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Research Article

The Schur Multiplier of Pairs of Groups of Order p^2q

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Let (G, N) be a pair of groups where G is a group and N is a normal subgroup of G. Then the Schur multiplier of pairs of groups (G, N) is a functorial abelian group $M(G, N)$. In this paper, $M(G, N)$ for groups of order p^2q where p and q are prime numbers are determined.

1. Introduction

The Schur multiplier was introduced by Schur [1] in 1904. The Schur multiplier of a group G, $M(G)$, is isomorphic to $R \cap [F, F]/[R, F]$ in which G is a group with a free presentation $1 \to R \to F \to G \to 1$. He also computed $M(G)$ for many different kinds of groups: for example, the dihedral group, metacyclic group, alternating group, and quaternion group. All computations of $M(G)$ were then compiled by Karpilovsky [2] in a book entitled "The Schur Multiplier."

In 1998, Ellis [3] extended the notion of the Schur multiplier of a group to the Schur multiplier of a pair of group, (G, N), where N is a normal subgroup of G. The Schur multiplier of a pair of groups, (G, N), is a functorial abelian group $M(G, N)$ whose principal feature is natural exact sequence

$$H_3(G) \xrightarrow{\eta} H_3\left(\frac{G}{N}\right) \longrightarrow M(G,N) \longrightarrow M(G) \xrightarrow{\mu} M\left(\frac{G}{N}\right) \longrightarrow \frac{N}{[N,G]} \longrightarrow (G)^{ab} \xrightarrow{\alpha} \left(\frac{G}{N}\right)^{ab} \longrightarrow 1, \quad (1)$$

in which $H_3(-)$ denotes some finiteness-preserving functor from groups to abelian groups (to be precise, $H_3(-)$ is the third homology of a group with integer coefficients). The homomorphisms η, μ, α are those due to the functorial of $H_3(-)$, $M(-)$, and $(-)^{ab}$. Ellis [3] also stated that, for any pair (G, N) of groups, $M(G, N) \cong \ker(N \wedge G \to G)$ where $N \wedge G$ is the exterior product of N and G. The exterior product $N \wedge G$ is obtained from $N \otimes G$ by imposing the additional relation $n \otimes g = 1$ for all $(n, g) \in N \wedge G$ and the image of a general element $n \otimes g$ in $N \wedge G$ is denoted by $n \wedge g$ for all $n \in N$ and $g \in G$.

The nonabelian tensor product, $G \otimes H$, was introduced by Brown and Loday [4] in 1987. $G \otimes H$ is the group generated by the symbols $g \otimes h$ subject to the relations

$$\begin{aligned} gg' \otimes h &= \left({}^{g}g' \otimes {}^{g}h\right)(g \otimes h), \\ g \otimes hh' &= (g \otimes h)\left({}^{h}g \otimes {}^{h}h'\right), \end{aligned} \quad (2)$$

for all g, g' in G and h, h' in H. $G \otimes H$ is used in computing the Schur multiplier of the direct product of two groups, $M(G \times H)$. Some computations of the nonabelian tensor product of cyclic group of p-power order have been done by Visscher [5] in 1998.

The nonabelian tensor square and Schur multiplier of groups of order p^2q, pq^2, and p^2qr has been computed by Jafari et al. [6]. In this paper, the Schur multiplier of pairs of groups of order p^2q where p and q are primes is determined.

In 2007, Moghaddam et al. [7] showed that $M(G, N) \cong R \cap [S, F]/[R, F]$ if S is a normal subgroup of F such that $N \cong S/R$. In 2012, Rashid et al. [8] determined the commutator subgroups of groups of order $8q$. The Schur multiplier, nonabelian tensor square, and capability of groups of order p^2q have been considered by Rashid et al. in [9], where p

and q are distinct primes. In [10], they also computed the nonabelian tensor square and capability of groups of order $8q$, where q is an odd prime.

2. Preliminaries

This section includes some preliminary results that are used in proving our main theorems.

Definition 1 (see [2]). A normal subgroup N of G is called a normal Hall subgroup of G if the order of N is coprime to its index in G.

Definition 2 (see [2]). $M(N)^T$ is defined as the T-stable subgroup of $M(N)$; that is, $M(N)^T = \{f \in M(N) \mid \mathrm{Con}_N^t(f) = f$ for all $t \in T\}$ where T is a subgroup of G in which G is the semidirect product of a normal subgroup N and a subgroup T, and $\mathrm{Con}_N^t(f)$ is the conjugation of t on f.

Proposition 3 (see [11]). *Let p and q be distinct primes and let G be a finite group of order p^2q. Then one of the following holds:*

(i) *$p > q$ and G has a normal Sylow p-subgroup;*

(ii) *$p < q$ and G has a normal Sylow q-subgroup;*

(iii) *$p = 2$, $q = 3$, $G \cong A_4$, and G has a normal 2-subgroup.*

Proposition 4 (see [9]). *Let G be a nonabelian group of order p^2q where p and q are distinct primes. Then exactly one of the following holds:*

(i) $G' \cong \mathbb{Z}_p$ *and* $G^{ab} \cong \mathbb{Z}_{pq}$;

(ii) $G' \cong \mathbb{Z}_{p^2}$ *and* $G^{ab} \cong \mathbb{Z}_q$;

(iii) $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *and* $G^{ab} \cong \mathbb{Z}_q$;

(iv) $G' \cong \mathbb{Z}_q$ *and* $G^{ab} \cong \mathbb{Z}_{p^2}$;

(v) $G' \cong \mathbb{Z}_q$ *and* $G^{ab} \cong \mathbb{Z}_p \times \mathbb{Z}_p$;

(vi) $G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition 5 (see [12]). *The factor group G/G' is abelian. If K is a normal subgroup of G such that G/K is abelian, then $G' \subseteq K$.*

Proposition 6 (see [5]). *Let $G \cong \mathbb{Z}_m$ and $H \cong \mathbb{Z}_n$ be cyclic groups that act trivially on each other. Then $G \otimes H \cong \mathbb{Z}_{(m,n)}$.*

Proposition 7 (see [2]). *Let G be a finite group. Then*

(i) *$M(G)$ is a finite group whose elements have order dividing the order of G.*

(ii) *$M(G) = 1$ if G is cyclic.*

Proposition 8 (see [2]). *If the Sylow p-subgroups of G are cyclic for all $p \mid |G|$, then $M(G) = 1$.*

Proposition 9 (see [2]). *Let N be a normal Hall subgroup of G and T a complement of N in G. Then*

$$M(G) \cong M(T) \times M(N)^T. \tag{3}$$

Proposition 10 (see [2]). *If G_1 and G_2 are finite groups, then*

$$M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1 \otimes G_2). \tag{4}$$

Proposition 11 (see [6]). *Let G be a finite nonabelian group. If G is a group of order p^2q, then*

$$M(G) = \begin{cases} 1, & if\ G' = \mathbb{Z}_q,\ G^{ab} = \mathbb{Z}_{p^2}, \\ \mathbb{Z}_p, & if\ G' = \mathbb{Z}_q,\ G^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p, \\ \mathbb{Z}_2, & if\ G^{ab} = \mathbb{Z}_2 \times \mathbb{Z}_2. \end{cases} \tag{5}$$

The following propositions are some of the basic results of the Schur multiplier of a pair deduced by Ellis [3], assuming only the existence of the natural exact sequence in (1) and the existence of a certain transfer homomorphism.

Proposition 12 (see [3]). *Let $N = 1$; then $M(G, N) = 1$.*

Proposition 13 (see [3]). *Let $N = G$; then $M(G, G) = M(G)$.*

Proposition 14 (see [3]). *Suppose that G is a finite group. Let the order of the normal subgroup N be coprime to its index in G and T a complement of N in G. Then $G \cong N \rtimes T$ and $M(G, N) \cong M(N)^T$.*

3. Main Result

In the following two theorems, the Schur multipliers of pairs of groups of order p^2q are stated and proved. We assume that the group is nonabelian.

Theorem 15. *Let G be a group of order p^2q where p and q are distinct primes, and $p < q$. If $N \triangleleft G$, then the Schur multiplier of pairs of G*

$$M(G, N) = \begin{cases} 1, & if\ G^{ab} \cong \mathbb{Z}_{p^2}\ or\ G^{ab} \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ & when\ N = 1\ or\ \mathbb{Z}_q, \\ \mathbb{Z}_p, & if\ G^{ab} \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ & when\ N = G, \mathbb{Z}_p, \mathbb{Z}_{pq}, \mathbb{Z}_p \times \mathbb{Z}_p\ or\ \mathbb{Z}_{p^2}, \end{cases} \tag{6}$$

where $G^{ab} = G/G'$.

Proof. Let G be a group of order p^2q where p and q are distinct primes, and $p < q$. Since $p < q$, then by Proposition 3 G has a normal Sylow q-subgroup: call it Q. Moreover, $[G : Q] = p^2$ so G/Q is abelian. Then by Proposition 5, we have $G' \subseteq Q$; that is, $G' = \mathbb{Z}_q$. Thus, by Proposition 4, $G^{ab} \cong \mathbb{Z}_{p^2}$ or $G^{ab} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Suppose $N \triangleleft G$; then the Schur multiplier of pairs of G is computed below.

Case 1. If $G^{ab} \cong \mathbb{Z}_{p^2}$ then by Proposition 11, $M(G) = 1$.

Since $M(G) = 1$, for all normal subgroups N of G, $M(G, N) \leq M(G) = 1$.

Case 2. If $G^{ab} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then by Proposition 11, $M(G) = \mathbb{Z}_p$.

(i) If $N = 1$ then by Proposition 12, $M(G,N) = M(G,1) = 1$.

(ii) If $N = G$ then by Proposition 13, $M(G,N) = M(G,G) = M(G)$. By Proposition 11, $M(G) = \mathbb{Z}_p$.

(iii) If $N = \mathbb{Z}_q$ then G is the semidirect product of $\mathbb{Z}_q$ and H in which $|N|$ and $[G : N]$ are coprimes, and N is a normal Hall subgroup of G (refer to Definition 1). Therefore by Proposition 14, $M(G,N) = M(N)^H = 1$ since $M(\mathbb{Z}_q) = 1$ (refer to Proposition 7). Note that, for this case, $G/N = G/G' \neq \mathbb{Z}_{p^2}$; that is, $G/N \neq \mathbb{Z}_{p^2}$.

(iv) If $N = \mathbb{Z}_p$ then G/N is nonabelian group of order pq. (If $G/N \cong \mathbb{Z}_{pq}$ then by Proposition 5, $G' \subseteq N$; that is, $\mathbb{Z}_q \subseteq \mathbb{Z}_p$ and this statement is a contradiction). Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong \mathbb{Z}_p$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Then $M(G,N) = \mathbb{Z}_p$.

(v) If $N = \mathbb{Z}_{pq}, \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ then by similar way as in (iv), $M(G,N) = \mathbb{Z}_p$.

□

Theorem 16. *Let G be a group of order p^2q where p and q are distinct primes, and $p > q$. If $N \lhd G$, then the Schur multiplier of pairs of G*

$$M(G,N) = \begin{cases} 1, & \text{if } G' \cong \mathbb{Z}_p \text{ or } G' \cong \mathbb{Z}_{p^2}, \text{ or } G' \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ & \text{when } N = 1 \text{ or } \mathbb{Z}_q, \\ \mathbb{Z}_p, & \text{if } G' \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ & \text{when } N = G,\ \mathbb{Z}_p \text{ or } \mathbb{Z}_p \times \mathbb{Z}_p, \end{cases} \quad (7)$$

where $G^{ab} = G/G'$.

Proof. Let G be a group of order p^2q where p and q are distinct primes, and $p > q$. Since $p > q$, G has a normal Sylow p-subgroup, namely, P (refer to Proposition 3). $[G : P] = q$ so G/P is abelian. Hence, $G' \subseteq P$ (refer to Proposition 5); that is, $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p$. Suppose $N \lhd G$; then the Schur multiplier of pairs of G is computed below.

(In this case $N = 1, \mathbb{Z}_q, \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p$ and G.)

Case 1. If $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $|G'| = p^2$ and $[G : G'] = q$ are coprimes. Then, by Definition 1, G' is a normal Hall subgroup of G. Therefore by Proposition 9, $M(G) = M(T) \times M(G')^T$ where T is a complement of G' and $T \cong \mathbb{Z}_q$. Thus, $M(G) = M(T) \times M(\mathbb{Z}_p \times \mathbb{Z}_p)^T$. $M(T) = 1$ (refer to Proposition 7). Hence, $M(G) = \mathbb{Z}_p$ (refer to Propositions 10, 6, and 7).

(i) If $N = 1$ then $M(G,N) = M(G,1) = 1$ (refer to Proposition 12).

(ii) If $N = G$ then $M(G,N) = M(G,G) = M(G)$ (refer to Proposition 13). Then, $M(G) = \mathbb{Z}_p$.

(iii) If $N = \mathbb{Z}_q$ then N is a normal Hall subgroup of G (refer to Definition 1) and G is the semidirect product of N and H in which H is a complement of N in G. Therefore by Proposition 14, $M(G,N) = M(N)^H = 1$ since $M(\mathbb{Z}_q) = 1$ (refer to Proposition 7).

(iv) If $N = \mathbb{Z}_p$ then G/N is nonabelian group of order pq. (If $G/N \cong \mathbb{Z}_{pq}$ then by Proposition 5, $G' \subseteq N$; that is, $\mathbb{Z}_p \times \mathbb{Z}_p \subseteq \mathbb{Z}_p$ and this statement is a contradiction). Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong \mathbb{Z}_p$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Then $M(G,N) = \mathbb{Z}_p$.

(v) If $N = \mathbb{Z}_p \times \mathbb{Z}_p$ then by similar way as in (iv), $M(G,N) = \mathbb{Z}_p$.

(vi) If $N = \mathbb{Z}_{pq}$ or $\mathbb{Z}_{p^2}$ then G/N is abelian group and $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p \subseteq N \cong \mathbb{Z}_{pq}$ or $\mathbb{Z}_{p^2}$ but this statement is a contradiction. So $M(G,N)$ when $N = \mathbb{Z}_{pq}$ or $\mathbb{Z}_{p^2}$ are not considered.

Case 2. If $G' \cong \mathbb{Z}_{p^2}$ then $G/G' \cong G^{ab} \cong \mathbb{Z}_q$. Hence, all Sylow subgroups of G are cyclic. Therefore, by Proposition 8, $M(G) = 1$. Thus, for all normal subgroups N of G, $M(G,N) \leq M(G) = 1$.

Case 3. If $G' \cong \mathbb{Z}_p$ then $M(G) = 1$ since $M(G) = M(G') \times M(K)$ where K is a group of order pq. By Propositions 7 and 8, $M(G') = 1$ and $M(K) = 1$. Thus, for all normal subgroups N of G, $M(G,N) \leq M(G) = 1$. □

4. Conclusion

For a group G of order p^2q where p and q are prime numbers, Q is the unique normal Sylow q-subgroups of G if $p < q$, while P is the unique normal Sylow p-subgroups of G if $p > q$. In this paper, we determined the Schur multiplier of pairs of groups of order p^2q. Our proofs show that $M(G,N)$ for groups of order p^2q is either 1 or $\mathbb{Z}_p$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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