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On the Schur Multiplier of Groups of Order $8q$

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ABSTRACT

In this paper we determine the Schur multiplier, $M(G)$ of finite nonabelian groups of order $8q$, where q is an odd prime.

Keywords: Groups of order $8q$; Schur multiplier.

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1 Introduction

The Schur multiplier $M(G)$ was introduced in Schur's work (Schur, 1904) on projective representation of groups. (Karpilovsky, 1987) has shown various results on the Schur multiplier of many groups.

Let a group G be presented as a quotient of a free group F by a normal subgroup R. Then the Schur multiplier of G is defined to be

$$M(G) = (R \cap [F, F])/[F, R].$$

The Schur multiplier of a group G is isomorphic to the $H_2(G, \mathbb{Z})$, the second homology group of G with coefficients. Moreover, for a finite group G, $M(G) \cong H^2(G, \mathbb{C}^*)$.

(Schur, 1907) obtained the Schur multiplier for finite abelian group $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} ... \oplus \mathbb{Z}_{n_k}$, where $n_{i+1}|n_i$ and $1 \leq i \leq k-1$ as follows:

$$M(G) = \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3}^{(2)} ... \oplus \mathbb{Z}_{n_k}^{(k-1)}.$$

(Steinberg, 1968) computed the Schur multiplier for symplectic groups, projective symplectic groups and some linear groups. The computation for the Schur multiplier of special linear groups and general linear groups can be found in (Hannebauer, 1990; Huppert, 1967).

(Niroomand, 2009) obtained a bound for the Schur multiplier of nonabelian p-group of order p^n, where this bound is related to the derived subgroup of the group.

It is clear that the Schur multiplier of cyclic group is trivial and the Schur multiplier of a group of orders p^2 and p^3 can be obtained easily. The Schur multiplier of the nonabelian groups of order p^2q have been computed by (Rashid, Sarmin, Erfanian and Mohd Ali, 2011), where p and q are distinct primes. In this paper we focus on the Schur multiplier of nonabelian groups of order $8q$, where q is an odd prime.

Groups, Algorithms and Programming (GAP, 2005) software has been used to verify the hand calculation of the Schur multiplier of groups of order $8q$, where $q = 3, 5, 7, 11, 13$ and 17.

In the following theorem, the classification of groups of order $8q$ which play an important rule for proving our main theorem is stated.

Theorem 1.1. $(Miah,\ 1975)$ *Let G be a nonabelian group of order $8q$, where q is an odd prime. Then G is isomorphic to exactly one group of the following types:*

$(1.1.1)\ D_4 \times \mathbb{Z}_q,$ $(1.1.2)\ Q_2 \times \mathbb{Z}_q,$

$(1.1.3)\ D_{2q} \times \mathbb{Z}_2,$ $(1.1.4)\ Q_q \times \mathbb{Z}_2,$

$(1.1.5)\ D_q \times \mathbb{Z}_4,$ $(1.1.6) < a, b | a^8 = b^q = 1, a^{-1}ba = b^{-1} >,$

$(1.1.7)\ D_{4q},$ $(1.1.8)\ Q_{2q},$

$(1.1.9) < a, b, c | a^4 = b^2 = c^q = 1, b^{-1}ab = a^{-1}, a^{-1}ca = c^{-1}, bc = cb >,$

$(1.1.10) < a, b | a^8 = b^q = 1, a^{-1}ba = b^{\alpha} >$*, where α is a primitive root of $\alpha^4 \equiv 1 \pmod q$, 4 divides $q-1$,*

$(1.1.11) < a, b | a^4 = b^2 = c^q = 1, ab = ba, a^{-1}ca = c^{\alpha}, bc = cb >$*, where α is a primitive root of $\alpha^4 \equiv 1 \pmod q$, 4 divides $q-1$,*

$(1.1.12) < a, b | a^8 = b^q = 1, a^{-1}ba = b^{\alpha} >$*, where α is a primitive root of $\alpha^8 \equiv 1 \pmod q$, 8* *divides* $q-1$,

(1.1.13) $\mathbb{Z}_2 \times A_4$, (1.1.14) $SL(2,3)$, (1.1.15) S_4,

(1.1.16) $< a,b,c,d | a^4 = b^2 = c^2 = d^q = 1, ab = ba, ac = ca, bc = cb, d^{-1}ad = b, d^{-1}bd = c, d^{-1}cd = ab >$.

For the Schur multiplier of groups of order $8q$ we have the following theorem:

Theorem 1.2. *Let G be a nonabelian group of order $8q$, where q is an odd prime. Then*

$$M(G) = \begin{cases} 1 & ; G \text{ is of type (1.1.2), (1.1.6), (1.1.8), (1.1.10),} \\ & \text{(1.1.12), (1.1.14) or (1.1.16),} \\ \mathbb{Z}_2 & ; G \text{ is of type (1.1.1), (1.1.4), (1.1.5), (1.1.7),} \\ & \text{(1.1.9), (1.1.11), (1.1.13) or (1.1.15),} \\ \mathbb{Z}_2^{(3)} & ; G \text{ is of type (1.1.3).} \end{cases}$$

2 Preliminaries

This section includes some results on the Schur multiplier of groups which are used for proving our main theorem.

Theorem 2.1. $(Karpilovsky,\ 1987)$ *For $n \geq 1$,*

$$M(Q_n) = 1, \qquad M(D_n) = \begin{cases} 1 & ; n \text{ is odd} \\ \mathbb{Z}_2 & ; n \text{ is even} \end{cases}$$

where Q_n and D_n are generalized quaternion and dihedral groups of orders $4n$ and $2n$, respectively.

Theorem 2.2. $(Karpilovsky,\ 1987)$ *Let N be a normal Hall subgroup of G, i.e., $(|G|, |G/N|) = 1$ and T be a complement of N in G. Then $M(G) \cong M(T) \times M(N)^T$.*

Theorem 2.3. $(Karpilovsky,\ 1987)$ *Let G be a finite metacyclic group $< a,b | a^m = e, b^s = a^t, bab^{-1} = a^r >$, where the positive integers m, r, s and t satisfy $r^s \equiv 1 \pmod{m}$ and $m | t(r-1)$ and $t|m$. Then $M(G) \cong \mathbb{Z}_n$, where $n = \frac{(r-1,m)(1+r+r^2+\ldots+r^{s-1},t)}{m}$.*

Theorem 2.4. *(Zassenhaus-Burnside-Holder)* $(Robinson,\ 1982)$
If G is a finite group all of whose Sylow subgroups are cyclic, then G has a presentation

$$G = < a,b | a^m = b^n = 1, b^{-1}ab = a^r >$$

where m is odd, $m | r^n - 1$, $0 \leq r \leq m-1$ and $(m, n(r-1)) = 1$. Conversely, in a group with such a presentation all Sylow subgroups are cyclic. In these groups G' and G^{ab} are cyclic .

Theorem 2.5. $(Brownand\ Loday,\ 1987)$ *If G and H act themselves by conjugation and trivially upon each other, then $G \otimes H \cong G/G' \otimes H/H'$.*

Theorem 2.6. $(Karpilovsky,\ 1987)$ *If G_1 and G_2 are finite groups, then*

$$M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1 \otimes G_2).$$

3 The Proof of Main Theorem

In this section we prove our main theorem as mentioned in Section 1, namely Theorem 1.2. The classification in Theorem 1.1 is used to compute the Schur multiplier of groups of order $8q$.

Proof: Let G be a group of type (1.1.1). Then $G \cong D_4 \times \mathbb{Z}_q$. Therefore, by Theorems 2.5 and 2.6 we have the following computations:

$$D_4 \otimes \mathbb{Z}_q \cong D_4/D_4' \otimes \mathbb{Z}_q \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \otimes \mathbb{Z}_q = 1,$$

$$M(G) = M(D_4) \times M(\mathbb{Z}_q) \times (D_4 \otimes \mathbb{Z}_q) = \mathbb{Z}_2.$$

For a group of type (1.1.2), we have the following computations:

$$Q_2 \otimes \mathbb{Z}_q = Q_2/Q_2' \otimes \mathbb{Z}_q = 1,$$

$$M(G) = M(Q_2) \times M(\mathbb{Z}_q) \times (Q_2 \otimes \mathbb{Z}_q) = 1.$$

If $G \cong D_{2q} \times \mathbb{Z}_2$ is a group of type (1.1.3), then

$$D_q \otimes (\mathbb{Z}_2)^2 = D_q/D_q' \otimes (\mathbb{Z}_2)^2 = \mathbb{Z}_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$M(G) = M(D_q) \times M(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (D_q \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2)) = \mathbb{Z}_2^{(3)}.$$

For a group of type (1.1.4), $G \cong Q_q \times \mathbb{Z}_2$. Then

$$M(G) = M(Q_q) \times M(\mathbb{Z}_2) \times (Q_q \otimes \mathbb{Z}_2) = \mathbb{Z}_4 \otimes \mathbb{Z}_2 = \mathbb{Z}_2.$$

For a group of type (1.1.5), $G \cong D_q \times \mathbb{Z}_4$. The proof is similar to type (1.1.4).

By choosing $m = q$, $n = 8$ and $r = q - 1$, the proof of type (1.1.6) is straightforward using Theorem 2.4.

For a group of types (1.1.7) and (1.1.8), $G \cong D_{4q}$ and $G \cong Q_{2q}$, respectively. Then by Theorem 2.1, $M(D_{4q}) = \mathbb{Z}_2$ and $M(Q_{2q}) = 1$.

For a group of type (1.1.9), $G \cong \mathbb{Z}_q \rtimes D_2$. Then by Theorem 2.2,

$$M(G) = M(D_2) \times M(\mathbb{Z}_q)^{D_2} = \mathbb{Z}_2.$$

For a group of type (1.1.10), by choosing $m = q$, $s = 8$, $t = q$, $r = a$ and by Theorem 2.3 the proof of this type is clear.

For a group of type (1.1.11), G' is a cyclic group of order q and $(|G'|, |G^{ab}|) = 1$. Then by Theorem 2.2, $M(G) = M(\mathbb{Z}_4 \times \mathbb{Z}_2) \times M(G')^{\mathbb{Z}_4 \times \mathbb{Z}_2}$. Thus $M(G) = \mathbb{Z}_2$.

For a group of type (1.1.12), the proof is similar to that of type (1.1.10).

The Schur multiplier of groups of type (1.1.13) can be found in (Brown, Johnson and Robertson, 1987), while for the Schur multiplier of types (1.1.14) and (1.1.15) have been computed in (Karpilovsky, 1987).

For a group of type (1.1.16), $(|G'|, |G^{ab}|) = 1$. Then $G \otimes G = G' \times (G^{ab} \otimes G^{ab}) = (\mathbb{Z}_2)^3 \times \mathbb{Z}_7$. Since $|G \otimes G| = |G||M(G)|$, then $M(G) = 1$. □

4 Conclusion

In this paper, we have proved that if G is a nonabelian group of order $8q$, where q is an odd prime, then $M(G) \cong 1, \mathbb{Z}_2$ or $\mathbb{Z}_2^{(3)}$.

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