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Some Matrix Representations for Dihedral Group of Order Twelve

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ABSTRACT

Let G be a finite group and V a vector space over K. Then, a matrix representation of G of degree n is a homomorphism $T : g \to T(g)$ *of G into GL(n, K), where GL(n, K) stands for the group of invertible n×n matrices over K. By using definition of matrix representation of G, the matrix representations for dihedral group of order twelve are provided in this paper. It has been computed that the representations of dihedral group of order twelve can be presented in some n×n matrices. Furthermore, it has also been proved that the matrix representations found are equivalent to each other. The method used in this paper can be applied to find matrix representations for dihedral group of other orders.*

Keywords : Representation, matrix representation, dihedral group.

Mathematics Subject Classification: 06BI5, 20C99

1. INTRODUCTION

A representation of a group realizes the elements of the group concretely as geometric symmetries. The same group may have many different such representations. A group that arises naturally as a specific set of symmetries may have representations as geometric symmetries at different levels. Recent researches on representations of groups have been done by Sharkawy and Hadidi in 2010 and Mohd Adnan in 2013. Many researchers have also studied on the representations of dihedral groups.

In 1942, Miller studied on the automorphisms of the dihedral groups. Next, Lee developed some new theorems on the integral representations of dihedral groups in 1964. The representations of quantum double of dihedral groups have been given by Dong in 2009. In 2010, Ishguro studied the invariant rings and dual representations of dihedral groups whereas Kohls and Sezer developed some results on the invariant of the dihedral group in characteristic two.

The objective of this paper is to find the matrix representations for dihedral groups, D_n when n = 6 which is a dihedral group of order 12. The paper is structured as follows. In the next section, we fix the notations and state a few definitions which are useful for the proof of our main results. In the third section, we list the matrix representations for dihedral group of order 12 together with the proof.

2. PRELIMINARIES

In this paper, G is denoted as an arbitrary finite multiplicative group with identity element 1, and K is a field. It is understood that all vector spaces considered are finite dimensional over K, (for a vector space V, the notation (V:K) is used to denote the dimension of V over K). $GL(V)$ denotes the group of all invertible linear transformations of a vector space V onto itself, and that $GL(n, K)$ stands for the group of invertible $n \times n$ matrices over K.

2.1. Representation of *G*

Let G be a finite group and V a vector space over K. A representation of G with representation space V is a homomorphism $T : g \rightarrow T(g)$ of G into $GL(V)$. Two representations T and T' with spaces V and V' are said to be equivalent if there exists a K-isomorphism S of V onto V' such that

$$T'(g)S = ST(g) \text{ for } g \in G,$$

that is,

$$T'(g)Sv = ST(g)v \text{ for all } v \in V \text{ and } g \in G.$$

The dimension $(V:K)$ of V over K is called the degree of T and will be denoted by deg T.

2.2. Matrix representation

A matrix representation of G of degree n is a homomorphism $\boldsymbol{T}: g \rightarrow \boldsymbol{T}(g)$ of G into $GL(n,K)$. Two matrix representations $\boldsymbol{T}$ and $\boldsymbol{T}'$ are equivalent if they have the same degree, say n, and if there exists a fixed matrix S in $GL(n,K)$ such that

$$T'(g) = ST(g)S^{-1} \text{ for } g \in G.$$

If $\boldsymbol{T}$ is a representation of G with space V, then from the homomorphism property, we have

$$\boldsymbol{T}(ab) = \boldsymbol{T}(a)\boldsymbol{T}(b), \qquad a, b \in G,$$

$$\boldsymbol{T}(a) - 1 = \boldsymbol{T}(a - 1),$$

$$\boldsymbol{T}(1) = 1_V,$$

where 1_V denotes the identity mapping on V. The corresponding statements hold, of course, for matrix representations.

3. RESULTS

It has been mentioned before that the goal of this paper is to find the matrix representation for dihedral group of order twelve. In this section, the results together with their proofs are stated.

Let G be the group of symmetries of hexagon, that is, dihedral group of order 12. Then G has two generators a and b which satisfies the following relations

$$a^6 = 1, \qquad b^2 = 1, \qquad ba^{-1}b = a^{-1} \qquad \text{or} \qquad (ab)^2 = 1.$$

Because all other relations involving a and b are consequences of those we have listed, in order to specify a matrix representation of G it is sufficient to find matrices $\boldsymbol{T}(a)$ and $\boldsymbol{T}(b)$ such that

$$\boldsymbol{T}(a)^6 = 1, \qquad \boldsymbol{T}(b)^2 = 1, \qquad \text{and} \qquad \left(\boldsymbol{T}(a)\boldsymbol{T}(b)\right)^2 = 1. \qquad (1)$$

Now, we want to find $\boldsymbol{T}(a)$ and $\boldsymbol{T}(b)$ which satisfy all relation in (1), then

$$a^i b^j \rightarrow \boldsymbol{T}(a)^i \boldsymbol{T}(b)^j$$

is a homomorphism of G.

Let $\boldsymbol{T}(a)$ and $\boldsymbol{T}(b)$ as in the following:

$$T(a)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix},\qquad T(b)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}.$$

By applying the rules of matrix multiplication, we can verify that this pair of ***T*** (*a*) and ***T*** (*b*) satisfies (1), namely:

$$T(a)^6=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix}^6=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad T(b)^2=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$T(a)T(b)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \left(T(a)T(b)\right)^2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Thus, ***T*** (*a*) and ***T*** (*b*) satisfy the relations in (1). Next, we can find another pair of matrices that satisfy relations in (1), given in the following:

$$W(a)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0\\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}}\\ 0 & 0 & -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix},\qquad W(b)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0\\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}}\\ 0 & 0 & \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}.$$

We can verify that these matrices also satisfy (1). We list now all the pairs of matrices which satisfy (1) and define representations of *G*.

$$T(a)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix},\qquad T(b)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}.$$

$$W(a)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0\\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}}\\ 0 & 0 & -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix},\qquad W(b)=\begin{pmatrix}\frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0\\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}}\\ 0 & 0 & \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}.$$

Next, we want to prove that these matrix representations are equivalent. By Definition 2, we can verify that there exists an invertible 4×4 matrix *S* such that

$$SW(a)S^{-1} = \begin{bmatrix} \boldsymbol{T}(a) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}(a) \end{bmatrix}.$$

or equivalently as

$$SW(a) = \begin{bmatrix} \boldsymbol{T}(a) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}(a) \end{bmatrix} S. \tag{2}$$

and

$$SW(b)S^{-1} = \begin{bmatrix} \boldsymbol{T}(b) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}(b) \end{bmatrix}.$$

or equivalently as

$$SW(b) = \begin{bmatrix} \boldsymbol{T}(b) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}(b) \end{bmatrix} S. \tag{3}$$

Let $S = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$. Here, $|S| = \left|\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\right| = 4 \neq 0$.

So, S is invertible with $S^{-1} = \begin{pmatrix} 2/4 & 0 & 2/4 & 0 \\ 0 & 2/4 & 0 & 2/4 \\ -2/4 & 0 & 2/4 & 0 \\ 0 & -2/4 & 0 & 2/4 \end{pmatrix}$. Now we can verify that this S satisfies the relation in (2).

The left hand side (LHS) of (2) gives

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0 \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ 0 & 0 & -\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} \\ -\frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}$$

while the right hand side (RHS) of (2) gives

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0 \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ 0 & 0 & -\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} \\ -\frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}.$$

Thus, the relation in (2) is satisfied. We proved that the matrix representations of G namely, $\boldsymbol{W}(a)$ and $\begin{bmatrix} \boldsymbol{T}(a) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}(a) \end{bmatrix}$ are equivalent to each other. Next, we show that $\boldsymbol{S}$ also satisfies the relation in (3).

The LHS of (3) gives

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0 \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ 0 & 0 & \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & \sqrt{\frac{3}{4}} & -\frac{1}{2} \\ -\frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix},$$

while the RHS of (3) gives

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0 \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ 0 & 0 & \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} & \sqrt{\frac{3}{4}} & -\frac{1}{2} \\ -\frac{1}{2} & -\sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} & \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix}.$$

Thus, (3) is satisfied by S. This showed that the matrix representation $\boldsymbol{W}(b)$ are equivalent to $\begin{bmatrix} \boldsymbol{T}(b) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}(b) \end{bmatrix}$.

4. CONCLUSION

The matrix representations for dihedral group of order twelve is provided and proven in this paper. We also proved that two matrix representations listed in this paper are equivalent to each other. The method of calculations used in this paper can be applied in finding matrix representations of different order of dihedral groups.

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