Neural Comput & Applic (2012) 21 (Suppl 1):S295–S305
DOI 10.1007/s00521-012-0843-3

ORIGINAL ARTICLE

New types of fuzzy bi-ideals in ordered semigroups

Asghar Khan · Nor Haniza Sarmin · Bijan Davvaz · Faiz Muhammad Khan

Received: 14 March 2011 / Accepted: 12 January 2012 / Published online: 10 February 2012

Abstract In Jun et al. (Bull Malays Math Sci Soc (2) 32(3):391–408, 2009), (α, β)-fuzzy bi-ideals are introduced and some characterizations are given. In this paper, we generalize the concept of (α, β)-fuzzy bi-ideals and define $(\in, \in \vee q_k)$-fuzzy bi-ideals in ordered semigroups, which is a generalization of the concept of an (α, β)-fuzzy bi-ideal in an ordered semigroup. Using this concept, some characterization theorems of regular, left (resp. right) regular and completely regular ordered semigroups are provided. In the last section, we give the concept of upper/lower parts of an $(\in, \in \vee q_k)$-fuzzy bi-ideal and investigate some interesting results of regular and intra-regular ordered semigroups.

Keywords Regular, left (resp. right) regular and completely regular ordered semigroups · Bi-ideals · Fuzzy bi-ideals · $(\in, \in \vee q_k)$-fuzzy bi-ideals

A. Khan (✉)
Department of Mathematics, COMSAT Institute of Information Technology, Khyber Pukhtoon Khwa, Abbottabad, Pakistan
e-mail: azhar4set@yahoo.com

N. H. Sarmin · F. M. Khan
Department of Mathematical Sciences,
Faculty of Science, Universiti Teknologi Malaysia,
81310 Johor Bahru, Johor, Malaysia
e-mail: nhs@utm.my

F. M. Khan
e-mail: faiz_zady@yahoo.com

B. Davvaz
Department of Mathematics, Yazd University, Yazd, Iran
e-mail: bdavvaz@yahoo.com

1 Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [21], provides a natural frame-work for generalizing several basic notions of algebra. A new type of fuzzy subgroup, that is, the (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1, 2] by using the combined notions of "belongingness" and "quasi-coincidence" of a fuzzy point and a fuzzy set. In particular, the concept of an $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroup [17]. Davvaz [5] introduced the concept of $(\in, \in \vee q)$-fuzzy sub-near-ring (R-subgroups, ideals) of a near-ring and investigated some of their properties. Davvaz and Khan [3] discussed some characterizations of regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals of ordered semigroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Also Khan et al. introduced the concept of an (α, β)-fuzzy interior ideals and gave some basic properties of ordered semigroups in terms of this notion (see [12]). In [18], regular semigroups are characterized by the properties of $(\in, \in \vee q)$-fuzzy ideals. Jun [8], introduced the concept of $(\in, \in \vee q_k)$-fuzzy subalgebras of a BCK/BCI-algebra and gave some basic properties of BCK-algebras. In [19], Shabir et al. gave the concept of more generalized forms of (α, β)-fuzzy ideals and defined $(\in, \in \vee q_k)$-fuzzy ideals of semigroups, by generalizing the concept of $[x; t]qF$ and defined $[x; t]q_kF$, as $F(x) + t + k > 1$, where $k \in [0, 1)$, (also see [9, 10, 20]). For further reading regarding (α, β)-fuzzy subsets and its generalization, we refer the reader to [4, 7, 11, 13, 15, 16].

The topic of these investigations belongs to the theoretical soft computing (fuzzy structure). Indeed, it is well known that semigroups are basic structures in many applied branches like automata and formal languages,

coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings (see e.g., monograph [14]).

Our aim in this paper is to introduce a new sort of fuzzy bi-ideals and fuzzy left (resp. right)-ideals in ordered semigroup, called $(\in, \in \vee q_k)$-fuzzy bi-ideals and characterize regular, left and right regular, and completely regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy bi-ideals. We define the lower/upper parts of $(\in, \in \vee q_k)$-fuzzy bi-ideals. Then some results are given in terms of $(\in, \in \vee q_k)$-fuzzy bi-ideals in regular and intra-regular ordered semigroups.

2 Basic definitions and preliminaries

By an *ordered semigroup* (or *po-semigroup*), we mean a structure $(S, \cdot, \leq)$ in which the following conditions are satisfied:

(*OS1*) $(S, \cdot)$ is a semigroup,
(*OS2*) $(S, \leq)$ is a poset,
(*OS3*) $a \leq b \longrightarrow ax \leq bx$ and $xa \leq xb$ for all $x \in S$.

For $A, B \subseteq S$, we denote by,

$$AB := \{ab | a \in A, \text{ and } b \in B\}.$$

If S is an ordered semigroup, and A a *subset* of S, we denote by $(A]$ the subset of S defined as follows:

$$(A] := \{t \in S | t \leq a \text{ for some } a \in A\}.$$

If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. The operator "(]" is a closure operator, and therefore

- extensive (that is, $A \subseteq (A]$),
- isoton (that is, $A \subseteq B$ implies $(A] \subseteq (B]$),
- idempotent (that is, $((A]] \subseteq (A]$ and therefore $((A]] = (A]$).

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-empty subset A of S is called *left* (resp. *right*) *ideal* [11] of S if

(i) $(\forall a \in S)(\forall b \in A)(a \leq b \longrightarrow a \in A)$,
(ii) $SA \subseteq A$ (resp. $AS \subseteq A$).

A non-empty subset A of S is called an ideal if it is both a left and a right ideal of S.

A non-empty subset A of an ordered semigroup S is called a *bi-ideal* [11] of S if

(i) $(\forall a \in S)(\forall b \in A)(a \leq b \longrightarrow a \in A)$,
(ii) $A^2 \subseteq A$,
(iii) $ASA \subseteq A$.

An ordered semigroup S is *regular* [11] if for every $a \in S$ there exists, $x \in S$ such that $a \leq axa$, or equivalently, we have (i) $a \in (aSa] \forall a \in S$ and (ii) $A \subseteq (ASA] \forall A \subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *regular* if for every $a \in S$ there exists $x \in S$, such that $a \leq xa^2$(resp. $a \leq a^2x$), or equivalently, (i) $a \in (Sa^2]$(resp. $a \in (a^2S]$)$\forall a \in S$ and (ii) $A \subseteq (SA^2]$(resp. $A \subseteq (A^2S]$)$\forall A \subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *simple* [11] if for every left (resp. right) ideal A of S we have $A = S$ and S is called *simple* [11] if it is both left and right simple. An ordered semigroup S is called *completely regular*, if it is left regular, right regular and regular.

Now, we give some fuzzy logic concepts.

A function $F : S \longrightarrow [0, 1]$ is called a *fuzzy subset* of S.

The study of fuzzification of algebraic structures has been started in the pioneering paper of Rosenfeld [17] in 1971. Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups in the theory of fuzzy groups. Kuroki [13] studied fuzzy ideals, fuzzy bi-ideals and semiprime fuzzy ideals in semigroups.

If F_1 and F_2 are fuzzy subsets of S then $F_1 \preceq F_2$ means $F_1(x) \leq F_2(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets:

$$F_1 \wedge F_2 : S \longrightarrow [0, 1] | x \longmapsto (F_1 \wedge F_2)(x) = F_1(x) \wedge F_2(x) = \min\{F_1(x), F_2(x)\}$$
$$F_1 \vee F_2 : S \longrightarrow [0, 1] | x \longmapsto (F_1 \vee F_2)(x) = F_1(x) \vee F_2(x) = \max\{F_1(x), F_2(x)\},$$

for all $x \in S$.

A fuzzy subset F of S is called a *fuzzy subsemigroup* if $F(xy) \geq \min\{F(x), F(y)\}$ for all $x, y \in S$.

A fuzzy subset F of S is called a *fuzzy left (resp. right)-ideal* [11] of S if

(i) $x \leq y \longrightarrow F(x) \geq F(y)$,
(ii) $F(xy) \geq F(y)$ (resp. $F(xy) \geq F(x)$) for all $x, y \in S$.

A fuzzy subset F of S is called a *fuzzy ideal* if it is both a fuzzy left and a fuzzy right ideal of S.

A fuzzy subsemigroup F is called a *fuzzy bi-ideal* [11] of S if

(i) $x \leq y \longrightarrow F(x) \geq F(y)$,
(ii) $F(xyz) \geq \min\{F(x), F(z)\}$ for all $x, y, z \in S$.

Let S be an ordered semigroup and F is a fuzzy subset of S. Then, for all $t \in (0, 1]$, the set $U(F; t) = \{x \in S | F(x) \geq t\}$ is called a *level set* of F.

Theorem 2.1 [6] *A fuzzy subset F of an ordered semigroup S is a fuzzy bi-ideal of S if and only if $U(F; t)(\neq \emptyset)$ is a bi-ideal of S, for all $t \in (0, 1]$.*

Theorem 2.2 [6] *A non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0,1] | x \longmapsto \chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

is a fuzzy bi-ideal of S.

If $a \in S$ and A is a non-empty subset of S. Then,

$A_a = \{(y,z) \in S \times S | a \leq yz\}$.

If F_1 and F_2 are two fuzzy subsets of S. Then the product $F_1 \circ F_2$ of F_1 and F_2 is defined by:

$$F_1 \circ F_2 : S \longrightarrow [0,1] | a \longmapsto (F_1 \circ F_2)(a) = \begin{cases} \bigvee_{(y,z) \in A_a} (F_1(y) \wedge F_2(z)) & \text{if } A_a \neq \emptyset, \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

Let F be a fuzzy subset of S, then the set of the form:

$$F(y) := \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by $[x; t]$. A fuzzy point $[x; t]$ is said to *belong to* (resp. *quasi-coincident* with) a fuzzy set F, written as $[x;t] \in F$ (resp. $[x; t]qF$) if $F(x) \geq t$ (resp. $F(x) + t > 1$). If $[x;t] \in F$ or $[x; t]qF$, then we write $[x;t] \in \vee qF$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Generalizing the concept of $[x; t]qF$, in BCK/BCI-algebras, Jun [8] defined $[x; t]q_kF$, as $F(x) + t + k > 1$, where $k \in [0,1)$.

3 $(\in, \in \vee q_k)$-fuzzy bi-ideals

In what follows, let S denote an ordered semigroup unless otherwise specified. In this section, we define a more generalized form of (α, β)-fuzzy bi-ideals of an ordered semigroups S, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}, \alpha \neq \in \wedge q$, and introduce $(\in, \in \vee q_k)$-fuzzy bi-ideals S, where k is an arbitrary element of [0,1) unless otherwise stated.

Definition 3.1 A fuzzy subset F of S is called an $(\in, \in \vee q_k)$-*fuzzy bi-ideal* of S if it satisfies the conditions:

1. $(\forall x, y \in S)(\forall t \in (0,1])(x \leq y, [y;t] \in F \longrightarrow [x;t] \in \vee q_kF)$,
2. $(\forall x, y \in S)(\forall t, r \in (0,1])([x;r] \in F, [y;t] \in F \longrightarrow [xy; r \wedge t] \in \vee q_kF)$,
3. $(\forall x, y, z \in S)(\forall t, r \in (0,1])([x;r] \in F, [z;t] \in F \longrightarrow [xyz; r \wedge t] \in \vee q_kF)$.

Theorem 3.2 *Let A be a bi-ideal of S and F a fuzzy subset in S defined by:*

$$F(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then

1. *F is a $(q, \in \vee q_k)$-fuzzy bi-ideal of S.*
2. *F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof (1) Let $x, y \in S, x \leq y$ and $t \in (0,1]$ be such that $[y; t]qF$. Then $y \in A, F(y) + t > 1$. Since A is a bi-ideal of S and $x \leq y \in A$, we have $x \in A$. Thus $F(x) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq t$ and so $[x;t] \in F$. If $t > \frac{1-k}{2}$, then $F(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x; t]q_kF$. Therefore $[x;t] \in \vee q_kF$.

Let $x, y \in S$ and $t, r \in (0,1]$ be such that $[x; t]qF$ and $[y; r]qF$. Then $F(x) + t > 1, F(y) + t > 1$ and $x, y \in A$, we have $xy \in A$. Thus $F(xy) \geq \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $F(xy) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xy; t \wedge r]q_kF$. If $t \wedge r \leq \frac{1-k}{2}$, then $F(xy) \geq t \wedge r$ and so $[xy; t \wedge r] \in F$. Therefore $[xy; t \wedge r] \in \vee q_kF$.

Let $x, y, z \in S$ and $t, r \in (0,1]$ be such that $[x; t]qF$ and $[z; r]qF$. Then $F(x) + t > 1$, $F(z) + t > 1$ and $x, z \in A$, hence $xyz \in A$. Thus $F(xyz) \geq \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $F(xyz) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xyz; t \wedge r]q_kF$. If $t \wedge r \leq \frac{1-k}{2}$, then $F(xyz) \geq t \wedge r$ and so $[xyz; t \wedge r] \in F$. Therefore $[xyz; t \wedge r] \in \vee q_kF$.

(2) Let $x, y \in S, x \leq y$ and $t \in (0,1]$ be such that $[y; t] \in F$. Then $F(y) \geq t$ and $y \in A$. Since A is a bi-ideal of S and $x \leq y \in A$, we have $x \in A$. Thus $F(x) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq t$ and so $[x;t] \in F$. If $t > \frac{1-k}{2}$, then $F(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x; t]q_kF$. Therefore $[x;t] \in \vee q_kF$.

Let $x, y \in S$ and $t, r \in (0,1]$ be such that $[x;t] \in F$ and $[y;r] \in F$. Then $x, y \in A$ and we have $xy \in A$. Thus $F(xy) \geq \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $F(xy) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xy; t \wedge r]q_kF$. If $t \wedge r \leq \frac{1-k}{2}$, then $F(xy) \geq t \wedge r$ and so $[xy; t \wedge r] \in F$. Therefore $[xy; t \wedge r] \in \vee q_kF$.

Let $x, y, z \in S$ and $t, r \in (0,1]$ be such that $[x;t] \in F$ and $[z;r] \in F$. Then $x, z \in A$, and so $xyz \in A$. Thus $F(xyz) \geq \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $F(xyz) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xyz; t \wedge r]q_kF$. If $t \wedge r \leq \frac{1-k}{2}$, then $F(xyz) \geq t \wedge r$ and so $[xyz; t \wedge r] \in F$. Thus, $[xyz; t \wedge r] \in \vee q_kF$. Consequently, F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. □

If we take $k = 0$ in Theorem 3.2, then we get the following corollary:

Corollary 3.3 [6] *Let A be a bi-ideal of S and F a fuzzy subset in S defined by:*

$$F(x) = \begin{cases} \geq \frac{1}{2} & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then

1. *F is a* $(q, \in \vee q)$*-fuzzy bi-ideal of S.*
2. *F is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of S.*

Theorem 3.4 *Let F be a fuzzy subset of S. Then F is an* $(\in, \in \vee q_k)$*-fuzzy bi-ideal of S if and only if*

1. $(\forall x, y \in S)(x \leq y \longrightarrow F(x) \geq F(y) \wedge \frac{1-k}{2})$,
2. $(\forall x, y \in S)\big(F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2}\big)$,
3. $(\forall x, y, z \in S)\big(F(xyz) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2}\big)$.

Proof Let F be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. On the contrary assume that, there exist $x, y \in S$, $x \leq y$ such that $F(x) < F(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $F(x) < t \leq F(y) \wedge \frac{1-k}{2}$. Then $[y; t] \in F$, but $F(x) < t$ and $F(x) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[x; t]\overline{\in \vee q_k}F$, which is a contradiction. Hence $F(x) \geq F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ with $x \leq y$.

If there exist $x, y \in S$ such that $F(xy) < F(x) \wedge F(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $F(xy) < t \leq F(x) \wedge F(y) \wedge \frac{1-k}{2}$. Then $[x; t] \in F, [y; t] \in F$ but $F(xy) < t$ and $F(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[xy; t]\overline{q}_k F$. Thus, $[xy;t]\overline{\in \vee q_k}F$, which is a contradiction. Therefore, $F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$.

Now if there exist $x, y, z \in S$ such that $F(xyz) < F(x) \wedge F(z) \wedge \frac{1-k}{2}$. Then, for $t \in (0, 1]$ such that $F(xyz) < t \leq F(x) \wedge F(z) \wedge \frac{1-k}{2}$, we have $[x;t] \in F$ and $[z;t] \in F$ but $F(xyz) < t$ and $F(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[xyz;t]\overline{q}_k F$. Thus, $[xyz;t]\overline{\in \vee q_k}F$, which is a contradiction. Therefore $F(xyz) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2}$ for all $x, y, z \in S$.

Conversely, let $[y; t] \in F$ for some $t \in (0, 1]$. Then $F(y) \geq t$. Now, $F(x) \geq F(y) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2}$. If $t > \frac{1-k}{2}$, then $F(x) \geq \frac{1-k}{2}$ and $F(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, it follows that $[x; t]q_k F$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq t$ and so $[x; t] \in F$. Thus, $[x; t] \in \vee q_k F$.

Let $[x; t] \in F$ and $[y; r] \in F$, then $F(x) \geq t$ and $F(y) \geq r$. Thus $F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2} \geq t \wedge r \wedge \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $F(xy) \geq \frac{1-k}{2}$ and $F(xy) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xy; t \wedge r]q_k F$. If $t \wedge r \leq \frac{1-k}{2}$, then $F(xy) \geq t \wedge r$ and hence, $[xy; t \wedge r] \in F$, it follows that $[xy; t \wedge r] \in \vee q_k F$. Now let $[x; t] \in F$ and $[z; r] \in F$, then $F(x) \geq t$ and $F(z) \geq r$. Therefore $F(xyz) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2} \geq t \wedge r \wedge \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $F(xyz) \geq \frac{1-k}{2}$ and $F(xyz) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xyz; t \wedge r]q_k F$. If $t \wedge r \leq \frac{1-k}{2}$, then $F(xyz) \geq t \wedge r$ and hence, $[xyz; t \wedge r] \in F$. Thus $[xyz; t \wedge r] \in \vee q_k F$ and consequently, F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. □

If we take $k = 0$ in Theorem 3.4, we have the following corollary:

Corollary 3.5 [6] *Let F be a fuzzy subset of S. Then F is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of S if and only if*

1. $(\forall x, y \in S)(x \leq y \longrightarrow F(x) \geq F(y) \wedge 0.5)$,
2. $(\forall x, y \in S)(F(xy) \geq F(x) \wedge F(y) \wedge 0.5)$,
3. $(\forall x, y, z \in S)(F(xyz) \geq F(x) \wedge F(z) \wedge 0.5)$.

Theorem 3.6 *A fuzzy subset F of S is an* $(\in, \in \vee q_k)$*-fuzzy bi-ideal of S if and only if* $U(F; t)(\neq \emptyset)$ *is a bi-ideal of S for all* $t \in \big(0, \frac{1-k}{2}\big]$.

Proof Suppose that F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S and $x, y \in S$ be such that $x \leq y \in U(F; t)$ for some $t \in \big(0, \frac{1-k}{2}\big]$. Then $F(y) \geq t$ and by hypothesis

$$F(x) \geq F(y) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t.$$

Hence $x \in U(F; t)$.

Now, let $x, y \in S$ be such that $x, y \in U(F; t)$ for some $t \in \big(0, \frac{1-k}{2}\big]$. Then $F(x) \geq t$ and $F(y) \geq t$ and by hypothesis

$$F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2} \geq t \wedge t \wedge \frac{1-k}{2} = t.$$

Hence $xy \in U(F; t)$.

For $x, z \in U(F; t)$, we have

$$F(xyz) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2} \geq t \wedge t \wedge \frac{1-k}{2} = t.$$

Thus, $xyz \in U(F; t)$.

Conversely, assume that $U(F; t)(\neq \emptyset)$ is a bi-ideal of S for all $t \in \big(0, \frac{1-k}{2}\big]$.

Let $x, y \in S$ with $x \leq y$ be such that $F(x) < F(y) \wedge \frac{1-k}{2}$. Choose $r \in \big(0, \frac{1-k}{2}\big]$ such that $F(x) < r \leq F(y) \wedge \frac{1-k}{2}$ then $F(y) \geq r$ implies that $[y; r] \in F$ but $[x; r]\overline{\in}F$. Now $F(x) + r + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $[x; r]\overline{q}_k F$, contradiction. Hence $F(x) \geq F(y) \wedge \frac{1-k}{2}$. If there exist $x, y \in S$ such that $F(xy) < F(x) \wedge F(y) \wedge \frac{1-k}{2}$. Then choose $t \in \big(0, \frac{1-k}{2}\big]$ such that $F(xy) < t \leq F(x) \wedge F(y) \wedge \frac{1-k}{2}$. Thus $x, y \in U(F; t)$ but $xy \notin U(F; t)$, a contradiction. Hence $F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ and $k \in [0, 1)$. If there exist $x, y, z \in S$ such that $F(xyz) < F(x) \wedge F(z) \wedge \frac{1-k}{2}$. Then choose $t \in \big(0, \frac{1-k}{2}\big]$ such that $F(xyz) < t \leq F(x) \wedge F(z) \wedge \frac{1-k}{2}$. Thus $x, z \in U(F; t)$ but $xyz \notin U(F; t)$, a contradiction. Hence $F(xyz) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2}$ for all

$x, y, z \in S$ and $k \in [0, 1)$. Therefore F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. □

Example 3.7 Consider the ordered semigroup $S = \{a, b, c, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$.

Then $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$ and $\{a, b, c, d\}$ are bi-ideals of S. Define a fuzzy subset F of S as follows:

$$F : S \longrightarrow [0, 1] | x \longmapsto F(x) = \begin{cases} 0.8 & \text{if } x = a \\ 0.7 & \text{if } x = d \\ 0.4 & \text{if } x = c \\ 0.3 & \text{if } x = b \end{cases}$$

Then

$$U(F; t) = \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a, c, d\} & \text{if } 0.3 < t \leq 0.4 \\ \{a, d\} & \text{if } 0.4 < t \leq 0.7 \\ \emptyset & \text{if } 0.8 < t \leq 1 \end{cases}$$

Then, by Theorem 3.6, F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S for $t \in \left(0, \frac{1-k}{2}\right]$ with $k = 0.4$.

Proposition 3.8 *If F is a nonzero $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Then the set $F_0 = \{x \in S | F(x) > 0\}$ is a bi-ideal of S .*

Proof The proof is straight forward. □

Lemma 3.9 *A non-empty subset A of S is a bi-ideal if and only if the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof The proof is straight forward. □

Proposition 3.10 *If $\{F_i : i \in I\}$ is a family of $(\in, \in \vee q_k)$-fuzzy bi-ideals of an ordered semigroup S. Then $\bigcap_{i \in I} F_i$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof Let $\{F_i\}_{i \in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy bi-ideals of S. Let $x, y \in S$, $x \leq y$. Then

$$\left(\bigcap_{i \in I} F_i\right)(x) = \bigwedge_{i \in I} F_i(x) \geq \bigwedge_{i \in I} \left(F_i(y) \wedge \frac{1-k}{2}\right) = \left(\bigcap_{i \in I} F_i\right)(y) \wedge \frac{1-k}{2}$$

and

$$\left(\bigcap_{i \in I} F_i\right)(xy) = \bigwedge_{i \in I} F_i(xy) \geq \bigwedge_{i \in I} \left(F_i(x) \wedge F_i(y) \wedge \frac{1-k}{2}\right) = \left(\bigwedge_{i \in I} \left(F_i(x) \wedge \frac{1-k}{2}\right) \wedge \bigwedge_{i \in I} \left(F_i(y) \wedge \frac{1-k}{2}\right)\right) = \left(\bigcap_{i \in I} F_i\right)(x) \wedge \left(\bigcap_{i \in I} F_i\right)(y) \wedge \frac{1-k}{2}$$

Let $x, y, z \in S$. Then,

$$\left(\bigcap_{i \in I} F_i\right)((xyz) = \bigwedge_{i \in I} F_i((xyz) \geq \bigwedge_{i \in I} \left(F_i(x) \wedge F_i(z) \wedge \frac{1-k}{2}\right) = \left(\bigwedge_{i \in I} \left(F_i(x) \wedge \frac{1-k}{2}\right) \wedge \bigwedge_{i \in I} \left(F_i(z) \wedge \frac{1-k}{2}\right)\right) = \left(\bigcap_{i \in I} F_i\right)(x) \wedge \left(\bigcap_{i \in I} F_i\right)(z) \wedge \frac{1-k}{2}.$$

Thus $\bigcap_{i \in I} F_i$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. □

4 Upper and lower parts of $(\in, \in \vee q_k)$-fuzzy bi-ideals

In this section, we define the upper/lower parts of an $(\in, \in \vee q_k)$-fuzzy bi-ideal and characterize regular and intra-regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy bi-ideals.

Definition 4.1 Let F_1 and F_2 be a fuzzy subsets of S. Then the fuzzy subsets $\overline{F_1}^k$, $(F_1 \wedge^k F_2)^-$, $(F_1 \vee^k F_2)^-$, $(F_1 \circ^k F_2)^-$, $\overset{+}{F_1}^k$, $(F_1 \wedge^k F_2)^+$, $(F_1 \vee^k F_2)^+$ and $(F_1 \circ^k F_2)^+$ of S are defined as follows:

$$\overline{F_1}^k : S \longrightarrow [0, 1] | x \longmapsto F_1^k(x) = F_1(x) \wedge \frac{1-k}{2},$$

$$(F_1 \wedge^k F_2)^- : S \longrightarrow [0, 1] | x \longmapsto (F_1 \wedge^k F_2)(x) = (F_1 \wedge F_2)(x) \wedge \frac{1-k}{2},$$

$$(F_1 \vee^k F_2)^- : S \longrightarrow [0, 1] | x \longmapsto (F_1 \vee^k F_2)(x) = (F_1 \vee F_2)(x) \wedge \frac{1-k}{2},$$

$$(F_1 \circ^k F_2)^- : S \longrightarrow [0, 1] | x \longmapsto (F_1 \circ^k F_2)(x) = (F_1 \circ F_2)(x) \wedge \frac{1-k}{2},$$

and

$$\overset{+}{F}{}_1^k : S \longrightarrow [0,1] | x \longmapsto F_1^k(x) = F_1(x) \vee \frac{1-k}{2},$$
$$(F_1 \wedge^k F_2)^+ : S \longrightarrow [0,1] | x \longmapsto (F_1 \wedge^k F_2)(x)$$
$$= (F_1 \wedge F_2)(x) \vee \frac{1-k}{2},$$
$$(F_1 \vee^k F_2)^+ : S \longrightarrow [0,1] | x \longmapsto (F_1 \vee^k F_2)(x)$$
$$= (F_1 \vee F_2)(x) \vee \frac{1-k}{2},$$
$$(F_1 \circ^k F_2)^+ : S \longrightarrow [0,1] | x \longmapsto (F_1 \circ^k F_2)(x)$$
$$= (F_1 \circ F_2)(x) \vee \frac{1-k}{2},$$

for all $x \in S$.

Lemma 4.2 *Let F_1 and F_2 be fuzzy subsets of S. Then the following hold:*

(i) $\left(F_1 \wedge^k F_2\right)^- = \left(\overline{F_1}^k \wedge \overline{F_2}^k\right)$,

(ii) $\left(F_1 \vee^k F_2\right)^- = \left(\overline{F_1}^k \vee \overline{F_2}^k\right)$,

(iii) $\left(F_1 \circ^k F_2\right)^- = \left(\overline{F}_1^k \circ \overline{F}_2^k\right)$.

Proof The proof is same as that of [19]. □

Let A be a non-empty subset of S, then the upper and lower parts of the characteristic function χ_A are defined as follows:

$$\overline{\chi}_A^k : S \longrightarrow [0,1] | x \longmapsto \overline{\chi}_A^k(x) = \begin{cases} \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$
$$\overset{+}{\chi}{}_A^k : S \longrightarrow [0,1] | x \longmapsto \overset{+}{\chi}{}_A^k(x) = \begin{cases} 1 & \text{if } x \in A \\ \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

Lemma 4.3 [10]. *Let A and B be non-empty subset of S. Then the following hold:*

1. $(\chi_A \wedge^k \chi_B)^- = \overline{\chi}_{A\cap B}^k$,
2. $(\chi_A \vee^k \chi_B)^- = \overline{\chi}_{A\cup B}^k$,
3. $(\chi_A \circ^k \chi_B)^- = \overline{\chi}_{(AB]}^k$.

Lemma 4.4 *The lower part $\overline{\chi}_A^k$ of the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if A is a bi-ideal of S.*

Proof The proof follows from Lemma 3.9. □

Lemma 4.5 *The lower part $\overline{\chi}_A^k$ of the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal of S if and only if A is a left (resp. right)-ideal of S.*

Proof The proof follows from Lemma 4.4. □

In the following Proposition, we show that, if F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, then $\overline{F}^k$ is a fuzzy bi-ideal of S.

Proposition 4.6 *If F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, then $\overline{F}^k$ is a fuzzy bi-ideal of S.*

Proof Let $x, y \in S$, $x \leq y$. Since F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S and $x \leq y$, we have $F(x) \geq F(y) \wedge \frac{1-k}{2}$. It follows that $F(x) \wedge \frac{1-k}{2} \geq F(y) \wedge \frac{1-k}{2}$, and hence $\overline{F}^k(x) \geq \overline{F}^k(y)$. For $x, y \in S$, we have $F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2}$. Then $F(xy) \wedge \frac{1-k}{2} \geq F(x) \wedge F(y) \wedge \frac{1-k}{2} = \left(F(x) \wedge \frac{1-k}{2}\right) \wedge \left(F(y) \wedge \frac{1-k}{2}\right)$, and so $\overline{F}^k(xy) \geq \overline{F}^k(x) \wedge \overline{F}^k(y)$.

Now for $x, y, z \in S$, we have $F(xyz) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2}$ and hence, $F(xyz) \wedge \frac{1-k}{2} \geq F(x) \wedge F(z) \wedge \frac{1-k}{2} = \left(F(x) \wedge \frac{1-k}{2}\right) \wedge \left(F(z) \wedge \frac{1-k}{2}\right)$, and so $\overline{F}^k(xyz) \geq \overline{F}^k(x) \wedge \overline{F}^k(z)$. Consequently, $\overline{F}^k$ is a fuzzy bi-ideal of S. □

In [6], regular and intra-regular ordered semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy left (resp. right) and $(\in, \in \vee q)$-fuzzy bi-ideals. In the following, we characterize regular, left and right simple and completely regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals.

Lemma 4.7 [11] *An ordered semigroup S is completely regular if and only if for every $A \subseteq S$, we have, $A \subseteq (A^2SA^2]$, or equivalently, $a \in (a^2Sa^2]$ for every $a \in S$.*

Theorem 4.8 *An ordered semigroup S is completely regular if and only if for every $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, we have*

$\overline{F}^k(a) = \overline{F}^k(a^2)$ for every $a \in S$.

Proof Let $a \in S$. Since S is completely regular, by Lemma 4.7, $a \in (a^2Sa^2]$. Then there exists $x \in S$, such that $a \leq a^2xa^2$. Since F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, we have

$$\begin{aligned} F(a) &\geq F(a^2xa^2) \wedge \frac{1-k}{2} \\ &\geq \left(F(a^2) \wedge F(a^2) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} \\ &= \left(F(a^2) \wedge \frac{1-k}{2}\right) \\ &\geq \left(F(a) \wedge F(a) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} \\ &= \left(F(a) \wedge \frac{1-k}{2}\right). \end{aligned}$$

Thus, $F(a) \wedge \frac{1-k}{2} \geq F(a^2) \wedge \frac{1-k}{2} \geq F(a) \wedge \frac{1-k}{2}$, and it follows that $\overline{F}^k(a) \geq \overline{F}^k(a^2) \geq \overline{F}^k(a)$. Thus $\overline{F}^k(a) = \overline{F}^k(a^2)$ for every $a \in S$.

Conversely, let $a \in S$ and we consider the bi-ideal $B(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$ of generated by a^2. Then by Lemma 3.4,

$$\chi_{B(a^2)} : S \longrightarrow [0,1] | x \longmapsto \chi_{B(a^2)}(x) = \begin{cases} \frac{1-k}{2} & \text{if } x \in B(a^2), \\ 0 & \text{if } x \notin B(a^2), \end{cases}$$

is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. By hypothesis, $\overline{\chi}^k_{B(a^2)}(a) = \overline{\chi}^k_{B(a^2)}(a^2)$. Since $a^2 \in B(a^2)$, we have $\overline{\chi}^k_{B(a^2)}(a^2) = \frac{1-k}{2}$ and hence, $\overline{\chi}^k_{B(a^2)}(a) = \frac{1-k}{2}$. Hence, $a \in B(a^2)$ and we have $a \leq a^2$ or $a \leq a^4$ or $a \leq a^2xa^2$ for some $x \in S$. If $a \leq a^2$, then, $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$ and $a \in (a^2Sa^2]$. Similarly, for $a \leq a^4$ or $a \leq a^2xa^2$, we get $a \in (t^2Ss^2]$ for some $s, t \in S$. Therefore, S is completely regular. □

An equivalence relation σ on S is called *congruence* if $(a,b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* [11] if $(a, a^2) \in \sigma$ and $(ab, ba) \in \sigma$. An ordered semigroup S is called a *semilattice of left and right simple semigroups*if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$, or equivalently, there exists a semilattice Y and a family $\{S_i : i \in Y\}$ of left and right simple subsemigroups of S such that

$$S_i \cap S_j = \emptyset, \forall i,j \in Y, i \neq j, S = \bigcup_{i \in Y} S_i, S_iS_j \subseteq S_{ij} \forall i,j \in Y.$$

A subset T of S is called *semiprime* [6], if for every $a \in S$ such that $a^2 \in T$, we have $a \in T$, or equivalently, for each subset A of S, such that $A^2 \subseteq T$, we have $A \subseteq T$.

Lemma 4.9 [11] *For an ordered semigroup S, the following are equivalent:*

(i) *$(x)_\mathcal{N}$ is a left (resp. right) simple subsemigroup of S, for every $x \in S$,*
(ii) *Every left (resp. right) ideal of S is a right (resp. left) ideal of S and semiprime.*

Lemma 4.10 [11] *An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for all bi-ideals A and B of S, we have*

$(A^2] = A$ and $(B^2] = B$.

Theorem 4.11 *An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for every $(\in, \in \vee q_k)$-fuzzy bi-ideal F of S, we have*

$\overline{F}^k(a) = \overline{F}^k(a^2)$ and $\overline{F}^k(ab) = \overline{F}^k(ba)$ for all $a, b \in S$.

Proof Suppose that F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal and by hypothesis, there exists a semilattice Y and a family $\{S_i : i \in Y\}$ of left and right simple subsemigroups of S such that:

$$S_i \cap S_j = \emptyset, \forall i,j \in Y, i \neq j, S = \bigcup_{i \in Y} S_i, S_iS_j \subseteq S_{ij} \forall i,j \in Y.$$

(i) To prove that $\overline{F}^k(a) = \overline{F}^k(a^2)$, for every $a \in S$, by Theorem 4.8 and Lemma 4.7, it is enough to prove that $a \in (a^2Sa^2]$. Let $a \in S$. Then, there exists Y such that $a \in S_i$. Since S_i is left and right simple, we have $(S_ia] = S_i$ and $(aS_i] = S_i$, and so $S_i = (aS_i] = (a(S_ia]] = (aS_ia]$. Since $a \in (aS_ia]$, there exists $x \in S_i$ such that $a \leq axa$. Since $x \in S_i$, we have $x \leq aya$ for some $y \in S_i$. Thus, $a \leq axa \leq a(aya)a = a^2ya^2 \in aS_ia \subseteq aSa$, and $a \in (a^2Sa^2]$.
(ii) Let $a, b \in S$. Then, by (i), we have

$$\overline{F}^k(ab) = \overline{F}^k((ab)^2) = \overline{F}^k((ab)^4).$$

Moreover, by Lemma 4.10, we have

$$\begin{aligned}
&(ab)^4 = (ab)^2(ab)^2 = (ab)(ab)(ab)(ab) \\
&= (aba)(babab) \in B(aba)B(babab) \\
&\subseteq (B(aba)B(babab)] \\
&= (B(babab)B(aba)] = (B(babab)(B(aba)^2]] \\
&\text{(by Lemma 4.10)} \\
&= (B(babab)(B(aba)B(aba)]] \\
&= (B(babab)B(aba)B(aba)] (\text{by } ((A]] = (A]) \\
&\subseteq (((babab)S(aba)(aba)]] \subseteq (bababSaba].
\end{aligned}$$

Then, $(ab)^4 \leq (babab)z(aba)$, for some $z \in S$. Since F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, we have

$$\begin{aligned}
&F((ab)^4) \geq F((babab)z(aba)) \wedge \frac{1-k}{2} \\
&= F((ba)(babza)(ba)) \wedge \frac{1-k}{2} \\
&\geq \left\{F(ba) \wedge F(ba) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2} \\
&= \left\{F(ba) \wedge \frac{1-k}{2}\right\}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\{F((ab)^4) \wedge \frac{1-k}{2}\right\} \geq \left\{F(ba) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2} \\
&= \left\{F(ba) \wedge \frac{1-k}{2}\right\},
\end{aligned}$$

and so $\overline{F}^k((ab)^4) \geq \overline{F}^k(ba)$, and $\overline{F}^k(ab) = \overline{F}^k((ab)^2) = \overline{F}^k((ab)^4) \geq \overline{F}^k(ba)$. In a similarly way one can see that, $\overline{F}^k(ba) \geq \overline{F}^k(ab)$.

Conversely, we know that $\mathcal{N}$ is a semilattice of left and right simple semigroups, so by Lemma 4.9, it is enough to prove that every left (resp. right) ideal of S is an ideal of S. Let L be a left ideal of S and let $a \in L$ and $t \in S$. Since L is a left ideal of S, by Lemma 4.4,

$$\overline{\chi_L}^k : S \longrightarrow [0,1] | ta \longmapsto \overline{\chi_L}^k(ta) = \begin{cases} \frac{1-k}{2} & \text{if } ta \in L \\ 0 & \text{if } ta \notin L \end{cases}$$

is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and by hypothesis, we have $\overline{\chi_L}^k(ta) = \overline{\chi_L}^k(at)$. Since $ta \in SL \subseteq L$, we have $\overline{\chi_L}^k(ta) = \frac{1-k}{2}$, then $\overline{\chi_L}^k(at) = \frac{1-k}{2}$ and so $at \in L$. That is, $LS \subseteq L$. Thus, L is a right ideal of S. If $a^2 \in L$, then $\overline{\chi_L}^k(a^2) = \frac{1-k}{2}$ and by hypothesis, $\overline{\chi_L}^k(a^2) = \overline{\chi_L}^k(a)$. Then, $\overline{\chi_L}^k(a) = \frac{1-k}{2}$ and so $a \in L$. Thus, L is semiprime. In a similar way, one can prove that a right ideal R is a left ideal of S and semiprime. □

In [6], regular and intra-regular ordered semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy bi-ideals. In the following, we characterize regular and intra-regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy bi-ideals.

Proposition 4.12 *If $\{F_i : i \in I\}$ is a family of $(\in, \in \vee q_k)$-fuzzy bi-ideals of an ordered semigroup S. Then $\bigcap_{i \in I} \overline{F}_i^k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Corollary 4.13 *If F and G are $(\in, \in \vee q_k)$-fuzzy bi-ideals of S. Then $(F \wedge^k G)^-$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Definition 4.14 An $(\in, \in \vee q_k)$-fuzzy bi-ideal F of S is called *idempotent* if $F_1 \wedge^k F_2 = F$.

Lemma 4.15 *For an $(\in, \in \vee q_k)$-fuzzy bi-ideal F of S, we have $(F \circ^k F)^- \preceq \overline{F}^k$.*

Proof Let F be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S and let $a \in S$. If $A_a = \emptyset$, then $(F \circ^k F)^-(a) = (F \circ F)(a) \wedge \frac{1-k}{2} = 0 \wedge \frac{1-k}{2} = 0 \leq \overline{F}^k(a)$.

Let $A_a \neq \emptyset$, then

$$\begin{aligned}(F \circ^k F)^-(a) &= \left[\bigvee_{(y,z) \in A_a} (F(y) \wedge F(z))\right] \wedge \frac{1-k}{2} \\ &\leq \left[\bigvee_{(y,z) \in A_a} F(yz)\right] \wedge \frac{1-k}{2} \leq \left[\bigvee_{(y,z) \in A_a} F(a)\right] \wedge \frac{1-k}{2} \\ &= F(a) \wedge \frac{1-k}{2} = \overline{F}^k(a).\end{aligned}$$ □

Lemma 4.16 *Every $(\in, \in \vee q_k)$-fuzzy one-sided ideal of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof It is obvious. □

For an ordered semigroup S, we define the fuzzy subsets "1" and "0" as follows:

$$1 : S \longrightarrow [0,1] | x \longrightarrow 1(x) = 1,$$
$$0 : S \longrightarrow [0,1] | x \longrightarrow 0(x) = 0,$$

for all $x \in S$.

Lemma 4.17 [10] *Let S be an ordered semigroup and F and G be fuzzy subsets of S. Then, $(F \circ^k G)^- \preceq (1 \circ^k G)^-$ (resp. $(F \circ^k G)^- \preceq (F \circ^k 1)^-$).*

Lemma 4.18 *Let S be an ordered semigroup and F an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Then $(F \circ^k 1 \circ^k F)^- \preceq \overline{F}^k$.*

Proof Let $a \in S$. If $A_a = \emptyset$, then

$$\begin{aligned}(F \circ^k 1 \circ^k F)^-(a) &= (F \circ 1 \circ F)(a) \wedge \frac{1-k}{2} = 0 \wedge \frac{1-k}{2} \\ &= 0 \leq \overline{F}^k(a).\end{aligned}$$

Let $A_a \neq \emptyset$, then

$$\begin{aligned}&(F \circ^k 1 \circ^k F)^-(a) \\ &= (F \circ 1 \circ F)(a) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} (F(y) \wedge (1 \circ F)(z)\right] \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} \left(F(y) \wedge \left\{\bigvee_{(p,q) \in A_z} (1(p) \wedge F(q))\right\}\right)\right] \wedge \frac{1-k}{2} \\ &= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_z} (F(y) \wedge 1(p) \wedge F(q)) \wedge \frac{1-k}{2} \\ &= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_z} (F(y) \wedge F(q)) \wedge \frac{1-k}{2} \\ &= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_z} \left(\left(F(y) \wedge \frac{1-k}{2}\right) \wedge \left(F(q) \wedge \frac{1-k}{2}\right)\right) \\ &\quad \wedge \frac{1-k}{2}.\end{aligned}$$

Since $a \leq yz \leq y(pq) = ypq$ and F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, so we have

$$\begin{aligned}F(a) &\geq F(ypq) \wedge \frac{1-k}{2} \geq \left(F(y) \wedge F(q) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} \\ &= \left\{\left(F(y) \wedge \frac{1-k}{2}\right) \wedge \left(F(q) \wedge \frac{1-k}{2}\right)\right\} \wedge \frac{1-k}{2}.\end{aligned}$$

Thus,

$$\bigvee_{(y,z)\in A_a}\bigvee_{(p,q)\in A_z}\left(\left(F(y)\wedge\frac{1-k}{2}\right)\wedge\left(F(q)\wedge\frac{1-k}{2}\right)\right)\wedge\frac{1-k}{2}$$
$$\leq\bigvee_{(y,pq)\in A_a}\left(\left(F(y)\wedge\frac{1-k}{2}\right)\wedge\left(F(q)\wedge\frac{1-k}{2}\right)\right)\wedge\frac{1-k}{2}$$
$$\leq\bigvee_{(y,pq)\in A_a}F(a)\wedge\frac{1-k}{2}=\overline{F}^k(a).$$ □

Lemma 4.19 [6] *Let S be an ordered semigroup. Then the following are equivalent:*

(i) *S is regular,*
(ii) $B=(BSB]$ *for all bi-ideals B of S,*
(iii) $B(a)=(B(a)SB(a)]$ *for every* $a\in S$.

Theorem 4.20 *An ordered semigroup S is regular if and only if for every* $(\in,\in\vee q_k)$*-fuzzy bi-ideal F of S, we have* $(F\circ^k 1\circ^k F)^-=\overline{F}^k$.

Proof Suppose that F is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S and let $a\in S$. Since S is regular, there exists $x\in S$ such that $a\leq axa\leq ax(axa)=a(xaxa)$. Then $(a,xaxa)\in A_a$, and $A_a\neq\emptyset$. Thus,

$$\begin{aligned}(F\circ^k 1\circ^k F)^-(a)&=(F\circ 1\circ F)(a)\wedge\frac{1-k}{2}\\&=\left[\bigvee_{(y,z)\in A_a}(F(y)\wedge(1\circ F)(z))\right]\wedge\frac{1-k}{2}\\&\geq\left(F(a)\wedge(1\circ F)(xaxa)\wedge\frac{1-k}{2}\right)\\&=\left[F(a)\wedge\bigvee_{(p,q)\in A_{xaxa}}\{1(p)\wedge F(q)\}\right]\wedge\frac{1-k}{2}\\&\geq(F(a)\wedge\{1(xax)\wedge F(a)\})\wedge\frac{1-k}{2}\\&=(F(a)\wedge\{1\wedge F(a)\})\wedge\frac{1-k}{2}\\&=(F(a)\wedge F(a))\wedge\frac{1-k}{2}=F(a)\wedge\frac{1-k}{2}\\&=\overline{F}^k(a).\end{aligned}$$

On the other hand, by Lemma 4.18, we have $(F\circ^k 1\circ^k F)^-(a)\leq\overline{F}^k(a)$. Therefore, $(F\circ^k 1\circ^k F)^-(a)=\overline{F}^k(a)$.

Conversely, assume that $(F\circ^k 1\circ^k F)^-=\overline{F}^k$ for every $(\in,\in\vee q_k)$-fuzzy bi-ideal F of S. To prove that S is regular, by Lemma 4.19, it is enough to prove that $B=(BSB]$ for all bi-ideals B of S.

Let $x\in B$. Since B is a bi-ideal of S, by Lemma 4.4, $\overline{\chi}_B$ is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S. By hypothesis, $(\chi_B\circ^k 1\circ^k\chi_B)^-(x)=\overline{\chi_B}^k(x)$. Since $x\in B$, we have $\overline{\chi_B}^k(x)=\frac{1-k}{2}$. Thus, $(\chi_B\circ^k 1\circ^k\chi_B)^-(x)=\frac{1-k}{2}$. But, by Lemma 4.3 (3), we have $(\chi_B\circ^k 1\circ^k\chi_B)^-=\overline{\chi}^k_{(BSB]}$, and $\overline{\chi}^k_{(BSB]}(x)=\frac{1-k}{2}$, hence we have $x\in(BSB]$ and so $B\subseteq(BSB]$. On the other hand, since B is a bi-ideal of S, we have $(BSB]\subseteq(B]=B$. □

Lemma 4.21 *Let F and G be* $(\in,\in\vee q_k)$*-fuzzy bi-ideals of S. Then* $(F\circ\wedge^k G)^-$ *is an* $(\in,\in\vee q_k)$*-fuzzy bi-ideal of S.*

Proof The proof is easy and so is omitted. □

Lemma 4.22 [6] *Let S be an ordered semigroup. Then the following are equivalent:*

(i) *S is both regular and intra-regular,*
(ii) $A=(A^2]$ *for every bi-ideals A of S,*
(iii) $A\cap B=(AB]\cap(BA]$ *for all bi-ideals A, B of S.*

Theorem 4.23 *Let S be an ordered semigroup. Then the following are equivalent:*

(i) *S is both regular and intra-regular,*
(ii) $(F\circ^k F)^-=\overline{F}^k$ *for every* $(\in,\in\vee q_k)$*-fuzzy bi-ideals F of S,*
(iii) $(F\wedge^k G)^-=\left((F\circ^k G)^-\wedge^k(G\circ^k F)^-\right)^-$ *for all* $(\in,\in\vee q_k)$*-fuzzy bi-ideals F and G of S.*

Proof (i)$\Longrightarrow$(ii) Let F be an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S and $a\in S$. Since S is regular and intra-regular, there exist $x,y,z\in S$ such that $a\leq axa\leq axaxa$ and $a\leq ya^2z$. Then, $a\leq axaxa\leq ax(ya^2z)xa=(axya)(azxa)$ and $(axya,azxa)\in A_a$. Thus,

$$\begin{aligned}(F\circ^k F)^-(a)&=(F\circ F)(a)\wedge\frac{1-k}{2}\\&=\left[\bigvee_{(y,z)\in A_a}(F(y)\wedge F(z))\right]\wedge\frac{1-k}{2}\\&\geq\{(F(axya)\wedge F(azxa))\}\wedge\frac{1-k}{2}\\&\geq\left\{\left(F(a)\wedge F(a)\wedge\frac{1-k}{2}\right)\right.\\&\left.\wedge\left(F(a)\wedge F(a)\wedge\frac{1-k}{2}\right)\right\}\wedge\frac{1-k}{2}\\&=\left(F(a)\wedge\frac{1-k}{2}\right)\wedge\frac{1-k}{2}\\&=\left(F(a)\wedge\frac{1-k}{2}\right)=\overline{F}^k(a).\end{aligned}$$

On the other hand, by Lemma 4.15, $(F\circ^k F)^-(a)\leq\overline{F}^k(a)$.

(ii)$\Longrightarrow$(iii) Let F and G be $(\in,\in\vee q_k)$-fuzzy bi-ideals of S. Then, by Corollary 4.13, $(F\wedge^k G)^-$ is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S. By (ii),

$$(F \wedge^k G)^- = \left((F \wedge^k G)^- \circ^k (F \wedge^k G)^-\right)^- \preceq (F \circ^k G)^-.$$

In a similar way, one can prove that $(F \wedge^k G)^- \preceq (G \circ^k F)^-$. Thus, $(F \wedge^k G)^- \preceq ((F \circ^k G)^- \wedge^k (G \circ^k F)^-)^-$. Moreover, $(F \circ^k G)^-$ and $(G \circ^k F)^-$ are $(\in, \in \vee q_k)$-fuzzy bi-ideals of S and hence, $(F \circ^k G)^- \wedge^k (G \circ^k F)^-$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Using (ii), we have

$$\begin{aligned}
&((F \circ^k G)^- \wedge^k (G \circ^k F)^-)^- \\
&= \left(((F \circ^k G)^- \wedge^k (G \circ^k F)^-) \circ^k ((F \circ^k G)^- \wedge^k (G \circ^k F)^-)\right)^- \\
&\preceq \left((F \circ^k G)^- \circ^k (G \circ^k F)^-\right)^- = (F \circ^k (G \circ^k G)^- \circ^k F)^- \\
&= (F \circ^k G \circ^k F)^- (\text{ as } (G \circ^k G)^- = \overline{G}^k \text{ by (i) above}) \\
&\preceq (F \circ^k 1 \circ^k F)^- = \overline{F}^k (\text{ as } (F \circ^k 1 \circ^k F)^- \\
&= \overline{F}^k \text{ by Theorem 4.20).}
\end{aligned}$$

In a similar way, one can prove that $((F \circ^k G)^- \wedge^k (G \circ^k F)^-)^- \preceq \overline{G}^k$. Consequently, $((F \circ^k G)^- \wedge^k (G \circ^k F)^-)^- \preceq \overline{F}^k \wedge \overline{G}^k = (F \wedge^k G)^-$. Therefore, we get $((F \wedge^k G)^- = ((F \circ^k G)^- \wedge^k (G \circ^k F)^-)^-$.

(iii)$\Longrightarrow$(i) To prove that S is both regular and intra-regular, by Lemma 4.22, it is enough to prove that $A \cap B = (AB] \cap (BA]$ for all bi-ideals A and B of S. Let $x \in A \cap B$. Then, $x \in A$ and $x \in B$. By Lemma 4.4, $\overline{\chi_A}^k$ and $\overline{\chi_B}^k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideals of S. Using (iii), we have

$$\begin{aligned}
&((\chi_A \circ^k \chi_B)^- \wedge^k (\chi_B \circ^k \chi_A)^-)^-(x) = (\chi_A \wedge^k \chi_B)^-(x) \\
&= \overline{\chi_A}^k(x) \wedge \overline{\chi_B}^k(x).
\end{aligned}$$

Since $x \in A$ and $x \in B$, we have $\overline{\chi_A}^k(x) = \frac{1-k}{2}$ and $\overline{\chi_B}^k(x) = \frac{1-k}{2}$. Thus, $\overline{\chi_A}^k(x) \wedge \overline{\chi_B}^k(x) = \frac{1-k}{2} \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $((\chi_A \circ^k \chi_B)^- \wedge^k (\chi_B \circ^k \chi_A)^-)^-(x) = \frac{1-k}{2}$. By Lemma 4.3, we have $((\chi_A \circ^k \chi_B)^- \wedge^k (\chi_B \circ^k \chi_A)^-)^- = \overline{\chi}^k_{(AB]} \wedge \overline{\chi}^k_{(BA]} = \overline{\chi}^k_{(AB] \cap (BA]}$. Thus, $\overline{\chi}^k_{(AB] \cap (BA]}(x) = \frac{1-k}{2}$ and $x \in (AB] \cap (BA]$. Moreover, if $x \in (AB] \cap (BA]$, then,

$$\begin{aligned}
\frac{1-k}{2} &= \overline{\chi}^k_{(AB] \cap (BA]}(x) \\
&= \left(\overline{\chi}^k_{(AB]} \wedge \overline{\chi}^k_{(BA]}\right)(x) \\
&= ((\chi_A \circ^k \chi_B)^- \wedge^k (\chi_B \circ^k \chi_A)^-)^-(x) \\
&= (\chi_A \wedge^k \chi_B)^-(x) \text{ (by (iii))} \\
&= \overline{\chi}^k_{A \cap B}(x).
\end{aligned}$$

Thus, $\overline{\chi}^k_{A \cap B}(x) = \frac{1-k}{2}$ and $x \in A \cap B$. Therefore, $A \cap B = (AB] \cap (BA]$, consequently, S is both regular and intra-regular. This completes the proof. □

5 Concluding remarks

Generalizing the concept of an (α, β)-fuzzy BCK/BCI-algebra, Jun [8], defined $(\in, \in \vee q_k)$-fuzzy subalgebras of a BCK/BCI-algebras, by generalizing the concept of $[x; t]qF$ and defined $[x; t]q_kF$, as $F(x) + t + k > 1$, where $k \in [0, 1)$. The theory of fuzzy sets on ordered semigroups can be developed. Since fuzzy ideals of ordered semigroups play an important role in the study of ordered semigroup structures, by using the idea of a more generalized from of quasi-coincidence of a fuzzy point with a fuzzy set, the concept of an $(\in, \in \vee q_k)$-fuzzy bi-ideal in an ordered semigroup S is introduced and different characterization theorems are provided. The idea of an upper/lower part of an $(\in, \in \vee q_k)$-fuzzy bi-ideal is provided and some interesting results are given by using the lower part of an $(\in, \in \vee q_k)$-fuzzy bi-ideal. In our future work, we will concentrate on (α, β)-fuzzy radical and (α, β)-fuzzy prime ideals of an ordered semigroup.

Acknowledgments We express our thanks to the referees for their valuable remarks and suggestions for the improvement of our paper.

References

1. Bhakat SK, Das P (1996) $(\in, \in \vee q)$-fuzzy subgroups. Fuzzy Sets Syst 80:359–368
2. Bhakat SK, Das P (1996) Fuzzy subrings and ideals redefined. Fuzzy Sets Syst 81:383–393
3. Davvaz B, Khan A (2011) Characterizations of regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals. Inform Sci 181:1759–1770
4. Davvaz B (2008) Fuzzy R-subgroups with thresholds of near-rings and implication operators. Soft Comput 12:875–879
5. Davvaz B (2006) $(\in, \in \vee q)$-fuzzy subnear-rings and ideals. Soft Comput 10:206–211
6. Jun YB, Khan A, Shabir M (2009) Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals. Bull Malays Math Sci Soc (2) 32(3):391–408
7. Jun YB (2005) On (α, β)-fuzzy subalgebras of BCK/BCI-algebras. Bull Korean Math Soc 42(4):703–711
8. Jun YB (2009) Generalizations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. Comput Math Appl 58:1383–1390
9. Jun YB, Dudek WA, Shabir M, Kang MS (2010) General form of (α, β)-fuzzy ideals of Hemirings. Honam Math J 32(3): 413–439
10. Khan A, Davvaz B, Sarmin NH, Khan FM Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals. Neural Comput Appl. doi:10.1007/s00521-011-0731-2
11. Kehayopulu N, Tsingelis M (2005) Fuzzy bi-ideals in ordered semigroups. Inf Sci 171:13–28
12. Khan A, Jun YB, Abbasi Z Characterizations of ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals. Neural Comput Appl. doi:10.1007/s00521-010-0463-8
13. Kuroki N (1981) On fuzzy ideals and fuzzy bi-ideals in semigroups. Fuzzy Sets Syst 5:203–215
14. Mordeson JN, Malik DS, Kuroki N (2003) Fuzzy semigroups. Studies in fuzziness and soft computing, vol 131. Springer, Berlin
15. Murali V (2004) Fuzzy points of equivalent fuzzy subsets. Inf Sci 158:277–288
16. Pu PM, Liu YM (1980) Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence. J Math Anal Appl 76:571–599
17. Rosenfeld A (1971) Fuzzy groups. J Math Anal Appl 35:512–517

18. Shabir M, Jun YB, Nawaz Y (2010) Characterization of regular semigroups by (α, β)-fuzzy ideals. Comput Math Appl 59: 161–175
19. Shabir M, Jun YB, Nawaz Y (2010) Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals. Comput Math Appl 60:1473–1493
20. Shabir M, Mahmood T (2011) Characterizations of hemirings by $(\in, \in \vee q_k)$-fuzzy ideals. Comput Math Appl 61:1059–1078
21. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353