

Conjugacy classes and commuting probability in finite metacyclic p -groups

K. Moradipour^a, N.H. Sarmin^{b,*}, A. Erfanian^c

^a Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

^b Department of Mathematical Sciences, Faculty of Science and Ibnu Sina Institute for Fundamental Science Studies, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

^c Department of Pure Mathematics and Centre of Excellence in Analysis on Algebraic Structure, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

*Corresponding author, e-mail: nhs@utm.my

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ABSTRACT: Let G be a finite non-abelian metacyclic p -group where p is any prime. We compute the exact number of conjugacy classes and the commutativity degree of G . In particular, we describe the number of conjugacy classes both in the split and non-split case.

KEYWORDS: split p -group, nilpotency class, commutativity degree

INTRODUCTION

We consider only finite groups. Recently many authors have investigated the number $k(G)$ of conjugacy classes of a group G . There are several papers on the conjugacy classes of finite p -groups^{1–3}. Many authors obtained significant results but only on the lower and upper bound of $k(G)$. For instance, Sherman⁴ proves that if G is a finite nilpotent group of nilpotency class m , then $k(G) > m|G|^{1/m} - m + 1$. Later Huppert⁵ proved that $k(G) > \log n$ for any nilpotent group G of order n . On the other hand, Liebeck and Pyber⁶ found an upper bound for $k(G)$ in terms of an arbitrary constant. Lopeze⁷ shows that a maximal abelian subgroup A of $|A| = p^\alpha$ of a nilpotent group G of $|G| = p^m$ and $|Z(G)| = p^\beta$ satisfies an equality of the form

$$k(G) = \frac{p^{2\alpha-m} + p^\beta(p+1)(p^{m-\alpha} - 1)}{p^{m-\alpha}} + \frac{k(p^2 - 1)(p - 1)}{p^{m-\alpha}}$$

where $k \geq 0$. For $k > 0$ this formula provides an upper bound by default but does not determine the exact number of conjugacy classes of G .

A group G is called metacyclic if it contains a normal cyclic subgroup N such that G/N is also cyclic. Concerning these groups, in Ref. 8 it was shown that if G is any finite split metacyclic p -group

for an odd prime p , that is, $G = H \rtimes K$ for subgroups H and K , and if $|H| = p^\alpha$ and $|K| = p^{\alpha+\beta}$, then there exist exactly

$$\frac{(\beta - \alpha + 1)(p^{\alpha+1} - 1)}{(p - 1)} + 4 \sum_{i=0}^{\alpha-1} p^i(\alpha + i)$$

conjugacy classes of subgroups of G .

The metacyclic p -groups of class 2 have been classified in Ref. 9 where homological methods are used. The case of $p = 2$ needs special attention and was the subject of Ref. 10. Moreover, Beuerle¹¹ classified the non-abelian metacyclic p -groups of class at least 3 where p is any prime. He showed there are four classes of such groups which have been called of *positive type*. We use these classifications in order to obtain the precise number of conjugacy classes of all non-abelian metacyclic p -groups of class 2 and class at least 3.

Each isomorphism class of metacyclic p -groups can be represented by five parameters p , α , β , ε , and γ . These parameters are used to measure the order, centre and abelianness of the groups, and also their nilpotency class, and whether the groups are split extension or not. We also use the parameters to compute the number of conjugacy classes of the