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The Conjugacy Classes, Conjugate Graph and Conjugacy Class Graph of Some Finite Metacyclic 2-Groups

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ABSTRACT

In this paper, G denotes a metacyclic 2-group of order at most 32 and Γ denotes a simple undirected graph. A conjugacy class is an equivalence relation, in which the group is partitioned into disjoint sets. The conjugate graph is a graph whose vertices are non-central elements of G in which two vertices are adjacent if they are conjugate. The conjugacy class graph is a graph whose vertices are non-central conjugacy classes of a group G in which two vertices are connected if their cardinalities are not coprime. In this paper, the conjugacy classes of metacyclic 2-groups of order at most 32 are computed. The results obtained are then applied to graph theory, more precisely to conjugate graph and conjugacy class graph. Some graph properties such as chromatic number, clique number, dominating number and independent number are also determined.

Keywords: Metacyclic groups, Conjugate graph, Conjugacy class graph.

INTRODUCTION

This section provides some backgrounds related to group theory and graph theory, starting with group theory. The concept of conjugacy classes is widely used by many researchers. The following is the definition of the conjugate between two elements of a group G.

Definition 1.1 (Dummit *et al*., 2004) Conjugate
Let a and b be two elements in a finite group G, then a and b are called conjugate if there exists an element g in G such that $gag^{-1} = b$.

The following definition is the definition of the conjugacy class, which will be used to find the conjugacy classes of a group.

Definition 1.2 (Goodman, 2003) Conjugacy Class
Let $x \in G$. The conjugacy class of x is the set $\mathrm{cl}(x) = \{axa^{-1} \mid a \in G\}$. In this research, the notation (G) is used for the number of conjugacy classes in G, while (G) is used for the center group G.

Proposition 1.1 (Fraleigh, 2002)
The conjugacy class of the identity element is its own class, namely cl(e) = $\{e\}$.

Proposition 1.2 (Fraleigh, 2002)
Let a and b be the two elements in a finite group G. The elements a and b are conjugate if they belong in one conjugacy class, that is cl(a) = cl(b) are the same.

Next, we provide some concepts related to metacyclic 2-groups, starting with the definition of a metacyclic group.

Definition 1.4 (Reis, 2011) Metacyclic Group

A group G is called metacyclic if it has a cyclic normal subrgroup H such that the quotient group G/H is also cyclic.

In 2005, Beuerle (Beuerle, 2005) separated the classification into two parts, namely for the non-abelian metacyclic p-groups of class two and class at least three. Based on (Beuerle, 2005), the metacyclic p- groups of nilpotency class two are then partitioned into two families of non-isomorphic p-groups stated as follows:

(1) $G \cong \langle a,b : a^{2^\alpha} = 1, b^2 = 1, [a,b] = a^{2^{\alpha-\lambda}} \rangle$, where $\alpha, \beta, \lambda \in \mathbb{N}, \alpha \geq 2\lambda, \beta \geq \lambda \geq 1$.

(2) $G \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$, a quaternion group of order 8, Q_8.

In this paper, we use the group (1) in which $3 \leq \alpha \leq 4, \lambda = 1$ and group (2). Hence the groups under the scope of this paper are given as follows:

(1) $G_1 \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$, a quaternion group of order 8, Q_8.

(2) $G_2 \cong \langle a,b : a^8 = 1, b^2 = 1, [a,b] = a^4 \rangle$,

(3) $G_3 \cong \langle a,b : a^{16} = 1, b^2 = 1, [a,b] = a^8 \rangle$.

The followings are some basic concepts of graph theory that are needed in this paper.

Definition 1.5 (Bondy and Murty, 1982) Graph
A graph is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively.

Definition 1.6 (Bondy and Murty, 1982) Subgraph
A subgraph of a graph Γ is a graph whose vertices and edges are subset of the vertices and edges of Γ.

Definition 1.7 (Godsil and Royle, 2001) Complete Graph
A complete graph is a graph where each order pair of distinct vertices are adjacent, and it is denoted by K_n.

Definition 1.8 (Bondy and Murty, 1982) Independent Set
A non-empty set S of $V(\Gamma)$ is called an independent set of Γ. While the independent number is the number of vertices in maximum independent set and it is denoted by $\alpha(\Gamma)$.

Definition 1.9 (Bondy and Murty, 1982) Chromatic Number
The chromatic number is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color, denoted by $\chi(\Gamma)$.

Definition 1.10 (Bondy and Murty, 1982) Clique
Clique is a complete subgraph in Γ, while the clique number is the size of the largest clique in Γ and it is denoted by $\omega(\Gamma)$.

Definition 1.11 (Bondy and Murty, 1982) Dominating Set
The dominating set $X \subseteq V(\Gamma)$ is a set where for each v outside X, $\exists x \in X$ such that v adjacent to x. The minimum size of X is called the dominating number and it's denoted by $\gamma(\Gamma)$.

In this section, some works that are related to metacyclic 2-groups, conjugate graph and conjugacy class graph are stated. The classifications of metacyclic 2-groups have been used to find the commutativity degree (Miller, (1944), Gustafson, (1973), McHale, (1974)), the probability that a group element fixes a set (Omer *et al*., 2013) and others. Next some related works on conjugate graph and conjugacy class graph are stated.

In 2013, Erfanian and Tolue (2013) introduced a new graph called the conjugate graph, here we denote as Γ_G^c. The vertices of this graph are non-central elements in which two vertices are adjacent if they are conjugate.

In 1990, a new graph was introduced by Bertram *et al*. (Bertram *et al.,* 1990), called the conjugacy class graph, here we denote as Γ_G^{cl}. The vertices of this graph are non-central conjugacy classes of a group in which two distinct vertices are connected by an edge if the greatest common divisor of the sizes between any two conjugacy classes is greater than one. Recently, Bianchi *et al.* (2012) studied the regularity of the graph related to conjugacy classes and provided some results. In 2013, Ilangovan and Sarmin (2013) found some graph properties of graph related to conjugacy classes of two-generator two-groups of class two.

This paper is divided into three sections. The first section focuses on some background topics in group theory and graph theory, while the second section provides some earlier and recent publications that are related to metacyclic 2-groups, conjugate graph and conjugacy class graph. In the third section, we present our main results, on which include the conjugacy classes of metacyclic 2-groups, conjugate graph and conjugacy class graph.

MAIN RESULTS

This section consists of two parts. The first part focuses on finding the conjugacy classes of metacyclic 2-groups of groups G_1, G_2 and G_3, while the second part applies the obtained results to graph theory, namely conjugate graph and conjugacy class graph.

The Conjugacy Classes of Metacyclic 2-groups of Order at Most 32.

In this section, the conjugacy classes of groups $G_{1,}$ G_2 and G_3 are determined, starting with the first group, namely G_1.

Theorem 3.1 Let G_1 be a quaternion group of order 8, $G_1 \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$. Then, the number of conjugacy class of G_1, $K(G_1) = 5$.

Proof The elements of G_1 are given as follows: $G_1 \cong \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Using Definition 1.2, that is $\mathrm{cl}(x) = \{axa^{-1} \mid a \in G\}$ for all elements $a \in G$, the conjugacy class of G_1 are determined. First let $x = e$, by Proposition 1.1, $\mathrm{cl}(e) = e$. Next, let $x = a$, then $\mathrm{cl}(a) = \{gag^{-1}\}$, where $g \in G_1$. Thus $\mathrm{cl}(a) = \{a, a^3\}$, and by using Proposition 1.2, then $\mathrm{cl}(a) = \mathrm{cl}(a^3)$. Using the same procedure, the conjugacy classes of G_1 are found follows: $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, a^3\}$, $\mathrm{cl}(a^2) = \{a^2\}$, $\mathrm{cl}(b) = \{a, a^2b\}$, and $\mathrm{cl}(ab) = \{ab, a^3b\}$, from which it follows that $|Z(G_1)| = 2$ and $K(G_1) = 5$, as desired. ■

Theorem 3.2 Let G_2 be a metacyclic 2-group of order 16, $G_2 \cong \langle a,b : a^8 = 1, b^2 = 1, [a,b] = a^4 \rangle$. Then the number of conjugacy classes, $K(G_2) = 10$.

Proof The elements of group G_2 are stated as follows: $G_2 \cong \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$. Using the Definition 1.2, the conjugacy classes of elements of G_2 are determined as follows: First let $x = e$, by Proposition 1.1, $\mathrm{cl}(e) = e$. Next, let $x = a$, then $\mathrm{cl}(a) = \{gag^{-1}, \forall g \in G_2\}$. Thus $\mathrm{cl}(a) = \{a, a^5\}$ using Proposition 1.2, then $\mathrm{cl}(a) = \mathrm{cl}(a^5)$. Using the same procedure, the conjugacy classes of G_2 are found and listed as follows: $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, a^5\}$, $\mathrm{cl}(a^2) = \{a^2\}$, $\mathrm{cl}(a^3) = \{a^3, a^7\}$, $\mathrm{cl}(a^4) = \{a^4\}$, $\mathrm{cl}(a^6) = \{a^6\}$, $\mathrm{cl}(b) = \{b, a^4b\}$, $\mathrm{cl}(ab) = \{ab, a^5b\}$, $\mathrm{cl}(a^2b) = \{a^2b, a^6b\}$ and $\mathrm{cl}(a^3b) = \{a^3b, a^7b\}$. Therefore, $K(G_2)10$. From the computations, it is found that $|Z(G_2)| = 4$, as claimed. ■

Theorem 3.3 Let G_3 be a metacyclic 2-group of order 32, $G_3 \cong \langle a,b : a^{16} = 1, b^2 = 1, [a,b] = a^8 \rangle$. Then the number of conjugacy classes, $K(G_3) = 20$.

Proof First the elements of G_3 are given as:

$$G_3 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2b, \\ a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b\}.$$

Using the Definition 1.2, the conjugacy classes of G_3 are determined. First let $x = e$, by Proposition 1.1, $\mathrm{cl}(e) = e$. Next, let $x = a$, then $\mathrm{cl}(a) = \{gag^{-1}, g \in G_3\}$. Thus $\mathrm{cl}(a) = \{a, a^9\}$ using Proposition 1.2, then $\mathrm{cl}(a) = \mathrm{cl}(a^9)$. Using the same procedure, the conjugacy classes of G_3 are found as follows: $\mathrm{cl}(e) = \{e\}$, $\mathrm{cl}(a) = \{a, a^9\}$, $\mathrm{cl}(a^2) = \{a^2\}$, $\mathrm{cl}(a^3) = \{a^3, a^{11}\}$, $cl(a^4) = \{a^4\}$, $\mathrm{cl}(a^5) = \{a^5, a^{13}\}$, $\mathrm{cl}(a^6) = \{a^6\}$, $\mathrm{cl}(a^7) = \{a^7, a^{15}\}$, $\mathrm{cl}(a^8) = \{a^8\}$, $\mathrm{cl}(a^{10}) = \{a^{10}\}$, $\mathrm{cl}(a^{12}) = \{a^{12}\}$, $\mathrm{cl}(a^{14}) = \{a^{14}\}$, $\mathrm{cl}(b) = \{b, a^8b\}$, $\mathrm{cl}(ab) = \{ab, a^9b\}$, $\mathrm{cl}(a^2b) = \{a^2b, a^{10}b\}$, $\mathrm{cl}(a^3b) = \{a^3b, a^{11}b\}$, $\mathrm{cl}(a^4b) = \{a^4b, a^{12}b\}$, $\mathrm{cl}(a^5b) = \{a^5b, a^{13}b\}$, $\mathrm{cl}(a^6b) = \{a^6b, a^{14}b\}$, and $\mathrm{cl}(a^7b) = \{a^7b, a^{15}b\}$, from which it follows that $|Z(G_3)| = 8$ and the number of conjugacy classes in G_3 is 20 i.e. $K(G_3) = 20$. ■

The Conjugate Graphs and Conjugacy Class Graphs of Three Metacyclic 2-Groups of Order at Most 32

In this section, the results on the conjugacy classes of the groups G_1, G_2 and G_3 are applied to conjugate graph and conjugacy class graph, starting with the conjugate graph of G_1.

Theorem 4.1 Let G_1 be the quaternion group of order 8, $G_1 \cong \langle a, b : a^4 = 1, b^2 = [b, a] = a^{-2} \rangle$. Then the conjugate graph of G_1 is $\Gamma_{G_1}^C = \bigcup_{i=1}^{3} K_2$, i.e. the union of three complete components K_2.

Proof Based on Theorem 3.1, there are 8 elements of G_1 and two central elements. Thus the number of vertices for the conjugate graph of group G_1 is six. Since, two vertices are adjacent if they are conjugate and by Proposition 1.2, thus there are three non-central conjugacy classes. Hence, Γ consists of three complete components of K_2. The conjugate graph of G_1 is represented in Figure 4.3. ■

Figure 4.1: The Conjugate Graph of G_1

Next, some properties of the conjugate graph of G_1 are given.

Corollary 4.1 Let $G_1 \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$ be a metacyclic 2-group of order 8. Then $\chi\left(\Gamma_{G_1}^C\right) = \omega\left(\Gamma_{G_1}^C\right) = 2$, $\alpha\left(\Gamma_{G_1}^C\right) = 3$ and $\gamma\left(\Gamma_{G_1}^C\right) = 3$.

Proof Based on Theorem 4.1, $\Gamma_{G_1}^C$ consists of three complete components of K_2. Using Definition 1.9, the chromatic number is equal to two since the two vertices in K_2 have a different color to each other. By Definition 1.10, clique number equals two since the largest subgraph in $\Gamma_{G_1}^C$ is K_2. Using Definition 1.8, the maximum independent set of conjugate graph of G_1 is $x = \{a, b, ab\}$, hence the independent number of G_1, $\alpha\left(\Gamma_{G_1}^C\right) = 3$. Using Definition 1.11, the dominating number is equal to three since three vertices are needed to dominate itself and another vertices in K_2. ■

In the next theorem, the conjugate graph is found for G_2.

Theorem 4.2 Let G_2 be a metacyclic 2-group of order 16, $G_2 \cong \langle a,b : a^8 = 1, b^2 = 1, [a,b] = a^4 \rangle$. Then, the conjugate graph, $\Gamma_{G_2}^C = \bigcup_{i=1}^{6} K_2$.

Proof Based on Theorem 3.2, there are 16 elements of G_2. Since the number of vertices in Γ is $|G| - |Z(G)|$, thus the number of vertices is 12. Two vertices are adjacent in Γ if they are conjugate. By Proposition 1.2, two elements are conjugate if they belong to the same conjugacy class. According to the Theorem 3.2, there are six non-central conjugacy classes. Thus, Γ consists of six complete components of K_2. The conjugate graph of G_2 is represented in Figure 4.2. ■

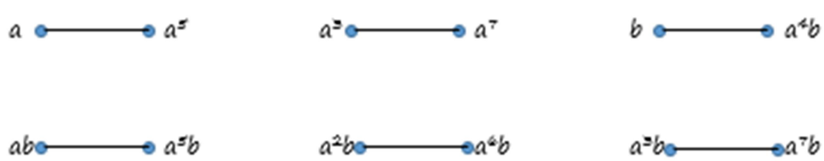

Figure 4.2: The Conjugate Graph of G_2.

Based on Theorem 4.2, some properties of the conjugate graph of G_2 are concluded.

Corollary 4.2 Let $G_2 \cong \langle a,b : a^8 = 1, b^2 = 1, [a,b] = a^4 \rangle$ be a metacyclic 2-group of order 16. Then $\chi\left(\Gamma_{G_2}^C\right) = \omega\left(\Gamma_{G_2}^C\right) = 2$, $\alpha\left(\Gamma_{G_2}^C\right) = 6$ and $\gamma\left(\Gamma_{G_2}^C\right) = 6$.

Proof Based on Theorem 4.2, there are six complete components of K_2. Thus by Definition 1.9, the chromatic number is equal to two since the two vertices can be colored by two different colors. Using Definition 1.10, the clique number, $\omega\left(\Gamma_{G_2}^C\right) = 2$ since the largest subgraph in $\Gamma_{G_2}^C$ is K_2. Using Definition 1.8, the maximum independent set of conjugate graph of G_2 is $x = \{a, a^3, b, ab, a^2b, a^3b\}$, hence the independent number of G_2, $\alpha\left(\Gamma_{G_2}^C\right) = 6$. Using Definition 1.11, the conjugate graph of G_2 has the dominating number, is $\gamma\left(\Gamma_{G_2}^C\right) = 6.$ since there are six vertex needed to dominate itself and another vertices in K_2. ■

In the next theorem, the conjugate graph is found for G_3.

Theorem 4.3 Let G_3 be a metacyclic 2-group of order 32, $G_3 \cong \langle a,b : a^{16} = 1, b^2 = 1, [a,b] = a^8 \rangle$. Then, the conjugate graph, $\Gamma_{G_3}^C = \bigcup_{i=1}^{12} K_2$.

Proof Based on Theorem 3.3, there are 32 elements of G_3 and eight are in the center of G_3. Thus the number of vertices of the conjugate graph of G_3 is 24. Based on the vertices adjacency of the conjugate graph, Γ consists of 12 complete components of K_2. The conjugate graph of G_3 is represented in Figure 4.3. ■

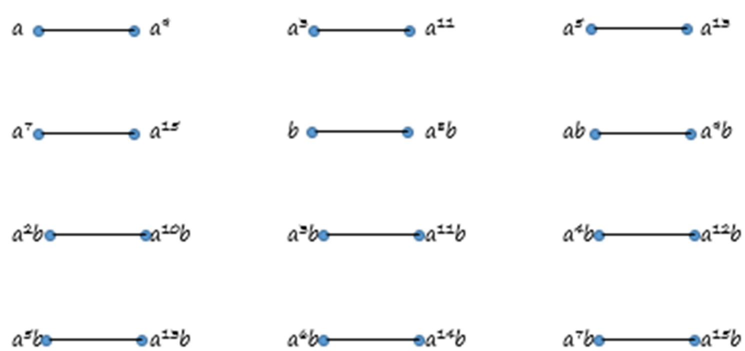

Figure 4.3: The Conjugate Graph of G_3

According to Theorem 4.3, some conjugate graph properties of G_3 are obtained.

Corollary 4.3 Let $G_3 \cong \langle a,b : a^{16} = 1, b^2 = 1, [a,b] = a^8 \rangle$ be a metacyclic 2-group of order 32. If $\Gamma_{G_3}^C = \bigcup_{i=1}^{12} K_2$, then $\Gamma_{G_3}^C = \omega\left(\Gamma_{G_3}^C\right) = 2$, $\alpha\left(\Gamma_{G_3}^C\right) = 12$ and $\gamma\left(\Gamma_{G_3}^C\right)$.

Proof Based on Theorem 4.3, $\Gamma^{C}_{G_3}$ consists of 12 complete components of K_2. Thus by Definition 1.9, the chromatic number is equal to two since the vertices in $\Gamma^{C}_{G_3}$ can be colored by two colors. Using Definition 1.10, the clique number, $\omega\left(\Gamma^{C}_{G_3}\right)$ since the largest subgraph in $\Gamma^{C}_{G_3}$ is K_2. Using Definition 1.8, the maximum independent set of conjugate graph of G_3 is $x=\left\{a,a^3,a^5,a^7,b,ab,a^2b,a^3b,a^4b,a^5b,a^6b,a^7b\right\}$, hence the independent number of G_3, $\alpha\left(\Gamma^{C}_{G_3}\right)=12$. Using Definition 1.11, the dominating number in $\Gamma^{C}_{G_3}$ is equal to 12 since there are 12 vertices needed to dominate itself and another vertices in K_2, as claimed. ■

In the following, the conjugacy class graph of metacyclic 2-groups of order at most 8 is determined. The results on the sizes of conjugacy classes of groups are used to get the conjugacy class graph. We start by finding the conjugacy class graph of G_1.

Theorem 4.4 Let G_1 be a metacyclic 2-group of order 8, $G_1 \cong \langle a,b : a^4=1, b^2=[b,a]=a^{-2}\rangle$. Then the conjugacy class graph of G_1, $\Gamma^{Cl}_{G_1}=K_3$.

Proof Based on Theorem 3.1, the number of conjugacy class of G_1 is five and the there two elements in the center of G_1. Since two vertices are adjacent in the conjugacy class graph if the gcd between the sizes of the conjugacy classes is greater that one. Thus since the size of all non-central conjugacy classes namely, $\mathrm{cl}(a)$, $\mathrm{cl}(b)$ and $\mathrm{cl}(ab)$ is two. Therefore, the three conjugacy classes then produce the conjugacy class graph of G_1 that is K_3, given by Figure 4.4. ■

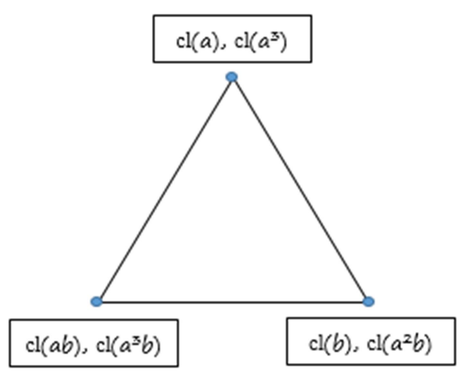

Figure 4.4: The Conjugacy Class Graph of G_1, K_3.

Next, some properties of the conjugacy class graph of G_1 are given.

Corollary 4.4 Let $G_1 \cong \langle a,b : a^4 = 1, b^2 = [b,a] = a^{-2} \rangle$ be a metacyclic 2-group of order 8 and let $\chi\left(\Gamma_{G_1}^{Cl}\right) = \omega\left(\Gamma_{G_1}^{Cl}\right) = 2,\ \alpha\left(\Gamma_{G_1}^{Cl}\right) = 1$ and $\gamma\left(\Gamma_{G_1}^{Cl}\right) = 1.$

Proof From theorem 4.4, $\Gamma_{G_1}^{Cl} = K_3$. Using Definition 1.9, the conjugacy class graph of G_1 has the chromatic number $\chi\left(\Gamma_{G_1}^{Cl}\right) = 3,$ since the three vertices in K_3 have a different color to each other. Using Definition 1.10, the the clique number, $\omega\left(\Gamma_{G_1}^{Cl}\right) = 3,$ since the largest subgraph in Γ is K_3. Using Definition 1.8, the conjugacy class graph of G_1 has the independent set number $\alpha\left(\Gamma_{G_1}^{Cl}\right) = 1,$ since 1 is the maximum number of the independent set in $V(\Gamma_{G_3})$. Using Definition 1.11, the conjugacy class graph of G_1 has the dominating number, is $\gamma\left(\Gamma_{G_1}^{Cl}\right) = 1,$ since one vertex can dominate itself and other vertices. ■

In the next theorem, the conjugacy class graph is found for G_2.

Theorem 4.5 Let G_2 be a metacyclic 2-group of order 16 $G_2 \cong \left\langle a,\ b : a^8 = e,\ b^2 = e,\ [a,b] = a^4 \right\rangle$. Then the conjugacy class graph of G_2, $\Gamma_{G_2}^{Cl} = K_6$.

Proof Based on Theorem 3.2, the number of conjugacy class of G_2 is 10 and the order of the center of G_2 is four. Since two vertices in conjugacy class graph are connected by an edge if the gcd between all the size of the conjugacy classes is greater than one, thus the sizes of all non-central conjugacy classes, namely $\mathrm{cl}(a)$, $\mathrm{cl}(a^3)$, $\mathrm{cl}(b)$, $\mathrm{cl}(ab)$, $\mathrm{cl}(a^2b)$ and $\mathrm{cl}(a^3b)$ is two. Therefore the vertices that adjacent in conjugacy class graph of G_2 are given as: $\{cl(a), cl(a^5)\}, \{cl(a^3), cl(a^7)\}, \{cl(b), cl(a^4b)\}, \{cl(a^2b), cl(a^6b)\}, \{cl(ab), cl(a^5b)\},$ $\{cl(a^3b), cl(a^7b)\}$. Thus the six conjugacy classes then produce the conjugacy class graph of G_2 that is K_6, given by Figure 4.5. ■

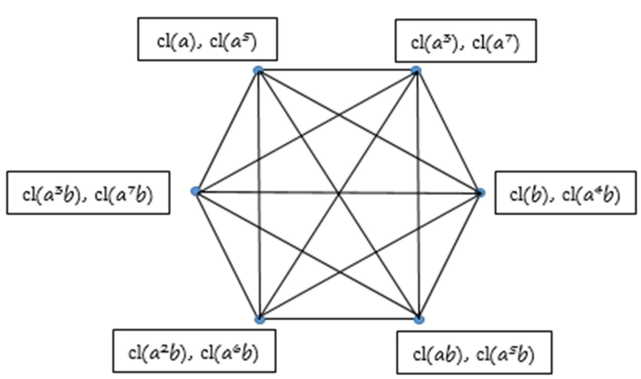

Figure 4.5: The Conjugacy Class Graph of G_2, K_6

Next, some properties of the conjugacy class graph of G_2 can be concluded.

Corollary 4.5 Let $G_2 \cong \langle a,b : a^8 = 1, b^2 = 1, [a,b] = a^4 \rangle$, be a metacyclic 2-group of order 16. Then $\chi\left(\Gamma_{G_2}^{Cl}\right) = \omega\left(\Gamma_{G_2}^{Cl}\right) = 6$, $\alpha\left(\Gamma_{G_2}^{Cl}\right) = 1$ and $\gamma\left(\Gamma_{G_2}^{Cl}\right) = 1$.

Proof From Theorem 4.5, $\Gamma_{G_2}^{Cl} = K_6$. Since $\Gamma_{G_2}^{Cl}$ consists of one complete graph of K_6, thus by Definition 1.9, the chromatic number in $\Gamma_{G_2}^{Cl}$ is equal to six since the six vertices in K_6 have a different color to each other. Using Definition 1.10, the clique number is equal to six, since the largest subgraph in $\Gamma_{G_2}^{Cl}$ is K_6. Using Definition 1.8, the conjugacy class graph of G_2 has the independent set number $\alpha\left(\Gamma_{G_2}^{Cl}\right) = 1$ since one is the maximum number of the independent set in $V(G_2)$. Using Definition 1.11, the dominating number $\gamma\left(\Gamma_{G_2}^{Cl}\right) = 1$, since one vertex can dominate itself and other vertices. ■

In the next theorem, the conjugacy class graph is found for G_3.

Theorem 4.6 Let G_3 be a metacyclic 2-group of order 32, with the presentation $G_3 \cong \langle a,b : a^{16} = 1, b^2 = 1, [a,b] = a^8 \rangle$. Then the conjugacy class graph of G_3, $\Gamma_{G_3}^{Cl} = K_{12}$.

Proof Based on Theorem 3.3, the number of conjugacy class of G_3 is 20 and the order of the center of G_3 is 8. Since two vertices are adjacent in Γ if the gcd between all the size of the conjugacy classes is greater that one. Hence, the sizes of all conjugacy classes of non-central elements namely, $\mathrm{cl}(a)$, $\mathrm{cl}(a^3)$, $\mathrm{cl}(a^5)$, $\mathrm{cl}(a^7)$, $\mathrm{cl}(b)$, $\mathrm{cl}(ab)$, $\mathrm{cl}(a^2b)$, $\mathrm{cl}(a^3b)$, $\mathrm{cl}(a^4b)$, $\mathrm{cl}(a^5b)$, $\mathrm{cl}(a^6b)$ and $\mathrm{cl}(a^7b)$ is 2. Therefore conjugacy class graph of G_3 consists of one complete graph of K_{12}. These twelve conjugacy classes then produce the conjugacy class graph of G_3 that is K_{12}, given by Figure 4.6. ■

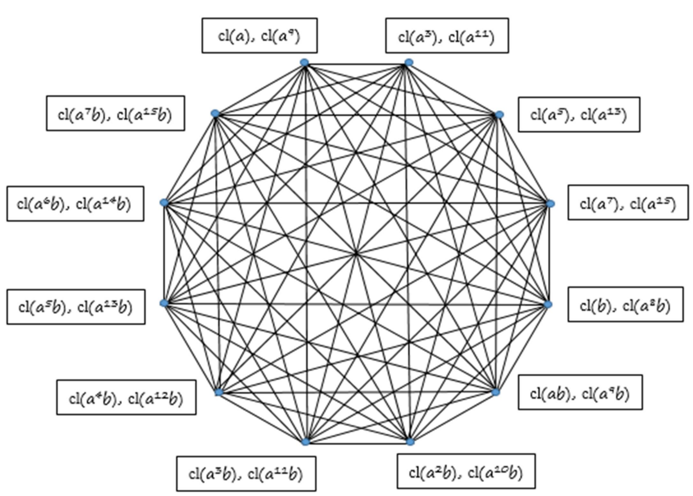

Figure 4.6: The Conjugacy Class Graph of G_3, K_{12}

Next, some properties of the conjugacy class graph of G_3 are concluded.

Corollary 4.6 Let $G_3 \cong \langle a,b : a^{16}=1, b^2=1, [a,b]=a^8 \rangle$ be a metacyclic 2-group of order 32. Then $\chi\left(\Gamma_{G_3}^{Cl}\right)=\omega\left(\Gamma_{G_3}^{Cl}\right)=12$, $\alpha\left(\Gamma_{G_3}^{Cl}\right)=1$ and $\gamma\left(\Gamma_{G_3}^{Cl}\right)=1$.

Proof From Theorem 4.6, $\Gamma_{G_3}^{Cl}=K_{12}$. Using Definition 1.9, the conjugacy class graph of G_3 has the chromatic number $\chi\left(\Gamma_{G_3}^{Cl}\right)=12$, since the twelve vertices in K_{12} have a different color to each other. Using Definition 1.10, the clique number, $\omega\left(\Gamma_{G_3}^{Cl}\right)=12$, since the largest subgraph in $\Gamma_{G_3}^{Cl}$ is K_{12}. Using Definition 2.8, the conjugacy class graph of G_3 has the independent set number $\alpha\left(\Gamma_{G_3}^{Cl}\right)=1$ since one is the only maximum number of the independent set in $V(G_3)$. Using Definition 1.11, the conjugacy class graph of G_3 has the dominating number $\gamma\left(\Gamma_{G_3}^{Cl}\right)=1$, since there is only one vertex that dominate itself and another vertex in K_2. ■

CONCLUSION

In this paper, the conjugacy classes of all metacyclic 2-groups of order at most 32 are computed. It is proven that for the group G_1 namely the quaternion of order 8 with $G_1 \cong a,b : a^4=e,\ b^2=[a,b]=a^{-2}$, it has five conjugacy classes, $K(G_1)=5$ and two elements in the center of the group, $|Z(G_1)|=2$. For the group G_2, of order 16 with, $G_2 \cong a,b : a^8=e,\ b^2=e,[a,b]=a^4$, it has ten conjugacy classes, i.e. $K(G_2)=10$ and four elements in the center of group, i.e. $|Z(G_2)|=4$. The third group G_3, group of order 32 with $G_3 \cong a,b : a^{16}=e, b^2=e,[a,b]=a^8$, the number of conjugacy class is 20, i.e. $K(G_3)=20$ and the number of elements in the center of group, $|Z(G_3)|=8$.

It is proven that the conjugate graphs of all three groups is a union of complete components of K_2. The conjugacy class graph of G_1 consists of one complete graph of K_3, while the conjugacy class graph of G_2 consists of a complete graph of K_6. In addition, the conjugacy class graph of

G_3 is K_{12}. Some graph properties are obtained for both types of graphs which include the chromatic number, the clique number, the independent number and the dominating number.

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