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A reversal model of fuzzy time series in regional load forecasting

Riswan Efendi[*,†], Zuhaimy Ismail[†,§], Nor Haniza Sarmin[†,¶] and Mustafa Mat Deris[‡,‖]

[*]*Department of Mathematics, Faculty of Science and Technology,*
State Islamic University of Sultan Syarif Kasim Riau,
28294 Panam, Pekanbaru, Riau, Indonesia

[†]*Department of Mathematical Sciences, Faculty of Science,*
Universiti Teknologi Malaysia, 81310 Skudai,
Johor Bahru, Johor, Malaysia

[‡]*Faculty of Computer Science, Universiti Tun Hussein Onn Malaysia,*
86400 Batu Pahat, Johor, Malaysia
[*]*riswan.efendi@uin-suska.ac.id*
[§]*zuhaimy@utm.my*
[¶]*nhs@utm.my*
[‖]*mmustafa@uthm.edu.my*

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In fuzzy time series forecasting, the weight of fuzzy logical relationships and the midpoint of interval values are extensively used in the product of defuzzified matrix and the transpose of the weight matrix into the final forecast model. An improved forecast can be achieved through this model. Additionally, because of its excellent performance, successful applications can be widely achieved by exploiting real life data such as enrollment, stock indices and exchange rates for example. However, the improvement of mean square error is still not significant when the midpoint and the weight values are strictly increased in both of the matrixes. In order to reduce the mean square error, a reversal model in which the weight elements are reversed in the transpose matrix is proposed in this paper. Moreover, a theorem is also proposed to justify this reversal model and the annual data of electricity load by regions of Taiwan from 1981 to 2000 is examined as further evidence. The result indicate a consistently significant advantage in the proposed reversal model, i.e. the mean square error is smaller than the non-reversal model for each region.

Keywords: Fuzzy time series; weight element; forecasting; reversal model; transpose matrix; electricity load.

Nomenclature

$ARIMA$	Autoregressive Integrated Moving Average.
ANN	Artificial Neural Network.
PSO	Particle Swarm Optimization.
FTS	Fuzzy time series.
$FLRs$	Fuzzy logical relationships.
FLG	Fuzzy logical group.
$TAIEX$	Taiwan Stock Index.
A	A fuzzy subset.
U	Universe of discourse.
μ_A	Membership function of A.
$\mu_A(u_A)$	The degree of membership of the element u_i in A.
$F(t)$	A fuzzy time series.
Σ	An operator.
p	The interval or subinterval number.
RHS	Right-hand side.
LHS	Left-hand side.
OWA	Ordered weighting averaging.
$F(A_i)_{NR}$	Non-reversal model.
$F(A_i)_R$	Reversal model.
$\times$	Matrix product operator.
$\mathbf{M}(A_i)$	$(1 \times k)$ matrix.
$\mathbf{W}(A_i)^T$	$(k \times 1)$ matrix.
NR	Non-reversal model.
R	Reversal model.
MSE	Mean Squared Error.

1. Introduction

Forecasting is a predictive analytical approach that deals with predicting the future, generally by using the past data set and corresponding models. It can be applied in various domains of management such as personnel management, resource management, finance management, and organizational management. The study on electric load forecasting models continues to remain a significant concern worldwide, especially among electricity companies and the output of these studies is very determinative for energy planning and power management [1]. The methods frequently used for electricity forecasting are ARIMA, regression time series, time series, genetic algorithm, artificial intelligence, and PSO. The fuzzy time series approach was first introduced in [1] and it is another alternative to resolve electricity forecasting problems. The concept of fuzzy time series is a combination between time series analysis and the fuzzy set theory [2].

In the applications of real scenarios, numerous models have been proposed by researchers in FTS forecasting to resolve problems in various areas such as university enrolment [2–14], stock prices index [15–18], temperature [19], financial sectors [20, 21], and electricity load consumptions [1, 22–25]. FTS is a novel approach, which was introduced in [2, 3] for resolving linguistic time series data problems. This approach is a combination of fuzzy logic and time series analysis. Additionally, the most critical part in the FTS forecasting is the assumption regarding the unneeded data, which is the main difference from the statistical approaches. In FTS approach, the final forecasted value has been modeled by using fuzzification, FLRs, FLG, and defuzzification.

Many different models have been proposed for fuzzy time series forecasting by researchers. For example, in [13], a study on heuristic models of the fuzzy time series for forecasting by using stock index data was initiated. A fuzzy time series model for stock index was then analyzed in [14]. The weighted fuzzy time series model has been suggested in [15] for forecasting of the TAIEX. This weight was assigned using the recurrent FLRs in the FLG. In addition, a trend-weighted fuzzy time series model is also discussed in [16] for TAIEX forecasting. A generalized approach for forecasting by using the fuzzy weights is presented in [17]. In [15], the final forecasted value was equal to the product of the defuzzified matrix and the transpose of the weight matrix. However, the model proposed by Yu [15] is yet to be improved to resolve the monotonic increasing of weight values if the chronological order occurred in the FLG.

This study introduces a new reversal model in fuzzy time series forecasting. Through this model, the monotonic (strictly) increase of the intervals midpoints and the weight values can be resolved if the chronological order of FLRs occurs in the FLG. Moreover, the position elements of weight will be reversed from maximum to minimum values respectively in the transpose matrix. Furthermore, by using the product rule, this transpose matrix and the defuzzified matrix can be implemented to determine the final forecasted value. To justify this proposed model, a theorem is also presented, supported with the proof.

The remainder of this paper is organized as follows: Section 2 presents the basic theory of fuzzy set and fuzzy time series; Section 3 describes the types of FLRS in the FLG and the importance of weighted fuzzy time series with some examples; The importance of weight fuzzy time series in forecasting is explained in Section 4; Section 5 explores the reversal model as a proposed model and an empirical study of electricity load forecasting by regions of Taiwan; some conclusions of this study are presented in the final section.

2. The Fundamental Theories in Fuzzy Time Series

2.1. *Fuzzy time series concept*

FTS is a new approach which was developed for resolving linguistic time series data problems in [2, 3]. This approach combines linguistic variables with the analysis process of applying fuzzy logic into time series to solve the fuzziness of the data. The most important part in fuzzy time series forecasting is the assumption regarding data that are not needed, which is what mainly differentiates it from the statistical approaches. For example, the number of observations does not need to be limited and the linearity assumption does not have to be considered. Overall, the model has been established by using fuzzification, FLRs, FLG, and defuzzification. The applications of this proposed model can be found in some domain problems as mentioned before in Section 1. Hence, this section describes the fuzzy set, fuzzy time series, other related terms, and the forecasting algorithm that is used in this paper.

2.2. *Fuzzy set definition*

We begin by defining A on U and in doing so we mainly follow [2]:

$$A = \{(u_i, \mu_i(u_i)) \,|\, u_i \in U\}, \tag{1}$$

where $\mu_A : U \to [0,1]$. If U is defined as finite and infinite sets, then A can be expressed as follows [2]:

$$A = \sum \frac{\mu_A(u_i)}{u_i} = \frac{\mu_A(u_1)}{u_1} + \frac{\mu_A(u_2)}{u_2} + \cdots + \frac{\mu_A(u_n)}{u_n}, \tag{2}$$

and

$$A = \int \frac{\mu_A(u_i)}{u_i} du, \quad \forall\, u_i \in U. \tag{3}$$

2.3. *FTS definitions and the forecasting algorithm*

Definition 1. Fuzzy time series [2]
Let $Y(t)$ $(t = 0, 1, 2, \ldots)$, a subset of real numbers, be the universe of discourse on which fuzzy sets $f_i(t)$ $(i = 1, 2, \ldots)$ are defined in the universe of discourse. $Y(t)$ and $F(t)$ is a collection of $f_i(t)$ $(i = 1, 2, \ldots)$. Then $F(t)$ is called a fuzzy time series defined on $Y(t)$ $(t = 0, 1, 2, \ldots)$. Therefore, $F(t)$ can be understood as a linguistics time series variable, where $f_i(t)$ $(i = 1, 2, \ldots)$, are possible linguistics values of $F(t)$.

Definition 2. Fuzzy relation [2]
If there exists a fuzzy relationship $R(t-1,t)$, such that $F(t) = F(t-1) \circ R(t-1,t)$, then $F(t)$ is said to be caused by $F(t-1)$ as denoted as

$$F(t-1) \to F(t). \tag{4}$$

Definition 3. First-order model in Fuzzy time series [2]
Suppose $F(t)$ is caused by $F(t-1)$ denoted by $F(t-1) \to F(t)$, then this relationship can be represented as:

$$F(t) = F(t-1) \circ R(t, t-1), \tag{5}$$

where $R(t, t-1)$ is a fuzzy relationship between $F(t)$ and $F(t-1)$ and is called the first-order model of $F(t)$.

Definition 4. FLRs [15]
Let $F(t-1) = A_i$ and $F(t) = A_j$. The relationship between two consecutive data (called a FLR), i.e., $F(t)$ and $F(t-1)$, can be denoted as $A_i \to A_j, i, j = 1, 2, \ldots, p$ is called the LHS, and A_j is the RHS of the FLR.

Definition 5. FLG [15]
Let $A_i \to A_j, A_i \to A_k, \ldots, A_i \to A_p$ are FLRs with the same LHS which can be grouped into an ordered FLG by putting all their RHS together as on the RHS of the FLG. It can be written as:

$$A_i \to A_j, A_i \to A_k, \ldots, A_i \to A_p; \quad i, j, k, \ldots, p = 1, 2, \ldots, n \ (n \in N). \tag{6}$$

Meanwhile, the forecasting algorithm can be delineated into several steps as suggested in [2]:

- U is defined and divided into several equal length of intervals.
- Each interval is fuzzified into linguistics time series values ($A_i, i = 1, 2, \ldots, p, p$ is partition number).
- The fuzzy logical relationships are established among linguistics time series values, $A_i \to A_j, i, j = 1, 2, \ldots, p$.
- The forecasting rule is established.
- The forecast value is determined.

3. The Fuzzy Logical Relationship Types

In order to know the types of FLRs in the FLG, we should define some related terms in fuzzy relation such as recurrence, non-recurrence, chronological order and non-chronological order of FLRs as bellow:

Definition 6. Recurrence and Non-recurrence of FLR
Let $A_i \to A_j, A_j, A_k, A_l, A_j, A_j, A_m$ $(i, j, k, l, m \leq p \in \mathbb{N})$ be an FLG. The occurrence of a particular FLR A_j represents the number of its appearances in the past. Thus, this condition is called as recurrence. While, the other condition is called as non-recurrence.

Definition 7. Chronological Order and Non-chronological Order of FLR
Let $A_i \to A_j, A_k, A_l, A_m$ $(j < k < l < m$ and $i, j, k, l, m \leq p \in \mathbb{N})$ be an FLG. The occurrence of each particular of A_j, A_k, A_l, A_m represents the strictly increasing of index number and follows the generated time series. Thus, this condition is called as chronological order. While, other conditions are called as non-chronological order.

In consideration of the previous studies and both Definitions 5 and 6, the types of FLRs can be classified as listed below:

a. *Recurrence with chronological order*

For this type, each FLR can occur more than one time and also be in chronological order in the FLG. For example, let $A_3 \to A_1, A_1, A_2, A_3, A_3, A_4, A_5$ be an FLG.

Through this example, the linguistics time series data of A_1 and A_3 can occur more than one time and the index numbers of these linguistics can increase monotonously.

b. *Recurrence with chronological order*

This type does not have too much differences compared with type (a) except that no chronological order can be found in the FLG among FLRs. For example, let $A_3 \to A_3, A_2, A_2, A_5, A_3, A_4, A_5$ be an FLG. This example shows that the linguistics time series data of A_2 and A_3 can occur more than one time, but the time events are not in chronological order respectively.

c. *Non-recurrence with chronological order*

For this type, no recurrence can be found among FLRs in the FLG, but the time events are in chronological order. For example, let $A_3 \to A_3, A_4, A_5, A_6, A_7$ be an FLG. Through this example, each FLR occurs only one time in FLG. Additionally, the time events of FLRs are in chronological order in FLG.

d. *Non-recurrence without chronological order*

For this type, the recurrence of FLRs cannot be found in the FLG. Moreover, the time event of each FLR occurs randomly in the FLG. For example, let $A_3 \to A_4, A_6, A_5, A_2, A_3$ be an FLG. This sample shows that no recurrence can be found among FLR. Furthermore, the time event of FLR does not occur in chronologically order in the FLG.

4. The Importance of Weight in Fuzzy Time Series Forecasting

In fuzzy time series, the forecasting model uses the fuzzy relationships among the linguistic time of series values. Two fuzzy types of relationships are (i) the same-fuzzy logical relationship and (ii) the different-fuzzy logical relationship. Both types of relationships may occur either recurrently or frequently. The occurrence of a particular fuzzy relationship explains the number of its appearances in the past. Some of the reasons for establishing the weight factor are:

(1) To compensate for the presence of bias especially when the events are frequently occurred [26].
(2) To raise the influence of the more accurate input data, and to reduce the influence of the less accurate ones [27].

Fundamentally, these are the reasons for finding the weights in the fuzzy relationships. Similar to the research findings in [15] and [16], the weight factors were denoted within the weight matrix given in the following definition.

Definition 8. Ordered weighting averaging [28].

Yager's OWA operator of dimension n is a mapping

$$\varnothing : \mathbb{R}^n \rightarrow \mathbb{R},$$

which has an associated weights $\mathbf{W} = (w_1 \ \ w_2 \ \ w_3 \ \ \cdots \ \ w_n)^T$ or can be written as:

$$\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \tag{7}$$

such that

(i) $w_i \in [0, 1]$
(ii) $\sum_{i=1}^{n} w_i = 1$

$$\varnothing(a) = \varnothing(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}, \tag{8}$$

where $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is a permutation function, such that $a_{\sigma(i)}$ is the highest value in the set $\{a_1, \ldots, a_n\colon a_{\sigma(i)} \geq a_{\sigma(i-1)}\}$.

The recurrence of FLRs in the FLG has been solved using weighted fuzzy time series in [15, 16], both weights rules are detailed as follows:

(a) Yu's weight rule
There were two reasons why the weight of FTS was suggested. The first reason was to resolve the recurrence of FLRs, which were not properly handled in previous related studies. The other one was to improve the forecast accuracy. The issues were elaborated by the following examples:
Let A_1, A_2, A_1, A_1, A_1, A_1 be linguistics time series values. Based on the Definitions 3, 4, and Yu's rule, then the FLRs, FLG and weights were described as follows:

- Establishing the FLRs: $A_1 \rightarrow A_2, A_2 \rightarrow A_1, A_1 \rightarrow A_1, A_1 \rightarrow A_1, A_1 \rightarrow A_1$. Thus, there were five relationships among the linguistic time series values.
- Establishing the FLG: $A_1 \rightarrow A_2, A_1, A_1, A_1$ was called as the first group and $A_2 \rightarrow A_1$ was called as the second group. A_1 had one relationship with A_2, but A_1 had 3 recurrent fuzzy relations with itself. On the other hand, there was no recurrence of A_2.
- Determining the weight values: $A_1 \rightarrow A_2, w_1 = 1/10, A_1 \rightarrow A_1, w_2 = 2/10, A_1 \rightarrow A_1, w_3 = 3/10, A_1 \rightarrow A_1, w_4 = 4/10$. The total value of w_2, w_3, and w_4 was more than w_1, because there were three recurrences of A_1, while there was no recurrence of A_2 in this group. Moreover, no weight was found for the second group. Yu proposed that the nominators of weights should be determined by using the natural number ($i = 1, 2, \ldots, n$). However, Yu's

rule indicates that weights increased following the number of relationships. In addition, these weights were applied to the forecasting approach.

(b) Cheng's *et al.* weight rule
Cheng *et al.* [16] also considered the trend-weighted recurrence of FLRs in the FLG. Moreover, by using the same example with part (a), the assigning of trend-weighted components for fuzzy relations were explained as follows:

- Establishing the FLRs: $A_1 \to A_2, A_2 \to A_1, A_1 \to A_1, A_1 \to A_1, A_1 \to A_1$. Thus, there were five relationships among the linguistics time series values.
- Establishing the FLG: $A_1 \to A_2, A_1, A_1, A_1$ was called as the first group and $A_2 \to A_1$ was called as the second group. A_1 had one relationship with A_2, but A_1 had 3 fuzzy relations with itself. On the other hand, there was no recurrence of A_2.
- Determining the weight values: $A_1 \to A_2, w_1 = 1/7, A_1 \to A_1, w_2 = 1/7, A_1 \to A_1, w_3 = 2/7, A_1 \to A_1, w_4 = 3/7$. The sum value of w_2, w_3, and w_4 was more than w_1, because that there were three recurrences of A_1, while there was no recurrence of A_2 in this group. According to this rule, the numerators of weights increased following the recurrence and the same left-hand sides of FLRs in FLG. Thus, these weights were called as a trend. The same as the findings in [15], which these numerators were also assigned with the natural numbers.

Definition 9. Monotonic sequences
The sequence (x_n) is said to be:

- monotonically increasing, or simply increasing, if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$;
- strictly increasing if $x_{n+1} > x_n$ for all $n \in \mathbb{N}$;
- monotonically decreasing, or decreasing, if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$;
- strictly decreasing if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.

5. The Reversal Model

The assigning weight fuzzy time series was introduced for handling the recurrence of relationships in the FLG [15]. In this paper, we consider the reversal method based on the recurrence and non-recurrence with chronological order of FLRs in the FLG. On the other hand, if Yu's rule is applied to these types of FLRs, then weight values would be increased monotonically in the weight transpose matrix. Moreover, the midpoint interval values would also be strictly increased in the defuzzified matrix. Therefore, the product of both matrixes would be increased sharply. In order to tackle this problem, we propose a new model named reversal model. Through this model, the weight elements are reversed respectively in the transpose matrix.

Suppose $A_i \to A_1, A_2, A_3, \ldots, A_p$ be an FLG ($i = 1, 2, 3, \ldots, p$). The corresponding weights for $A_1, A_2, A_3, \ldots, A_p$, say, $c_1, c_2, \ldots, c_p$ were specified. By applying Yu's model, the weights values were derived as follows:

Given $c_1 = 1, c_2 = 2, \ldots, c_p = p$ and $w_1 = \frac{c_1}{\sum_{i=1}^{p} c_i}, \ldots, w_p = \frac{c_p}{\sum_{i=1}^{p} c_i}, p \in \mathbb{N}$. Thus

$$\begin{aligned}\mathbf{W}(A_i) &= \left[\frac{c_1}{c_1 + \cdots c_p} \quad \frac{c_2}{c_1 + \cdots c_p} \quad \cdots \quad \frac{c_p}{c_1 + \cdots c_p}\right] \\ &= \left[\frac{c_1}{\sum_{i=1}^{p} c_i} \cdots \frac{c_p}{\sum_{i=1}^{p} c_i}\right] \\ &= \left[\frac{1}{\sum_{i=1}^{p} i} \ \frac{2}{\sum_{i=1}^{p} i} \cdots \frac{p}{\sum_{i=1}^{p} i}\right],\end{aligned} \tag{9}$$

where $\sum_{i=1}^{p} w_i = \frac{\sum_{i=1}^{p} c_i}{\sum_{i=1}^{p} c_i} = 1$ had satisfied both the condition and Definition 5. Furthermore, weight elements could also be presented in the weight matrix $\mathbf{W}$ as shown below:

$$\mathbf{W}(A_i) = [w_1 \ \ w_2 \ \ \cdots \ \ w_p]. \tag{10}$$

By using Eq. (9) and applying Definition 9, the weight elements are strictly increasing in this matrix, hence, it also satisfied the following condition:

$$w_1 < w_2 < \cdots < w_n. \tag{11}$$

Suppose the forecast of $F(A_i)$ is $A_1, A_2, A_3, \ldots, A_p$. The defuzzified matrix was equal to the matrix of the midpoints of $A_1, A_2, A_3, \ldots, A_p$:

$$\mathbf{M}(A_i) = [m_1 \ \ m_2 \ \ \cdots \ \ m_p]. \tag{12}$$

In this matrix, the midpoint values were also strictly increasing, written as:

$$m_1 < m_2 < \cdots < m_p. \tag{13}$$

The final forecast of $F(A_i)$ was equal to the product of defuzzified matrix and the transpose of the weight matrix [15].

$$F(A_i) = \mathbf{M}(A_i) \times \mathbf{W}(A_i)^T. \tag{14}$$

Furthermore, Eq. (14) is called as a NR. In this model, the forecasting values of A_i are always higher than the actual values of A_i because the interval midpoints (Eq. (13)) and the weights values (Eq. (11)) are strictly increasing. Additionally, these values would influence the level of forecasting accuracy significantly. To resolve these increasing values, a new model is proposed as follows:

Given

$$W(A_i)^T = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{bmatrix}. \tag{15}$$

The element weights were reversed in the transpose matrix as written below:

$$W(A_i)^T = \begin{bmatrix} w_p \\ w_{p-1} \\ \vdots \\ w_1 \end{bmatrix}. \tag{16}$$

Substituting Eq. (16) in Eq. (14) provides:

$$F(A_i) = \mathbf{M}(A_i) \times \mathbf{W}(A_i)^T,$$

$$F(A_i) = [m_1 \;\; m_2 \;\; \cdots \;\; m_p] \times \begin{bmatrix} w_p \\ w_{p-1} \\ \vdots \\ w_1 \end{bmatrix}. \tag{17}$$

Equation (17) is as R. Even though this equation looks similar with Eq. (14), both models are different in the weight transpose matrix. This model is proposed to be suitable for the scenario of strictly increasing weight values. Furthermore, in order to justify the capability of this model, a theorem is proposed as follows:

Theorem. *Non Reversal and Reversal Models*

Suppose $A_i \to A_1, A_2, A_3, \ldots, A_p$ be a non-recurrence in the FLG $(i = 1, 2, 3, \ldots, p)$ and corresponding weights for $A_1, A_2, A_3, \ldots, A_p$ are $w_1, w_2, \ldots, w_p$ respectively. The defuzzified value of the midpoint of $A_1, A_2, A_3, \ldots, A_p$ are $m_1, m_2, \ldots, m_p$ respectively. Let $\{m_1, m_2, \ldots, m_p\} \in \mathbb{R}^+$ and $\{w_1, w_2, \ldots, w_p\} \in \mathbb{R}^+$ which the values of the midpoint and the weight were strictly increasing. If the final forecast of $F(A_i)$ was equal to the product of defuzzified matrix and the transpose of the weight matrix, then the forecasted values from non-reversal model $\hat{F}(A_i)_{\mathrm{NR}}$ would be greater than the reversal model $\hat{F}(A_i)_{\mathrm{R}}$.

Proof. Let

$$\text{non-reversal model } \hat{F}(A_i) = [m_1 \;\; m_2 \;\; \cdots \;\; m_p] \times [w_1 \;\; w_2 \;\; \cdots \;\; w_p]^T$$

and

$$\text{reversal model } \hat{F}(A_i) = [m_1 \;\; m_2 \;\; \cdots \;\; m_p] \times [w_p \;\; w_{p-1} \;\; \cdots \;\; w_1]^T,$$

we would like to prove that $\hat{F}(A_i)_{\mathrm{NR}} > \hat{F}(A_i)_{\mathrm{R}}$. Since the values of the midpoint and the weight were strictly increasing, which indicate that $m_{p+1} > m_p$ and $w_{p+1} > w_p$ for all $p \in \mathbb{N}$, thus

$$m_1 \cdot w_1 < m_2 \cdot w_1$$

since $w_1 < w_2$,

$$m_1 \cdot w_1 < m_2 \cdot w_1 < m_2 \cdot w_2,$$

and since $w_2 < w_3$,

$$m_2 \cdot w_2 < m_3 \cdot w_2 < m_3 \cdot w_3. \tag{18}$$

If we added more of midpoint interval values until the numbers in Eq. (18) reaches p, so

$$w_{p-1} < w_p,$$

thus

$$m_{p-2} \cdot w_{p-1} < m_{p-1} \cdot w_{p-1} < m_p \cdot w_p,$$

then

$$m_1 \cdot w_1 < m_2 \cdot w_2 < m_3 \cdot w_3 < \cdots < m_{p-1} \cdot w_{p-1} < m_p \cdot w_p.$$

According to Eq. (18), it is clear that $m_1 \cdot w_1 < m_3 \cdot w_3$, by adding $m_2 \cdot w_2$ to both sides of the equation we obtain,

$$(m_1 \cdot w_1) + (m_2 \cdot w_2) < (m_2 \cdot w_2) + (m_3 \cdot w_3).$$

Since $w_1 < m_1, w_2 < m_2$, and $w_3 < m_3$, thus

$$(m_1 \cdot w_1) + (m_2 \cdot w_2) > (m_1 \cdot w_2) + (m_2 \cdot w_1), \tag{19}$$

$$(m_2 \cdot w_2) + (m_3 \cdot w_3) > (m_2 \cdot w_3) + (m_3 \cdot w_2). \tag{20}$$

This gives

$$(m_{p-1} \cdot w_{p-1}) + (m_p \cdot w_p) > (m_{p-1} \cdot w_p) + (m_p \cdot w_{p-1}). \tag{21}$$

According to (19)$\sim$(21), then

$$\begin{aligned}(m_1 \cdot w_1) + (m_2 \cdot w_2) + (m_3 \cdot w_3) + \cdots + (m_{p-1} \cdot w_{p-1}) + (m_p \cdot w_p) > (m_1 \cdot w_p) \\ + (m_2 \cdot w_{p-1}) + \cdots + (m_{p-1} \cdot w_2) + (m_p \cdot w_1),\end{aligned}$$

so

$$\begin{aligned}&[m_1 \;\; m_2 \;\; m_3 \;\; \ldots \;\; m_{p-1} \;\; m_p] \times [w_1 \;\; w_2 \;\; w_3 \;\; \ldots \;\; w_{p-1} \;\; w_p]^T \\ &\quad > [m_1 \;\; m_2 \;\; m_3 \;\; \ldots \;\; m_{p-1} \;\; m_p] \times [w_p \;\; w_{p-1} \;\; \ldots \;\; w_2 \;\; w_1]^T,\end{aligned}$$

which implies

$$\hat{F}(A_i)_{\mathrm{NR}} > \hat{F}(A_i)_{\mathrm{R}}. \tag{22}$$

Furthermore, we illustrate the implementation of this theorem via the following examples.

Example 1. Recurrence with chronological order (type a)

Let $A_3 \rightarrow A_1, A_1, A_2, A_4, A_3, A_3, A_5$ be an FLG. Suppose the midpoint values $m_1, m_1, m_2, m_4, m_3, m_3, m_5$ are 89, 89, 91, 95, 93, 93, 97 respectively.

Both non-reversal and reversal models are applied to get the final forecast of A_3 as comparison below.

Given an FLG with 7 FLRs from A_3, in which $c_1 = 1, c_2 = 2, \ldots, c_7 = 7$,

$$w_1 = \frac{1}{\sum_{i=1}^{7} c_i}, \ldots, w_7 = \frac{7}{\sum_{i=1}^{7} c_i}$$

$$w_1 = \frac{1}{28}, \ldots, w_7 = \frac{7}{28}.$$

Thus

$$\mathbf{W}(A_3) = [w_1 \;\; w_2 \;\; \cdots \;\; w_7] = \left[\frac{1}{28}\,\frac{2}{28} \cdots \frac{7}{28}\right] = [0.04 \;\; 0.07 \;\; 0.11 \;\; 0.14 \;\; 0.18 \;\; 0.21 \;\; 0.25]$$

- Non-Reversal Model
  $\mathbf{M}(A_3) = [89 \;\; 89 \;\; 91 \;\; 95 \;\; 93 \;\; 93 \;\; 97]$ and $\mathbf{W}(A_3) = [0.04 \;\; 0.07 \;\; 0.11 \;\; 0.14 \;\; 0.18 \;\; 0.21 \;\; 0.25]$.
  Thus

$$\begin{aligned}\hat{F}(A_3) &= \mathbf{M}(A_3) \times \mathbf{W}(A_3)^T \\ &= [89 \;\; 89 \;\; 91 \;\; 95 \;\; 93 \;\; 93 \;\; 97] \times [0.04 \;\; 0.07 \;\; 0.11 \;\; 0.14 \;\; 0.18 \;\; 0.21 \;\; 0.25]^T \\ &= 93.64.\end{aligned}$$

- Reversal Model

$$\begin{aligned}\hat{F}(A_3) &= \mathbf{M}(A_3) \times \mathbf{W}(A_3)^T \\ &= [89 \;\; 89 \;\; 91 \;\; 95 \;\; 93 \;\; 93 \;\; 97] \times [0.25 \;\; 0.21 \;\; 0.18 \;\; 0.14 \;\; 0.11 \;\; 0.07 \;\; 0.04]^T \\ &= 91.12.\end{aligned}$$

According to the non-reversal and reversal models, the final forecasted values of A_3 are 93.64 and 91.12 respectively. As the actual value of A_3 was 92.10, the difference between the actual value and the forecasted value by the non-reversal model is larger than the difference between actual value and forecasted value by the reversal method. Therefore, the forecasting error based on non-reversal model is also larger in comparison to the error from reversal model we propose in this paper.

Example 2. Non-Recurrence with chronological order (type c)

Let $A_3 \rightarrow A_1$, A_2, A_3, A_4, A_5 be an FLG. Suppose the midpoint of A_1, A_2, A_3, A_4, A_5 are 89, 91, 93, 95, 97 respectively. Then, the weights values and the final forecast of A_3 can be calculated by using both non-reversal and reversal models, the final forecast of A_3 are:

Given an FLG with 5 FLRs from A_3 which $c_1 = 1, c_2 = 2, \ldots, c_5 = 5$ and the weight values for A_3 as follow:

$$w_1 = \frac{1}{\sum_{i=1}^{5} c_i}, \ldots, w_5 = \frac{5}{\sum_{i=1}^{5} c_i}$$

$$w_1 = \frac{1}{15}, \ldots, w_5 = \frac{5}{15}$$

Thus,

$$\mathbf{W}(A_3) = [w_1 \;\; w_2 \;\; \cdots \;\; w_5] = \left[\frac{1}{15} \;\; \frac{2}{15} \cdots \frac{5}{15}\right] = [0.07 \;\; 0.13 \;\; 0.20 \;\; 0.27 \;\; 0.33]$$

Furthermore, the final forecast of A_3 is derived by using both non-reversal and reversal models as follows:

- Non-Reversal Model
  $\mathbf{M}(A_3) = [89 \;\; 91 \;\; 93 \;\; 95 \;\; 97]$ and $\mathbf{W}(A_3) = [0.07 \;\; 0.13 \;\; 0.20 \;\; 0.27 \;\; 0.33]$.
  Thus

$$\begin{aligned}\hat{F}(A_3) &= \mathbf{M}(A_3) \times \mathbf{W}(A_3)^T \\ &= [89 \;\; 91 \;\; 93 \;\; 95 \;\; 97] \times [0.07 \;\; 0.13 \;\; 0.20 \;\; 0.27 \;\; 0.33]^T \\ &= 94.32\end{aligned}$$

- Reversal Model

$$\begin{aligned}\hat{F}(A_3) &= \mathbf{M}(A_3) \times \mathbf{W}(A_3)^T \\ &= [89 \;\; 91 \;\; 93 \;\; 95 \;\; 97] \times [0.33 \;\; 0.27 \;\; 0.20 \;\; 0.13 \;\; 0.07]^T \\ &= 91.68\end{aligned}$$

The final forecasted values of A_3 by using non reversal and reversal models are 94.32 and 91.68 respectively. Compared to the actual value of A_3, which was 92.10, the

Table 1. Annual electric load data from 1981 to 2000 in Taiwan by regions (Mega-Watts).

Year	North	Central	Southern	Eastern
1981	3388	1663	2272	122
1982	3523	1829	2346	127
1983	3752	2157	2494	148
1984	4296	2219	2686	142
1985	4250	2190	2829	143
1986	5013	2638	3172	176
1987	5745	2812	3351	206
1988	6320	3265	3655	227
1989	6844	3376	3823	236
1990	7613	3655	4256	243
1991	7551	4043	4548	264
1992	8352	4425	4803	292
1993	8781	4594	5192	307
1994	9400	4771	5352	325
1995	10254	4483	5797	343
1996	11222	5061	6336	358
1997	10719	4935	6369	363
1998	11642	5061	6318	397
1999	11981	5233	6259	401
2000	12924	5633	6804	420

difference between actual value and the forecasted value by non-reversal method is larger. In other words, the reversal model again provides a better forecast as it shows smaller differences between the original value and the forecast. In general, the forecasting error by the reversal model is smaller than the non-reversal model.

Example 3. Comparison of mean square error (MSE) based on real data.

Both non-reversal and reversal models are applied in forecasting the yearly electric load of Taiwan by regions in [29], in which the data covers the period from 1981 to 2000 as presented in Table 1.

By using Yu's algorithm, the performance of forecasting measured by MSE for both models is illustrated in Table 2.

In order to evaluate the performances of both models more clearly, the time series plot between actual data (Eastern region) and the forecasted values by both models is also illustrated in Fig. 1.

By comparing the actual load of the Eastern region with forecasted values from the non-reversal model and reversal model, Fig. 1 indicates that reversal model

Table 2. Comparative MSE between non-reversal and reversal models.

MSE	Non-reversal	Reversal
North	36843.3217	24997.7174
Central	3195.0985	3066.1605
Southern	7751.0529	5819.9045
Eastern	30.8836	22.0894
Rank	2	1

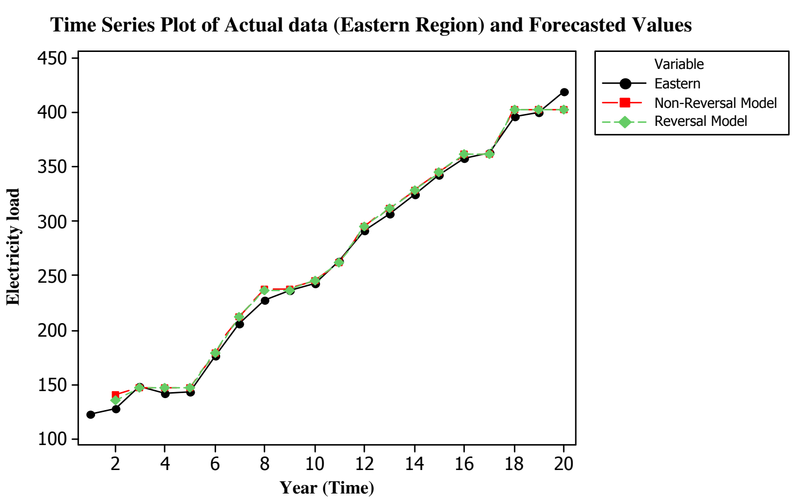

Fig. 1. Actual value and forecasted values by non-reversal and reversal models for the Eastern region in Taiwan.

Table 3. Comparative aspects between non-reversal and reversal models.

Non-reversal Model	Reversal Model
• Easier to be applied in determining the final forecasted values of A_i. • Ineffective to handle the strictly increasing values of the midpoint intervals and the weight FLRs. • Effective to handle the strictly decreasing values of the midpoint intervals and the weight FLRs. • Unpromising to reduce the forecasting error or mean square error especially for type (a, c) of FLRs occurred.	• Easier to be applied in determining the final forecasted values of A_i. • Effective to handle the strictly increasing values of the midpoint intervals and the weight FLRs especially for type (a, c). • Ineffective to handle the strictly decreasing values of the midpoint intervals and the weight FLRs. • Promising to reduce the forecasting error or mean square error especially for type (a, c) of FLRs occurred.

provides better performance than the non-reversal model in term of forecasting accuracy. According to the three examples listed so far, the comparisons of non-reversal and reversal models are summarized in detail in Table 3.

6. Conclusion

The primary concern of this study was proposing a new model in FTS forecasting named the reversal model. This model was applied and shown to be suitable for resolving the type (a) and (c) of FLRs, which were left unsolved in [15]. The most important outcome of this proposed model is that the error of the final forecasted values of A_i can be reduced significantly. Additionally, a theorem was also presented and proved to justify the capability of this reversal model. This paper lists the facts and proofs, also provides several examples to illustrate the implementation of this theorem. In summary, the final forecasts of A_i from the reversal model provide a lower error than the final forecasts from the non-reversal model. Therefore, the proposed reversal model outperforms the non-reversal model in term of forecasting accuracy.

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