} = l_1^{-1}, {}^{a_0}a_1 = a_1 l_1 l_2^{-1} \\ {}^{a_0}l_1 = l_1^{-1}, {}^{a_0}l_2 = l_2, {}^{a_0}l_3 = l_3, \\ {}^{a_1}l_1 = l_1, {}^{a_1}l_2 = l_2^{-1}, {}^{a_1}l_3 = l_3, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right\rangle \quad (1.2)$$

2.0 METHODOLOGY

The method developed by Blyth and Morse [8] for polycyclic groups is used in this paper to compute the central subgroups of the nonabelian tensor squares as one of the properties of the nonabelian tensor squares. In this section, some basic results that are used in this paper are given. We start with the definition of the group $\nu(G)$.

Definition 2.1 [9]
Let G be a group with presentation $\langle G | R \rangle$ and let G^{φ} be an isomorphism copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is as follows:
$\nu(G) = \langle G, G^{\varphi} \,|\, R, R^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x\varphi}[g, h^{\varphi}], \forall x, g, h \in G \rangle.$

Proposition 2.1 [9]
Let G be a group. The map $\sigma : G \otimes G \to [G, G^{\varphi}] \triangleleft \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.

The next proposition shows that there is a relation between the structure of $\nabla(G)$ and G^{ab}.

Proposition 2.2 [10]
Let G be any group such that G^{ab} is finitely generated. Assume G^{ab} is the direct product of the cyclic groups $\langle x_i G' \rangle$, for $i = 1, \ldots, s$ and let $E(G)$ be the subgroup of $\nu(G)$ defined by $E(G) = \langle [x_i, x_j^{\varphi}] \,|\, 1 \le i < j \le s \rangle [G, G'^{\varphi}]$. Then, the following hold:

(i) $\nabla(G)$ is generated by the elements of the set $\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] \,|\, 1 \le i < j \le s\}$;

(ii) $[G, G^{\varphi}] = \nabla(G) E(G)$.

The commutator subgroup of $\nu(G)$ is isomorphic to $G \otimes G$ by Proposition 2.1. Then, all the tensor computation can be done within the commutator subgroup of $\nu(G)$. Next, the list of commutator identities in $\nu(G)$ with left conjugation are given as in the following. Let x, y and z be in G, then,

$$[xy, z] = {}^{x}[y, z] \cdot [x, z]; \quad (1.3)$$
$$[x, yz] = [x, y] \cdot {}^{y}[x, z]; \quad (1.4)$$
$${}^{z}[x, y] = [{}^{z}x, {}^{z}y]. \quad (1.5)$$

The following propositions are some basic identities used in this paper.

Proposition 2.3 ([8], [11])
Let G be a group. Then the following relations hold in $\nu(G)$:

(i) $[g, g^{\varphi}]$ is central in $\nu(G)$ for all $g \in G$,

(ii) $[g, g^{\varphi}] = 1$ for all $g \in G'$.

Proposition 2.4 [10]
Let G be any group. Then the following hold:

(i) If A and B are two subgroups of G with $B \le G'$, then $[A, B^{\varphi}] = [B, A^{\varphi}]$. In particular, $[G, G'^{\varphi}] = [G', G^{\varphi}]$;

(ii) If $g_1 \in G'$ or $g_2 \in G'$, then $[g_1, g_2{}^{\varphi}]^{-1} = [g_2, g_1{}^{\varphi}]$.

Proposition 2.5 [8]
Let g and h be elements of G such that $[g,h]=1$. Then in $\nu(G)$:

(i) $[g^n, h^{\varphi}] = [g, h^{\varphi}]^n = [g, (h^{\varphi})^n]$ for all integers n;

(ii) $[g^n, (h^m)^{\varphi}][h^m, (g^n)^{\varphi}] = ([g, h^{\varphi}][h, g^{\varphi}])^{nm}$ for all integers m, n.

The following proposition will be used in determining the order of the generators of the groups.

Proposition 2.6 [2]
Let G and H be groups and let $g \in G$. Suppose ϕ is a homomorphism from G to H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise, the order of g equals the order of $\phi(g)$.

Proposition 2.7 [12]
Let A and B be abelian groups. The properties of the ordinary tensor product of two abelian groups are given as the following.

(i) $B_0 \otimes A \cong A$,

(ii) $B_0 \otimes B_0 \cong B_0$.

Proposition 2.8 [13]
Let G be a group and $H \le G$ and let $a, b \in G$. Then,

(i) $aH = H$ if and only if $a \in H$,

(ii) $aH = bH$ if and only if $a \in bH$,

(iii) $aH = bH$ or $aH \cap bH = \varnothing$,

(iv) $aH = bH$ if and only if $a^{-1}b \in H$.

Proposition 2.9 [14]
Let G and H be groups such that there is an epimorphism $\eta : G \to H$. Then, there exists an epimorphism $\alpha : G \otimes G \to H \otimes H$ such that $\alpha(g \otimes h) = \eta(g) \otimes \eta(h)$ for all $g, h \in G$.

Proposition 2.10 [15]
Let G be any group. Then the natural homomorphism $\mu : G \to G/G'$ induces the epimorphism

$$f : [G, G^{\varphi}] \to (G/G') \otimes (G/G')$$

with $[x, y^{\varphi}] \mapsto \mu(x) \otimes \mu(y)$ for all x and y in G.

3.0 RESULTS AND DISCUSSION

In this section, the abelianizations of the groups $S_1(3)$ and $S_2(3)$ are constructed first.

Lemma 3.1
The group $S_1(3)$ has the derived subgroup, $S_1(3)' = \left\langle l_1^{-2}, l_2^{-2}, l_1 l_2^{-1} l_3 \right\rangle$ and its abelianization, $S_1(3)^{ab} = \langle a_0 S_1(3)', a_1 S_1(3)' \rangle \cong C_4{}^2$.

Proof. From relation in (1.1), a_0 commutes with l_1 and a_1 commutes with l_2. Then, we have,

$[a_0, a_1] = a_0 a_1 a_0{}^{-1} a_1{}^{-1} = a_1 l_1{}^{-1} l_2 l_3{}^{-1} a_1{}^{-1} = l_1 l_2{}^{-1} l_3$,

$[a_0, l_2] = a_0 l_2 a_0{}^{-1} l_2{}^{-1} = l_2{}^{-1} l_2{}^{-1} = l_2{}^{-2}$,

$[a_0, l_3] = a_0 l_3 a_0{}^{-1} l_3{}^{-1} = l_3{}^{-1} l_3{}^{-1} = l_3{}^{-2}$,

$[a_1, l_1] = a_1 l_1 a_1{}^{-1} l_1{}^{-1} = l_1{}^{-1} l_1{}^{-1} = l_1{}^{-2}$,

$[a_1, l_3] = a_1 l_3 a_1{}^{-1} l_3{}^{-1} = l_3{}^{-1} l_3{}^{-1} = l_3{}^{-2}$, and

$[l_i, l_j] = 1$ for $1 \le i,\ j \le 3$.

Then, $S_1(3)' = \left\langle l_1{}^{-2}, l_2{}^{-2},\ l_3{}^{-2},\ l_1 l_2{}^{-1} l_3 \right\rangle$. However, some of the generators of $S_1(3)$ can be written as a product of other generators such as $l_3{}^{-2} = (l_1 l_2{}^{-1} l_3)^{-2} (l_1{}^{-2})^{-1} (l_2{}^{-2})$. $S_1(3)$ is a polycyclic group generated by polycyclic generating sequences a_0, a_1, l_1, l_2 and l_3. Therefore, $S_1(3)' = \left\langle l_1^{-2}, l_2^{-2}, l_1 l_2^{-1} l_3 \right\rangle$.

The factor group of $S_1(3)^{ab}$ is generated by $a_0 S_1(3)'$, $a_1 S_1(3)'$, $l_1 S_1(3)'$, $l_2 S_1(3)'$ and $l_3 S_1(3)'$. Since $l_1 l_2{}^{-1} l_3 \in S_1(3)'$, then by Proposition 2.8(iv) $l_3 S_1(3)' = l_1{}^{-1} l_2 S_1(3)' = a_0{}^2 a_1{}^{-2} S_1(3)'$. Based on the relations in (1.1), $a_0{}^2 = l_1{}^{-1}$ and $a_1{}^2 = l_2{}^{-1}$. By Proposition 2.8(iii), $a_0{}^2 S_1(3)' \cap l_1{}^{-1} S_1(3)'$ and $a_1{}^2 S_1(3)' \cap l_2{}^{-1} S_1(3)'$ is not trivial since $a_0{}^2 S_1(3)' = l_1{}^{-1} S_1(3)'$ and $a_1{}^2 S_1(3)' = l_2{}^{-1} S_1(3)'$. By Proposition 2.8(ii), $a_0{}^2 \in l_1{}^{-1} S_1(3)'$ and $a_1{}^2 \in l_2{}^{-1} S_1(3)'$ which implies $a_0 \in l_1{}^{-1} S_1(3)'$ and $a_1 \in l_2{}^{-1} S_1(3)'$. Therefore, it can be concluded that

$a_0S_1(3)'=l_1^{-1}S_1(3)'$ and $a_1S_1(3)'=l_2^{-1}S_1(3)'$. Hence, $S_1(3)^{ab}=\langle a_0S_1(3)', a_1S_1(3)'\rangle$.

Next, the order of $a_0S_1(3)'$ and $a_1S_1(3)'$ will be determined. By the relation in (1.1), $a_0^{\,2}=l_1^{-1}$, $a_1^{\,2}=l_2^{-1}$, then $a_0^{\,4}=l_1^{-2}$, $a_1^{\,4}=l_2^{-2}$. Since $l_1^{-2},l_2^{-2}\in S_1(3)'$, therefore, the cosets of $a_0S_1(3)'$ and $a_1S_1(3)'$ have order 4. Hence,

$$S_1(3)^{ab}=\langle a_0S_1(3)', a_1S_1(3)'\rangle \cong C_4\times C_4. \qquad \square$$

Lemma 3.2
The group $S_2(3)$ has the derived subgroup, $S_2(3)'=\left\langle l_1^{-2}, l_1^{-1}l_2\right\rangle$ and its abelianization, $S_2(3)^{ab}=\langle a_0S_2(3)', a_1S_2(3)'\rangle \cong C_0\times C_4$.

Proof. From relation in (1.2), a_0 commutes with l_2,l_3 and a_1 commutes with l_1,l_3. Then, we have,

$[a_0,a_1]=a_0a_1a_0^{-1}a_1^{-1}=a_1l_1^{-1}l_2^{-1}a_1^{-1}=l_1^{-1}l_2$,

$[a_0,l_1]=a_0l_1a_0^{-1}l_1^{-1}=l_1^{-1}l_1^{-1}=l_1^{-2}$,

$[a_1,l_2]=a_1l_2a_1^{-1}l_2^{-1}=l_2^{-1}l_2^{-1}=l_2^{-2}$, and

$[l_i,l_j]=1$ for $1\le i,\ j\le 3$.

Then, $S_2(3)'=\left\langle l_1^{-2}, l_2^{-2}, l_1^{-1}l_2\right\rangle$. However, some of the generators can be written as a product of other generators. Here, $l_1^{-2}=(l_1^{-1}l_2)^2(l_2^{-2})$. Thus, $S_2(3)$ is a polycyclic group generated by polycyclic generating sequences a_0, a_1, l_1, l_2 and l_3. Therefore, $S_2(3)'=\left\langle l_1^{-2}, l_1^{-1}l_2\right\rangle$.

The factor group of $S_2(3)^{ab}$ is generated by $a_0S_2(3)'$, $a_1S_2(3)'$, $l_1S_2(3)'$, $l_2S_2(3)'$ and $l_3S_2(3)'$. Since $l_1^{-1}l_2\in S_2(3)'$, then by Proposition 2.8(iv) $l_2^{-1}S_2(3)'=l_1^{-1}S_2(3)'$. However, based on the relations in (1.2), $a_1^{\,2}=l_1^{-1}$. By Proposition 2.8(iii), $a_1^{\,2}S_2(3)'\cap l_1^{-1}S_2(3)'$ is not trivial since $a_1^{\,2}S_2(3)'=l_1^{-1}S_2(3)'$. By Proposition 2.8(ii) and $a_1^{\,2}\in l_1^{-1}S_2(3)'$ which implies $a_1\in l_1^{-1}S_2(3)'$. Hence, $a_1S_2(3)'=l_1^{-1}S_2(3)'$. Besides, $a_0^{\,2}=l_2^{-1}l_3^{-1}$ and since a_0 commutes with l_2,l_3, then $l_3S_2(3)'=a_0^{-2}l_2^{-1}S_2(3)'=a_0^{-2}l_1^{-1}S_2(3)'=a_0^{-2}a_1^{\,2}S_2(3)'$. Therefore, $S_2(3)^{ab}=\langle a_0S_2(3)', a_1S_2(3)'\rangle$.

Next, the order of $a_0S_2(3)'$ and $a_1S_2(3)'$ will be determined. By the relation in (1.2), $a_1^{\,2}=l_1^{-1}$, then $a_1^{\,4}=l_2^{-2}$. Since $l_1^{-2}\in S_1(3)'$, then the cosets of $a_1S_2(3)'$ has order 4. Next we want to show that $a_0S_2(3)'$ has infinite order. Suppose that the order of $a_0S_2(3)'$ is finite. However, for any integer $r\in\mathbb{Z}$, there is no $a_0^{\,r}$ in $S_2(3)'$ which cannot be written as any element of $S_2(3)'$. So, the cosets of $a_0S_2(3)'$ has infinite order. Hence,

$$S_2(3)^{ab}=\langle a_0S_2(3)', a_1S_2(3)'\rangle \cong C_0\times C_4. \qquad \square$$

Theorem 3.1
The central subgroup of the nonabelian tensor squares of $S_1(3)$,

$$\nabla(S_1(3))=\langle [a_0,a_0^{\varphi}],[a_1,a_1^{\varphi}],[a_0,a_1^{\varphi}][a_1,a_0^{\varphi}]\rangle \cong C_4\times C_8^{\,2}.$$

Proof. By Lemma 3.1, $S_1(3)^{ab}$ is generated by the cosets $a_0S_1(3)'$ and $a_1S_1(3)'$. Then, by Proposition 2.2, $\nabla(S_1(3))$ is generated by $[a_0,a_0^{\varphi}]$, $[a_1,a_1^{\varphi}]$ and $[a_0,a_1^{\varphi}][a_1,a_0^{\varphi}]$. Next the order of each generator will be determined.

$$\begin{aligned}
[a_0,a_0^{\varphi}]^{16} &= [a_0^{\,4},a_0^{\,4\varphi}] && \text{by Proposition 2.5(i)}\\
&= [l_1^{-2},l_1^{-2\varphi}] && \text{since } a_0^2=l_1^{-1}\\
&= 1 && \text{since } l_1^{-2}\in S_1(3)'.
\end{aligned}$$

$$\begin{aligned}
[a_1,a_1^{\varphi}]^{16} &= [a_1^{\,4},a_1^{\,4\varphi}] && \text{by Proposition 2.5(i)}\\
&= [l_2^{-2},l_2^{-2\varphi}] && \text{since } a_1^2=l_2^{-1}\\
&= 1 && \text{since } l_2^{-2}\in S_1(3)'.
\end{aligned}$$

Hence, the orders of $[a_0,a_0^{\varphi}]$ and $[a_1,a_1^{\varphi}]$ divides 16.

However, since $a_0^{\,4}=l_1^{-2}\in S_1(3)'$ and $a_1^{\,4}=l_2^{-2}\in S_1(3)'$ then by Proposition 2.4(ii),

$$[a_0,(a_0^{\,4})^{\varphi}]=[a_0,(l_1^{-2})^{\varphi}];$$

$$[a_0,(l_1^{-2})^{\varphi}]=[l_1^{-2},a_0^{\varphi}]^{-1};$$

$$[a_0,(l_1^{-2})^{\varphi}][l_1^{-2},a_0^{\varphi}]=1;$$

$[a_0,(a_0{}^4)^{\varphi}][a_0{}^4,a_0{}^{\varphi}]=1$; and

$$[a_0,a_0{}^{\varphi}]^8=1.$$

By using similar arguments, $[a_1,a_1{}^{\varphi}]^8=1$. Then,the order of $[a_0,a_0{}^{\varphi}]$ and $[a_1,a_1{}^{\varphi}]$ are not 16.

We denote the abelianization of $S_1(3)$ by $S_1(3)^{ab}$ with natural homomorphism

$$\eta: S_1(3)\to S_1(3)^{ab} \qquad (3.1)$$

Since $S_1(3)^{ab}$ is finitely generated then its nonabelian tensor square is simply ordinary tensor product of two copies of $S_1(3)^{ab}\cong C_4\times C_4$. By Proposition 2.7,

$$S_1(3)^{ab}\otimes S_1(3)^{ab}\cong C_4\times C_4\times C_4\times C_4 \qquad (3.2)$$

By Lemma 3.1 and equation (3.1), the group $S_1(3)^{ab}$ is generated by $\eta(a_0)$ and $\eta(a_1)$ of order 4. Proposition 2.7 gives, $\langle \eta(a_0)\otimes\eta(a_0)\rangle\cong C_4$ and $\langle \eta(a_1)\otimes\eta(a_1)\rangle\cong C_4$. By Proposition 2.9, there is a natural epimorphism $\alpha: S_1(3)\otimes S_1(3)\to S_1(3)^{ab}\otimes S_1(3)^{ab}$. Therefore, the image $\alpha(a_0\otimes a_0)=\eta(a_0)\otimes\eta(a_0)$ and $\alpha(a_1\otimes a_1)=\eta(a_1)\otimes\eta(a_1)$ have order 4. By Proposition 2.6, the order of $\eta(a_0)\otimes\eta(a_0)$ divides the order of $(a_0\otimes a_0)$ and the order of $\eta(a_1)\otimes\eta(a_1)$ divides the order of $(a_1\otimes a_1)$. Therefore, the order of $(a_0\otimes a_0)\cong[a_0,a_0{}^{\varphi}]$ and $(a_1\otimes a_1)\cong[a_1,a_1{}^{\varphi}]$ must multiple of 4. It is either 4 or 8. However, the order of $[a_0,a_0{}^{\varphi}]$ and $[a_1,a_1{}^{\varphi}]$ cannot be 4 since:

$[a_0,a_0{}^{\varphi}]^4=[a_0{}^2,a_0{}^{2\varphi}]$	by Proposition 2.5(i)
$=[l_1{}^{-1},l_1{}^{-\varphi}]$	since $a_0{}^2=l_1{}^{-1}$
$\neq 1$	since $l_1{}^{-1}\notin S_1(3)'$

Also, since $l_2{}^{-1}\notin S_1(3)'$, then $[a_1{}^2,a_1{}^{2\varphi}]^4=[a_1{}^2,a_1{}^{2\varphi}]=[l_2{}^{-1},l_2{}^{-\varphi}]\neq 1$. Hence, the order of $[a_0,a_0{}^{\varphi}]$ and $[a_1,a_1{}^{\varphi}]$ are 8.

By using Proposition 2.4(ii) and Proposition 2.5(ii) we also have,

$$([a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}])^4=[a_0,a_1{}^{4\varphi}][a_1{}^4,a_0{}^{\varphi}]$$
$$=[a_0,l_2{}^{-2\varphi}][l_2{}^{-2},a_0{}^{\varphi}]$$
$$=[a_0,l_2{}^{-2\varphi}][a_0,l_2{}^{-2\varphi}]^{-1}$$
$$=1\ .$$

Hence, the order of $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ divides 4. (3.3)

By Lemma 3.1 and equation (3.1), the group $S_1(3)^{ab}$ is generated by $\eta(a_0)$ and $\eta(a_1)$ of order 4. Proposition 2.7 gives, $\langle \eta(a_0)\otimes\eta(a_1)\rangle\cong C_4$ and $\langle \eta(a_1)\otimes\eta(a_0)\rangle\cong C_4$.

By Proposition 2.9, there is a natural epimorphism $\alpha: S_1(3)\otimes S_1(3)\to S_1(3)^{ab}\otimes S_1(3)^{ab}$. Therefore, the image $\alpha((a_0\otimes a_1)(a_1\otimes a_0))=\alpha(a_0\otimes a_1)\alpha(a_1\otimes a_0)=(\eta(a_0)\otimes\eta(a_1))$ $(\eta(a_1)\otimes\eta(a_0))$ has order 4. By Proposition 2.6, the order of $(\eta(a_0)\otimes\eta(a_1))$ $(\eta(a_1)\otimes\eta(a_0))$ divides the order of $(a_0\otimes a_1)(a_1\otimes a_0)$. Therefore, the order of $(a_0\otimes a_1)(a_1\otimes a_0)\cong[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ must multiple of 4. However, by (3.3) the order of $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ divides 4. Hence, the order of $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ is exactly 4. Then,

$$\nabla(S_1(3))=\langle [a_0,a_0{}^{\varphi}],[a_0,a_0{}^{\varphi}],[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]\rangle$$
$$\cong C_4\times C_8{}^2.$$

□

Theorem 3.2
The central subgroup of the nonabelian tensor square of $S_2(3)$,
$\nabla(S_2(3))=\langle [a_0,a_0{}^{\varphi}],[a_1,a_1{}^{\varphi}],[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]\rangle\cong C_4\times C_8\times C_0$.

Proof. By Lemma 3.2, $S_2(3)^{ab}$ is generated by the cosets $a_0S_2(3)'$ and $a_1S_2(3)'$. Then, by Proposition 2.2, $\nabla(S_2(3))$ is generated by $[a_0,a_0{}^{\varphi}]$, $[a_1,a_1{}^{\varphi}]$ and $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$. Next the order of each generator will be determined.

$[a_1,a_1{}^{\varphi}]^{16}=[a_1{}^4,a_1{}^{4\varphi}]$	by Proposition 2.5(i)
$=[l_1{}^{-2},l_1{}^{-2\varphi}]$	since $a_1^2=l_1^{-1}$
$=1$	since $l_1^{-2}\in S_2(3)'$.

Hence, the order of $[a_1,a_1{}^{\varphi}]$ divides 16.

However, since $a_1{}^4=l_1{}^{-2}\in S_2(3)'$ and Proposition 2.4(ii), then

$$[a_1,(a_1{}^4)^{\varphi}]=[a_1,(l_1{}^{-2})^{\varphi}];$$
$$[a_1,(l_1{}^{-2})^{\varphi}]=[l_1{}^{-2},a_1{}^{\varphi}]^{-1};$$
$$[a_1,(l_1{}^{-2})^{\varphi}][l_1{}^{-2},a_1{}^{\varphi}]=1;$$
$$[a_1,(a_1{}^4)^{\varphi}][a_1{}^4,a_1{}^{\varphi}]=1; \text{ and}$$
$$[a_1,a_1{}^{\varphi}]^8=1.$$

Hence, the order of $[a_1,a_1{}^{\varphi}]$ is 8. Then, the order of $[a_1,a_1{}^{\varphi}]$ is not 16.

We denote the abelianization of $S_2(3)$ by $S_2(3)^{ab}$ with natural homomorphism

$$\eta: S_2(3)\to S_2(3)^{ab} \qquad (3.4)$$

Since $S_2(3)^{ab}$ is finitely generated then its nonabelian tensor square is simply ordinary tensor product of two copies of $S_2(3)^{ab}\cong C_4\times C_0$. By Proposition 2.7,

$$S_2(3)^{ab}\otimes S_2(3)^{ab}\cong C_4\times C_4\times C_4\times C_0 \qquad (3.5)$$

By Lemma 3.2 and equation (3.4), the group $S_2(3)^{ab}$ is generated by $\eta(a_0)$ and $\eta(a_1)$ of infinite order and order 4 respectively. Proposition 2.7 gives, $\langle \eta(a_1)\otimes\eta(a_1)\rangle\cong C_4$. By Proposition 2.9, there is a natural epimorphism $\alpha: S_2(3)\otimes S_2(3)\to S_2(3)^{ab}\otimes S_2(3)^{ab}$. Therefore, the image $\alpha(a_1\otimes a_1)=\eta(a_1)\otimes\eta(a_1)$ has order 4. By Proposition 2.6, the order of $\eta(a_1)\otimes\eta(a_1)$ divides the order of $(a_1\otimes a_1)$. Therefore, the order of $(a_1\otimes a_1)\cong[a_1,a_1{}^{\varphi}]$ must multiple of 4. It is either 4 or 8. However, the order of $[a_1,a_1{}^{\varphi}]$ cannot be 4 since:

$$\begin{aligned}[a_1,a_1{}^{\varphi}]^4&=[a_1{}^2,a_1{}^{2\varphi}] && \text{by Proposition 2.5(i)}\\ &=[l_1{}^{-1},l_1{}^{-\varphi}] && \text{since } a_1{}^2=l_1{}^{-1}\\ &\neq 1 && \text{since } l_1{}^{-1}\notin S_2(3)'\end{aligned}$$

Hence, the order of $[a_1,a_1{}^{\varphi}]$ is 8.

By Lemma 3.2, the group $S_2(3)^{ab}$ is generated by $\eta(a_0)$ and $\eta(a_1)$ of infinite order and order 4 respectively. By Proposition 2.7, gives $\langle\eta(a_0)\otimes\eta(a_0)\rangle\cong C_0$. Note that there is a natural homomorphism $\alpha: G\otimes G\to G^{ab}\otimes G^{ab}$. Therefore, the image $\alpha(a_0\otimes a_0)=\eta(a_0)\otimes\eta(a_0)$ has infinite order. By Proposition 2.8, $a_0\otimes a_0$ has infinite order and $[a_0,a_0{}^{\varphi}]$ also has infinite order. Next,

$$\begin{aligned}([a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}])^4&=[a_0,a_1{}^{4\varphi}][a_1{}^4,a_0{}^{\varphi}] && \text{by Proposition 2.5(i)}\\ &=[a_0,l_1{}^{-2\varphi}][l_1{}^{-2},a_0{}^{\varphi}] && \text{by relation } S_2(3)\\ &=[a_0,l_1{}^{-2\varphi}][a_0,l_1{}^{-2\varphi}]^{-1} && \text{by Proposition 2.4(ii)}\\ &=1\end{aligned}$$

Hence, the order of $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ divides 4. (3.6)

By Lemma 3.2 and equation (3.4), the group $S_2(3)^{ab}$ is generated by $\eta(a_0)$ and $\eta(a_1)$ of order infinite and order 4 respectively. Proposition 2.7 gives, $\langle\eta(a_0)\otimes\eta(a_1)\rangle\cong C_4$ and $\langle\eta(a_1)\otimes\eta(a_0)\rangle\cong C_4$.

By Proposition 2.9, there is a natural epimorphism $\alpha: S_2(3)\otimes S_2(3)\to S_2(3)^{ab}\otimes S_2(3)^{ab}$. Therefore, the image $\alpha((a_0\otimes a_1)(a_1\otimes a_0))=\alpha(a_0\otimes a_1)\alpha(a_1\otimes a_0)$ $=(\eta(a_0)\otimes\eta(a_1))(\eta(a_1)\otimes\eta(a_0))$ has order 4. By Proposition 2.6, the order of $(\eta(a_0)\otimes\eta(a_1))$ $(\eta(a_1)\otimes\eta(a_0))$ divides the order of $(a_0\otimes a_1)(a_1\otimes a_0)$. Therefore, the order of $(a_0\otimes a_1)(a_1\otimes a_0)\cong[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ must multiple of 4. However, by (3.6) the order of $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ divides 4. Hence, the order of $[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]$ is exactly 4. Therefore,

$$\begin{aligned}\nabla(S_2(3))&=\langle[a_0,a_0{}^{\varphi}],[a_0,a_0{}^{\varphi}],[a_0,a_1{}^{\varphi}][a_1,a_0{}^{\varphi}]\rangle\\ &\cong C_4\times C_8\times C_0.\end{aligned}$$

□

4.0 CONCLUSION

In this study, the central subgroups of the nonabelian tensor squares of two Bieberbach groups with elementary abelian 2-group point group, $C_2\times C_2$ of dimension 3, $S_1(3)$ and $S_2(3)$ are computed. The findings of this study, can be used for further research to compute the nonabelian tensor squares of Bieberbach group with point group $C_2\times C_2$.

Acknowledgements

The first author would like to express her appreciation to the Ministry of Education (MOE) for her MyMaster

Scholarship and for the financial funding through FRGS with vote no: 2016-0084-102-02. The fifth author would like to acknowledge the MOE for her MyPhD Scholarship. Also, the first author would to thank Miss Siti Afiqah for her helps in this paper.

References

[1] Brown, R. and Loday, J. L. 1987. Van Kampens Theorems for Diagrams of Spaces. *Topology*. 26(3): 311-355.

[2] Masri, R. 2009. The Nonabelian Tensor Squares of Certain Bieberbach Groups with Cyclic Point Group of Order Two. Ph.D. Thesis. Universiti Teknologi Malaysia.

[3] Mohd Idrus, N. 2011. Bieberbach Groups with Finite Point Groups. Ph.D. Thesis. Universiti Teknologi Malaysia.

[4] Wan Mohd Fauzi, W. N. F., Mohd Idrus, N., Masri, R., Tan, Y. T. and Sarmin, N. H. 2015. The Central Subgroup of the Nonabelian Tensor Square of the Second Bieberbach Group with Dihedral Point Group. *J. Sci. Math. Lett.* 3: 40-49.

[5] Tan, Y. T., Masri, R., Mohd Idrus, N., Wan Mohd Fauzi, W. N. F., Sarmin, N. H. and Mat Hassim, H. I. 2015. The Central Subgroup of the Nonabelian Tensor Square of Bieberbach Group of Dimension Six with Symmetric Point Group of Order Six. *Int. J. Appl. Math. Stat.* 53(4): 98-103.

[6] Mohammad, S. A., Sarmin, N. H. and Mat Hassim, H. I. 2016. The Central Subgroup of the Nonabelian Tensor Square of a Torsion Free Space Group. *AIP Conference Proceedings.* 1750: 1-7.

[7] Abdul Ladi, N. F., Masri, R., Mohd Idrus, N. and Tan, Y. T. 2016. On The Generalization of The Abelianizations of Two Families of Bieberbach Groups with Elementary Abelian 2-group Pint Group. *Proceeding of the 6th International Graduate Conference on Engineering, Science and Humanities*. 393-395. ISBN: 978-967-0194-67-7.

[8] Blyth, R. D. and Morse, R. F. 2009. Computing the Nonabelian Tensor Squares of Polycyclic Groups. *Journal of Algebra.* 321: 2139-2148.

[9] Rocco, N. R. 1991. On a Construction Related to The Nonabelian Tensor Square of a Group. *Bol. Soc. Brasil. Mat. (N.S.)*. 22(1): 63-79.

[10] Blyth,R. D., Fumagalli, F. and Morigi, M. 2010. Some Structural Results on Non-Abelian Tensor Square of Groups. *Journal of Group Theory.* 13: 83-94.

[11] Blyth, R. D., Moravec, P. and Morse R. F. 2008. On the Nonabelian Tensor Squares of Free Nilpotent Groups of Finite Rank. *Contemporary Mathematics*. 470: 27-44.

[12] Zomorodian, A. J. 2005. *Topology for Computing.* New York: Cambridge University Press.

[13] Gallian, J. A. 2010. *Contemporary Abstract Algebra.* 7th ed. USA: Cegage LEARNING.

[14] Brown, R., Johnson, D. L. and Robertson, E. F. 1987. Some Computations of Nonabelian Tensor Products of Groups. *Journal of Algebra*. 111: 177-202.

[15] Magidin, A. and Morse, R. F. 2010. Certain Homological Functors. *Contemporary Mathematics*. 511: 127-166.