World Applied Sciences Journal 32 (3): 459-464, 2014
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2014.32.03.887

The Probability that an Element of Metacyclic 2-Groups Fixes a Set

[1]Mustafa Anis El-Sanfaz, [2]Nor Haniza Sarmin and [3]Sanaa Mohamed Saleh Omer

[1,2,3]Department of Mathematical Sciences, Faculty of Science,
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
[1,3]Department of Mathematics, Faculty of Science, University of Benghazi, Benghazi, Libya

Abstract: In this paper, let G be a metacyclic 2-group of negative type of class two and class at least three. Let Ω be the set of all subsets of all commuting elements of size two in the form of (a, b) where a and b commute and $|a| = |b| = 2$. The probability that an element of a group fixes a set is considered as one of the extensions of the commutativity degree that can be obtained by some group actions on a set. In this paper, the probability that an element of G fixes the set Ω under conjugate action is computed.

Key words: Commutativity degree · Metacyclic group · Group action

INTRODUCTION

The commutativity degree is a concept that is used to determine the abelianness of a group. This concept was firstly introduced by Miller in 1944 [1]. The definition of the commutativity degree is given as follows:

Definition 1.1: Let G be a finite non-abelian group. Suppose that x and y are two random elements of G. The probability that x and y commute is defined as:

$$P(G) = \frac{\left|\{(x,y) \in G \times G : xy = yx\}\right|}{|G|^2}.$$

The commutativity degree of symmetric groups was investigated by Erdos and Turan [2]. Few years later, Gustafson [3] and MacHale [4] proved that this probability is less than or equal to 5/8. The above probability has been generalized and extended by several authors. In this paper, we used one of these extensions, namely the probability that a group element fixes a set denoted by $P_G(\Omega)$. This probability was firstly introduced by Omer *et al.* [5] in 2013.

Next, we state some basic concepts that are needed in this paper.

Definition 1.2: [6] A group G is called a metacyclic if it has a cyclic normal subgroup H such that the quotient group G/H is also cyclic.

Definition 1.3: [7] Let G be a finite group. A group G acts on itself if there is a function $G \times G \to G$ such that:

$i.(gh)x = g(hx), \forall\, g, h, x \in G.$
$ii.\, 1_G x = x, \forall x \in G.$

Definition 1.4: [8] Let G be any finite group and X be a set. A group G acts on X if there is a function $G \times X \to X$ such that:

(1) $(gh)x = g(hx)$, $\forall\, g, h \in G, x \in X$
(2) $1_G x = x$, $\forall\, x \in G$

In the following, some concepts related to metacyclic p-groups are provided.

In 1973, King [9] gave the presentation of metacyclic p-groups, as in the following:

$G \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = a^n, [a,b] = a^m \rangle$, where
$\alpha, \beta > 0, m > 0, n \le p^\alpha, p^\alpha \,|\, n(m-1).$

However, Beuerle [10] in 2005 separated the classification into two parts, namely for the non-abelian metacyclic p-groups of class two and of class at least three. Based on [10], the metacyclic p-groups of nilpotency class two are then partitioned into two families of non-isomorphic p-groups stated as follows:

Corresponding Author: Nor Haniza Sarmin, Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia.
Tel: + 607-5537839, Fax: +607-5537800.

(1) $G \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = 1, [a,b] = a^{p^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma \in \mathbb{Z}, \alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$.
(2) $G \cong Q_8$.

Meanwhile, the metacyclic p-groups of nilpotency class of at least three (p is an odd prime) are partitioned into the following groups:

(i) $G \cong \langle a,b : a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma \in \mathbb{Z}, \alpha - 1\gamma < 2$ and $\alpha \leq \beta$.
(ii) $G \cong \langle a,b : a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma, \varepsilon \in \mathbb{Z}, \alpha - 1\gamma < 2\gamma, \alpha \leq \beta$
and $\alpha \leq \beta + \in$.

Moreover, metacyclic p-groups can also be classified into two types, namely negative and positive [10]. The following notations are used in this paper represented as follows:

$G(\alpha, \beta, \in, \gamma, \pm) \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\in}}$,
$[b,a] = a^{t} \rangle$, where $\alpha, \beta, \gamma, \in \in \mathbb{Z}$,
$t = p^{\alpha-\gamma} \pm 1$.

If $t = p^{\alpha-\gamma} - 1$, then the group is called a metacyclic of negative type and it is of positive type if $t = p^{\alpha-\gamma}$. Thus, $G(\alpha, \beta, \in, \gamma, -)$ is denoted by the metacyclic group of negative type, while $G(\alpha, \beta, \in, \gamma)$ is denoted by the positive type. These two notations are shortened to $(Gp, +)$ and $G(p, -)$ for metacyclic p-group of positive and negative type respectively ([9] and [10]).

In addition, the metacyclic 2-groups of negative type of class at least three are partitioned into eight families [10]. The followings are the classifications of the negative types which are considered in the scope of this paper:

(3) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = a^{2^{\alpha-1}}, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3$,
(4) $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3$,
(5) $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3$,
(6) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle$, where $\alpha \geq 3, \beta > 1$,
(7) $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3, \beta > 1$,
(8) $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta > \gamma > 1$,
(9) $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta \geq \gamma > 1$.

Definition 1.5: [11] Let G act on a set S and $x \in S$. If G acts on itself by conjugation, the orbit $O(x)$ is defined as follows:

$$O(x) = \{ y \in G : y = axa^{-1} \text{ for some } a \in G \}.$$

In this case $O(x)$ is called the conjugacy classes of x in G. Throughout this paper, we use $K(G)$ as a notation for the number of conjugacy classes in G.

This paper is structured as follows: Section 1 provides some fundamental concepts in group theory, more precisely the classifications of metacyclic 2-groups which are used in this paper. In section 2, we state some of the previous works, which are related to the commutativity degree, in particular related to the probability that a group element fixes a set. The main results are presented in Section 3.

Preliminaries: In this section, we provide some previous works related to the commutativity degree, more precisely to the probability that an element of a group fixes a set.

In 1975, a new concept was introduced by Sherman [12], namely the probability of an automorphism of a finite group fixes an arbitrary element in the group. The definition of this probability is given in the following.

Definition 2.1: [12] Let G be a group. Let X be a non-empty set of G (G is a group of permutations of X). Then the probability of an automorphism of a group fixes a random element from X is defined as follows:

$$P_G(X) = \frac{\left| \{ (g,x) \mid gx = x \ \forall g \in G, x \in X \} \right|}{|X||G|}.$$

In 2011, Moghaddam [13] explored Sherman's definition and introduced a new probability, which is called the probability of an automorphism fixes a subgroup element of a finite group, the probability is stated as follows:

$$P_{A_G}(H,G) = \frac{\left| \{ (\alpha, h) \mid h^{\alpha} : h \in H, \alpha \in A_G \} \right|}{|H| \quad |G|},$$

where A_G is the group of automorphisms of a group G. It is clear that if $H = G$, then $P_{AG}(G, G) = P_{AG}$.

Recently, Omer *et al.* [5] generalized the commutativity degree by defining the probability that an element of a group fixes a set of size two. Their results are given as follows:

Definition 2.2: [5] Let G be a group. Let S be a set of all subsets of commuting elements of size two in G, G acts on S by conjugation .Then the probability of an element of a group fixes a set is defined as follows:

$$P_G(S)=\frac{\left|\{(g,s)\,|\,gs=s\ \forall g\in G, s\in S\}\right|}{|S||G|}.$$

Theorem 2.1: [5] Let G be a finite group and let X be a set of elements of G of size two in the form of (a, b) where a and b commute. Let S be the set of all subsets of commuting elements of G of size two and G acts on S by conjugation. Then the probability that an element of a group fixes a set is given by:

$$P_G(S)=\frac{K(S)}{|S|},$$

where $K(S)$ is the number of conjugacy classes of S in G.

Moreover, Omer *et al*. [14] found the probability that a symmetric group element fixes a set where their results are then applied to graph theory.

Recently, Mustafa *et al.* [15] has extended the work in [4] by restricting the order of Ω. The following theorem illustrates their results.

Theorem 2.2: [15] Let G be a finite group and let S be a set of elements of G of size two in the form of (a, b) where a, b commute and $|a| = |b| = 2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω. Then the probability that an element of a group fixes a set is given by $P_G(\Omega)=\frac{K(\Omega)}{|\Omega|}$, where $K(G)$ is the number of conjugacy classes of Ω in G.

Proposition 2.3: [15] If G is an abelian group, the probability that an element of a group fixes a set $P_G(\Omega)$ is equal to one.

Main Results: This section provides our main results, where the probability that a group element fixes a set is found for all metacyclic 2-group of negative type of class two and class at least three.

In the next theorems, let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $|a| = |b| = 2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. We start by finding the probability that an element of metacyclic 2-groups of negative type of nilpotency class at least three fixes a set.

Theorem 3.1: Let G be a group of type,

(3), $G\cong\langle a,b: a^{2^\alpha}=1,\ b^2=a^{2^{\alpha-1}},[b,a]=a^{-2}\rangle$, where $\alpha\geq 3$. Then $P_G(\Omega)=\frac{2}{|\Omega|}$.

Proof: If G acts on Ω by conjugation, then there exists Ψ: $G\times\Omega\to\Omega$ such that $\Psi_g(w)=gwg^{-1}$, $w\in\Omega$, $g\in a^{2\alpha-3i}b$, $a^{2\alpha-3i}$, $0\leq i\leq 2^\alpha$. The elements of order two in G are $a2^{\alpha-1}$, $a^{2\alpha-3i}b$, i is odd and $0\leq i\leq 2^\alpha$. The elements of G are stated as follows: There are four elements in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-3}i}b\right),0\leq i\leq 2^\alpha$ where i is odd and two elements in the form of $\left(a^{2^{\alpha-3}i}b,a^{2^{\alpha-3}i+2^{\alpha-1}}b\right),i\leq 0\leq 2^\alpha,i$ is odd. This gives that $|\Omega| = 6$. If G acts on Ω by conjugation then $\mathrm{cl}(w) = \{gwg^{-1}\}$. The conjugacy classes can be described as follows: One conjugacy class in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-3}i}b\right),0\leq i\leq 2^\alpha$ where i is odd and one conjugacy class in the form $\left(a^{2^{\alpha-3}i}b,\ a^{2^{\alpha-3}i+2^{\alpha-1}}\right),\ 0\leq i\leq 2^\alpha,$ where i is odd. Then, there are two conjugacy classes. Using Theorem 2.2, the probability that an element of a group fixes a set, $P_G(\Omega)$ is $\frac{2}{|\Omega|}$.

Theorem 3.2: Let G be a group of type,

(4), $G\cong\langle a,b: a^{2^\alpha}=b^2=1,[b,a]=a^{-2}\rangle$, $\alpha\geq 3$. Then $P_G(\Omega)=\frac{2}{|\Omega|}$.

Proof: If G acts on Ω by conjugation then there exists Ψ: $G\times\Omega\to\Omega$ such that $\Psi g(w)=gwg^{-1}$, $w\in\Omega$, $g\in a^{2\alpha-3i}b$, $a^{2\alpha-3i}$, $0\leq i\leq 2\alpha$. The elements of order two in G are $a^{2\alpha-1}$, $a^{2\alpha-2i}b$, $0\leq i\leq 2^\alpha$. Hence the elements of Ω can be described as follows: there are four elements in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-2}i}b\right),0\leq i\leq 2^\alpha$ and two elements in the form of $\left(a^{2^{\alpha-2}i}b,a^{2^{\alpha-2}i+2^{\alpha-1}}b\right),0\leq i\leq 2^\alpha$. From which it follows that $|\Omega| = 6$. If G acts on Ω by conjugation then $cl(w) = \{gwg^{-1}\}$. The conjugacy classes can be described as follows: one conjugate class in the form of $\left(a^{2^{\alpha-1}},a^{2^{\alpha-2}i}b\right),0\leq i\leq 2^\alpha$ and one conjugate class in the form $\left(a^{2^{\alpha-2}i}b,a^{2^{\alpha-2}i+2^{\alpha-1}}b\right),0\leq i\leq 2^\alpha$. Then we have two conjugacy classes, using Theorem 2.2, the probability that an elements of a group fixes a set $P_G(\Omega)=\frac{2}{|\Omega|}$.

Next, we find $P_G(\Omega)$ of the following presentation of metacyclic 2-groups of negative type of nilpotency class at least three.

Theorem 3.3: Let G be a group of type,

(5), $G \cong \langle a,b : a^{2^\alpha} = b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3$. Then,

$$P_G(\Omega) = \begin{cases} \frac{4}{|\Omega|}, & \text{if } \alpha=3, \\ \frac{2}{|\Omega|}, & \text{if } \alpha > 3. \end{cases}$$

Proof: We first find $P_G(\Omega)$ when $\alpha = 3$, then $G \cong D_{16}$. Using Theorem 3.1 in [15] $P_G(\Omega) = \frac{4}{|\Omega|}$ Second, if $\alpha > 3$ and G acts on Ω by conjugation then there exists Ψ: $G \times \Omega \to \Omega$ such that $\Psi_g(w) = gwg^{-1}$, $w \in \Omega$, $g \in \alpha^{2\alpha-3i}$, $a^{2\alpha-3i}b$, $0 \leq i \leq 2^\alpha$. The elements of oredr two in G are $a^{2\alpha-3i}b$, $a^{2\alpha-1i}$, $0 \leq i \leq 2^\alpha$. The elements of Ω can be described as follows: four elements in the form of $\left(a^{2^{\alpha-2}i}b, a^{2^{\alpha-1}}\right), 0 \leq i \leq 2^\alpha$ and two elements in the form $\left(a^{2^{\alpha-2}i}b, a^{2^{\alpha-2}i+2^{\alpha-1}}b\right), 0 \leq i \leq 2^\alpha$. Then $|\Omega| = 6$. If G acts on Ω by conjugation, then $\mathrm{cl}(w) = \{gwg^{-1}\}$. The conjugacy classes can be described as follows: one conjugacy class in the form $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-2}i}b\right), 0 \leq i \leq 2^\alpha$ and one conjugacy class in the form of $\left(a^{2^{\alpha-2}i}b, a^{2^{\alpha-2}i+2^{\alpha-1}}b\right), 0 \leq i \leq 2^\alpha$. Therefore, we have two conjugacy classes. Using Theorem 2.2, $P_G(\Omega) = \frac{2}{|\Omega|}$

Theorem 3.4: Let G be a group of type

(6), $G \cong \langle a,b : a^{2^\alpha} = b^{2^\beta} = 1, [b,a] = a^{-2} \rangle$, $\alpha \geq 3, \beta > 1$. Then

$$P_G(\Omega) = \begin{cases} \frac{2}{|\Omega|}, & \text{if } \alpha > 3, \\ 1, & \text{if } \alpha = 3. \end{cases}$$

Proof: If $\alpha = 3$ and G acts on Ω by conjugation, then there exists $\Psi : G \times \Omega \to \Omega$ such that $\Psi_g(w) = \left\{gwg^{-1}, w \in \Omega, g \in G\right\}$. The elements of order two are $a^{2^{\alpha-1}}b^{2^{\beta-1}}, b^{2^{\beta-1}}, a^{2^{\alpha-1}}$. Thus the elements of Ω can be described as follows: there are two elements in the form $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}}\right), 0 \leq i \leq 2^\alpha$ and one element in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$, from which it follows that $|\Omega| = 3$. If G acts on Ω by conjugation then $\mathrm{cl}(w) = \left\{gwg^{-1}, w \in \Omega, g \in G\right\}$. The conjugacy classes can be described as follows: One conjugacy class in the form $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$, one conjugacy class in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$, and one conjugacy class in the form $\left(a^{2^{\alpha-1}}, b^{2^{\beta-1}}\right)$. It follows that, there are three conjugacy classes. Using Theorem 2.2, the probability that an element of a group fixes a set $P_G(\Omega) = 1$. Now if $\alpha > 3$, the elements of Ω are similar to the elements in the case that $\alpha = 3$. Now if G acts on Ω by

conjugation, then $\mathrm{cl}(w) = gwg^{-1}, w \in \Omega$, $g \in a^{2^{\alpha-4}i}, a^{2^{\alpha-4}i}b^{2^{\beta-1}}, 0 \leq i \leq 2^\alpha$. Therefore,

the conjugacy classes can be described as follows: One conjugacy class of the form $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}}\right), 0 \leq i \leq 2^\alpha$ and one conjugacy class of the form $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}}\right)$. Therefore we have two conjugacy classes. Using Theorem 2.2, $P_G(\Omega) = \frac{2}{|\Omega|}$. ■

Theorem 3.5: Let G be a group of type (7), $G \cong \langle a,b : a^{2^\alpha} = 1, b^{2^\beta} = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3, \beta > 1$. Then

$$P_G(\Omega) = \begin{cases} \frac{2}{|\Omega|}, & \text{if } \alpha>3, \\ 1, & \text{if } \alpha = 3. \end{cases}$$

Proof: The proof is similar to the proof in Theorem 3.4. ■

Theorem 3.6: Let G be a group of type

(8), $G \cong \langle a,b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-1}}, [b,a]$

$= a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta > \gamma > 1$. Then

$$P_G(\Omega) = \frac{2}{|\Omega|}.$$

Proof: The proof is similar to the proof of Theorem 3.2. ■

Theorem 3.7: Let G be a group of type

(9), $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] =$

$a^{2^{\alpha-\gamma-2}} \rangle$, where $\alpha - \gamma > 1, \beta \geq \gamma > 1$. Then

$$P_G(\Omega) = \begin{cases} \frac{2}{|\Omega|}, & \text{if } \alpha > 4, \\ 1, & \text{if } \alpha = 4. \end{cases}$$

Proof: The proof is similar to the proof of Theorem 3.4. ■

In the following, we find the probability that a group element fixes a set of metacyclic 2-groups of nilpotency class two. We begin with type (1).

Theorem 3.8: Let G be a group of type

(1), $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma \in \square, \alpha > 2\gamma$, and $\beta \geq \gamma \geq 1$.

Then

$$P_G(\Omega) = \begin{cases} \frac{4}{|\Omega|}, & \text{if } \beta = \gamma = 1, \quad \alpha = 2, \\ 1, & \text{if } \beta > 1, \gamma = 1, \alpha \geq 3, \\ \frac{2}{|\Omega|}, & \text{if } \beta > 1, \gamma > 1, \alpha > 3. \end{cases}$$

Proof: Case I. If $\beta = \gamma = 1$ and $\alpha = 2$, then $G \cong D_8$ therefore the proof is similar to the proof of Theorem 3.1 in [15]. Case II. If $\beta > 1$, $\gamma = 1$ and $\alpha \geq 3$ and G acts on Ω by conjugation, then there exists $\Psi : G \times \Omega \to \Omega$ such that $\Psi_g(w) = \{gwg^{-1}, w \in \Omega, g \in G\}$. The elements of order two in G are $a^{2^{\alpha-1}}, b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}$. Thus the elements of Ω are described as follows: One element in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$, one element in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$, and one element in the form of $\left(a^{2^{\alpha-1}}, b^{2^{\beta-1}}\right)$. Therefore, $|\Omega| = 3$. If G acts on Ω by conjugation ,then the conjugacy classes can be described as follows: One conjugacy class in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$, one conjugate class in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$ and one conjugacy class in the form of $\left(a^{2^{\alpha-1}}, b^{2^{\beta-1}}\right)$. Therefore we have three conjugacy classes. Using Theorem 2.2, $P_G(\Omega) = 1$. Case III, If $\beta > 1$, $\gamma > 1$ and $\alpha > 3$. The elements of order two in G are $a^{2^{\alpha-1}}, b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}$. Thus the elements of Ω are described as follows: two elements in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$. and one element in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$. From which it follows that $|\Omega| = 3$. If G acts on Ω by conjugation, then the conjugacy classes are described as follows: One conjugacy class in the form of $\left(a^{2^{\alpha-1}}, a^{2^{\alpha-1}} i b^{2^{\beta-1}}\right), 0 \leq i \leq 2^{\alpha}$ and one conjugacy class in the form of $\left(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}}\right)$. Therefore we have two conjugacy classes. Using Theorem 2.2, $P_G(\Omega) = \frac{2}{|\Omega|}$. ■

Remark 3.1: The group of type

(2), $G \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$,

$P_G(\Omega)$ cannot be computed, since the only element in G of order two is a^2 thus there is no set Ω such that $|a| = |b| = 2$ where a and b commute.

CONCLUSION

In this paper, the probability that a group element fixes a set of metacyclic 2-groups of negative type of nilpotency class two and class at least three is found. However, it is proven that the probability of a group element fixes a set of metacyclic 2-groups of nilpotency class two of the second type, namely quaternion group of order eight cannot be computed.

ACKNOWLEDGEMENTS

The first author would like to acknowledge the Ministry of Higher Education in Libya for the scholarship.

REFERENCES

1. Miller, G.A., 1944. Relative Number of Non-invariant Operators in a Group. Proc. Nat. Acad. Sci. USA., 30(2): 25-28.

2. Erdos, P. and P. Turan, 1968. On Some Problems of a Statistical Group Theory. IV, Acta Math. Acad Sci. Hungaricae, 19: 413-435.
3. Gustafson, W.H., 1973. What is the Probability That Two Group Elements Commute?, The American Mathematical Monthly, 80(9): 1031-1034.
4. MacHale, D., 1974. How Commutative Can a Non-Commutative Group Be?, The Mathematical Gazette, 58(405): 199-202.
5. Omer, S.M.S., N.H. Sarmin, A. Erfanian and K. Moradipour, 2013. The Probability That an Element of a Group Fixes a Set and the Group Acts on Set by Conjugation, International Journal of Applied Mathematics and Statistics, 32(2): 111-117.
6. Reis, C., 2011. Abstract Algebra: An Introduction to Groups, Rings and Fields. World Scientific.
7. Nicholson, W.K., 2011. Introduction to Abstract Algebra. 4th ed, Wiley and sons. Inc.
8. Rotman, J.J., 2002. Advanced Modren Algebra. New Jersey 07458: Pearson Education,Inc. Upper Saddle River.
9. King, B.W., 1973. Presentation of Metacyclic Groups. Bull Aust Math Soc., pp: 103-131.
10. Beuerle, J.R., 2005. An Elementary Classification of Finite Metacyclic *p*-Groups of Class at Least Three. Algebra Colloq., 12(4): 553-562.
11. Goodman, F.M., 2003. Algebra abstract and concrete. Streesing Symmetry. 2nd Edition. New Jersey 07458: Pearson Education,Inc. Upper Saddle River.
12. Sherman, G.J., 1975. What is the Probability an Automorphsim Fixes a Group Element? The American Mathematical Monthly, 82(3): 261-264.
13. Moghaddam, M.R.R., F. Saeedi and E. Khamseh, 2011. The Probability of an Automorphism Fixing a Subgroup Element of a Finite Group. Asian-European Journal of Mathematics, 4(2): 301-308.
14. Omer, S.M S., N.H. Sarmin and A. Erfanian, 2013. The Probability that an Element of a Symmetric Group Fixes a Set and Its Application in Graph Theory, World Applied Sciences Journal, 27(12): 1637-1642.
15. El-sanfaz, M.A., N.H. Sarmin and S.M.S. Omer, 2014. The Probability That an Element of the Dihedral Groups Fixes a Set. International Journal of Applied Mathematics and Statistics, 52(1): 1-6.