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On the nonabelian tensor square and capability of groups of order p^2q

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Abstract. A group G is said to be capable if it is isomorphic to the central factor group $H/Z(H)$ for some group H. Let G be a nonabelian group of order p^2q for distinct primes p and q. In this paper, we compute the nonabelian tensor square of the group G. It is also shown that G is capable if and only if either $Z(G) = 1$ or $p < q$ and $G^{\mathrm{ab}} = \mathbb{Z}_p \times \mathbb{Z}_p$.

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1. Introduction. The nonabelian tensor square $G \otimes G$ of the group G is a group generated by the symbols $g \otimes h$

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h), \quad g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$$

for all $g, g', h, h' \in G$, where ${}^g g' = gg'g^{-1}$. The nonabelian tensor square is a special case of the nonabelian tensor product, which has its origin in homotopy theory and was introduced by Brown and Loday [6,7]. The exterior square $G \wedge G$ is obtained by imposing the additional relations $g \otimes g = 1_\otimes$ for all $g \in G$ on $G \otimes G$, with $1_\otimes$ being the identity of $G \otimes G$. The commutator map induces homomorphisms $\kappa : G \otimes G \to G$ and $\kappa' : G \wedge G \to G$, sending $g \otimes h$ and $g \wedge h$, respectively, to $[g, h] = ghg^{-1}h^{-1}$and $J_2(G)$ denotes the kernel of κ. The results in [6,7] give the commutative diagram given as in Figure 1 with exact rows and central extensions as columns, where G' is the commutator subgroup of G, $M(G)$ is the multiplicator of G and Γ is Whitehead's quadratic function [23].

In 1987, Brown et al. [5] computed the nonabelian tensor square of some groups such as dihedral, quaternionic, symmetric and all groups of order at most 30. The determination of $G \otimes G$ for groups of orders p^2q, pq^2, pqr and

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$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\Gamma(G^{\mathrm{ab}}) \arrow[r] & J_2(G) \arrow[r] \arrow[d] & M(G) \arrow[r] \arrow[d] & 0 \\
\Gamma(G^{\mathrm{ab}}) \arrow[r] & G \otimes G \arrow[r] \arrow[d, "\kappa"'] & G \wedge G \arrow[r] \arrow[d, "\kappa'"] & 1 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\
 & 1 & 1 &
\end{tikzcd}$$

Figure 1. The commutative diagram

p^2qr, where p, q, r are distinct primes and $p < q < r$ was mentioned by Jafari et al. [13].

Hall [12], in his paper on the classification of prime-power groups, remarked:

> The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group $H, G \cong H/Z(H)$, is an interesting one. But while it is easy to write down a number of necessary conditions, it is not so easy to be sure that they are sufficient.

Later, Hall and Senior [11] called a central factor group as a capable group.

A group G is capable if there exists a group H such that $G \cong H/Z(H)$. Ellis (see [10, Proposition 16]) proved that a group G is capable if and only if its exterior center $Z^{\wedge}(G)$ is trivial, where

$$Z^{\wedge}(G) = \{g \in G | g \wedge x = 1_{\wedge} \quad \text{for all } x \in G\}.$$

Here, $1_{\wedge}$ denotes the identity in $G \wedge G$.

In 1979, Beyl et al. (see [3, Corollary 2.3]) established a necessary and sufficient condition for a group to be capable, that is, a group is capable if and only if the epicenter $Z^*(G)$ of the group is trivial, where

$$Z^*(G) = \bigcap \{\phi Z(E) | (E, \phi) \text{ is a central extension of } G\}.$$

For the case of a group G of order pq, it can easily be shown that $G \otimes G \cong \mathbb{Z}_{pq}$ and G is capable if and only if $G = < a, b | a^p = b^q = 1, ba = a^s b >$, where $p \nmid s - 1$ and $p \mid s^q - 1$. For a group of order p^2,

$$G \otimes G \cong \begin{cases} \mathbb{Z}_{p^2} & ; G \cong \mathbb{Z}_{p^2}, \\ (\mathbb{Z}_p)^4 & ; G \cong \mathbb{Z}_p \times \mathbb{Z}_p, \end{cases}$$

and G is capable if and only if $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. For p-groups capability is closely related to their classification. Baer [2] characterized the capable groups which are direct sums of cyclic groups; the capable extra-special p-groups were characterized by Beyl et al. [3] (only the dihedral group of order 8 and the extra-special groups of order p^3 and exponent p are capable); they also described the metacyclic groups which are capable. Magidin [15,16], characterized the 2-generated capable p-groups of class two (for odd p, independently obtained in part by Bacon and Kappe [1]).

In this paper we focus on the multiplicator, nonabelian tensor square and capability of nonabelian groups of orders p^2q, where p, q are distinct primes.

In the next theorem, the nonabelian tensor square for groups of order p^2q is stated:

Theorem 1.1. *Let G be a nonabelian group of order p^2q, where p, q are distinct primes. Then exactly one of the following holds:*

$$G \otimes G \cong \begin{cases} \mathbb{Z}_{pq} \times \mathbb{Z}_p & ;\, G' \cong \mathbb{Z}_p, \\ \mathbb{Z}_q \times T & ;\, G' \cong \mathbb{Z}_p \times \mathbb{Z}_p, \\ \mathbb{Z}_{p^2q} & ;\, G^{\mathrm{ab}} \cong \mathbb{Z}_{p^2} \text{ or } G' \cong \mathbb{Z}_{p^2}, \\ \mathbb{Z}_q \times (\mathbb{Z}_p)^4 & ;\, G^{\mathrm{ab}} \cong \mathbb{Z}_p \times \mathbb{Z}_p, \end{cases}$$

where $T \cong (\mathbb{Z}_p)^3$, nonabelian group of order p^3 or Q_2, quaternion group of order 8.

In the following theorem, the capability of groups of order p^2q can be determined with given conditions:

Theorem 1.2. *Let G be a nonabelian group of order p^2q for distinct primes p and q. Then G is capable if and only if either*

(i) $Z(G) = 1$, *or*

(ii) $p < q$ *and* $G^{\mathrm{ab}} = \mathbb{Z}_p \times \mathbb{Z}_p$.

2. Preliminaries. This section includes some results on the commutator subgroup, multiplicator, nonabelian tensor square and capability which play an important rule for proving our main theorems.

Theorem 2.1. (see [20, Proposition 3.9]) *Let G be a group of order p^2q where p and q are distinct primes. Then exactly one of the following holds:*

1. *If $p > q$, then G has a normal Sylow p-subgroup.*
2. *If $q > p$, then G has a normal Sylow q-subgroup.*
3. *If $p = 2, q = 3$, then $G \cong A_4$ and G has a normal Sylow 2-subgroup.*

By this theorem, the commutator subgroup of nonabelian groups of order p^2q is isomorphic to $\mathbb{Z}_p, \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_q$.

Schur and Zassenhaus (see [21, Theorem 9.1.2]) stated conditions for the existence of any complement for a group G as follows:
Let N be normal subgroup of G. Assume that $|N| = n$ and $[G : N] = m$ are relatively prime. Then G contains subgroups of order m and any two of them are conjugate in G.
This theorem asserts that the complement of G exists.

Let N be a normal subgroup of G. We say that N is a normal Hall subgroup of G if the order of N is coprime with its index in G. If N is a normal Hall subgroup of G, then G is a semidirect product of N and a subgroup T of G. Any such subgroup T is referred as a complement of N in G. In this case, we can obtain $M(G)$ by the next theorem:

Theorem 2.2. (see [14, Corollary 2.2.6]) *Let N be a normal Hall subgroup of G and T be a complement of N in G. Then $M(G) \cong M(T) \times M(N)^T$.*

By the use of Theorem 2.1, the multiplicator for groups of order p^2q is computed in the next lemma.

Lemma 2.3. *Let G be a group of order p^2q where p and q are distinct primes. Then $M(G) = 1$ or $\mathbb{Z}_p$.*

Proof. Suppose P be a normal Sylow p-subgroup of G. Since $(|P|, |G/P|) = 1$, thus P is a normal Hall subgroup of G. Then $G \cong P \rtimes T$, where T is a subgroup of G of order q and $P \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.
Therefore, by Theorem 2.2,

$$M(G) = M(T) \times M(P)^T = M(P)^T = 1 \text{ or } \mathbb{Z}_p.$$

If Q is a normal Sylow q-subgroup of G, then Q is a normal Hall subgroup of G. Therefore, $G \cong Q \rtimes T$, where T is a subgroup of G of order p^2. Therefore,

$$M(G) = M(T) \times M(Q)^T = M(T) = \begin{cases} 1 & ; T = \mathbb{Z}_{p^2}, \\ \mathbb{Z}_p & ; T = \mathbb{Z}_p \times \mathbb{Z}_p. \end{cases}$$

If $G \cong A_4$, then $M(G)$ has been computed in (see [5, Table 1]). □

The following five theorems will be used to compute the nonabelian tensor square of groups of order p^2q.

Theorem 2.4. (see [5, Proposition 8]) *Let G be a group in which G' has a cyclic complement C. Then $G \otimes G \cong (G \wedge G) \times G^{\mathrm{ab}}$ and $|G \otimes G| = |G||M(G)|$.*

For a solvable group of derived length 2, the following theorem can be used:

Theorem 2.5. (see [17, Theorem 3.3]) *Let G be a finite solvable group of derived length 2. Then $|G \otimes G|$ divides $|G^{\mathrm{ab}} \otimes_{\mathbb{Z}} G^{\mathrm{ab}}||G' \wedge G'||G' \otimes_{\mathbb{Z}[G^{\mathrm{ab}}]} I(G^{\mathrm{ab}})|$, where $I(G^{ab})$ is the kernel of $\mathbb{Z}[G^{\mathrm{ab}}] \to \mathbb{Z}$.*

Theorem 2.6. (see [18, Theorem C]) *If G is a finite group such that the derived subgroup G' is cyclic and $(|G'|, |G/G'|) = 1$, then $G \otimes G \cong G' \times (G^{\mathrm{ab}} \otimes_{\mathbb{Z}} G^{\mathrm{ab}})$.*

Theorem 2.7. (see [4, Proposition 2.2]) *Let G be a group such that G^{ab} is finitely generated. If G^{ab} has no element of order two or if G' has a complement in G then $G \otimes G \cong \Gamma(G^{\mathrm{ab}}) \times G \wedge G$.*

Theorem 2.8. (see [13, Theorem 2.2]) *Let G be a group such that $G^{ab} = \prod_{i=1}^{n} \prod_{j=1}^{k_i} \mathbb{Z}_{{p_i}^{e_{ij}}}$ where $1 \leq e_{i1} \leq e_{i2} \leq \cdots \leq e_{ik_i}$ for all $1 \leq i \leq n, k_i \in \mathbb{N}$ and $p_i \neq 2$. Then*

$$|G \otimes G| = \prod_{i=1}^{n} {p_i}^{d_i} |G||M(G)|$$

in which $d_i = \sum_{j=1}^{k_i} (k_i - j) e_{ij}$.

The last part of this section includes two theorems that will be used in determining the capability of groups of order p^2q.

Theorem 2.9. (see [9, Proposition 1]) *Let G be a finitely generated capable group. Then every central element z in G has order dividing* $\exp\left((G/{<}z{>})^{\mathrm{ab}}\right)$.

Theorem 2.10. (see [19, Corollary 3.6]) *Assume that G is a group with trivial Schur multiplier and finite $d(G/Z^\wedge(G))$. Then $C_G(x) = C_G^\wedge(x)$ for every element x of G. In particular, such a group has $Z(G) = Z^\wedge(G)$.*

3. The Proof of Main Theorems. In this section we prove our main theorems as mentioned in Section 1, namely Theorems 1.1 and 1.2.

3.1. Proof of Theorem 1.1.

Proof. The proof of this theorem is based on the commutator subgroup of groups of order p^2q as mentioned in Section 2.

If $G' \cong \mathbb{Z}_p$ then by Theorem 2.5, it is clear that $|G \otimes G|$ divides p^2q, where $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \cong \mathbb{Z}_{pq}, G' \wedge G' = 1$ and $G' \otimes_{\mathbb{Z}[G^{\mathrm{ab}}]} I(G^{\mathrm{ab}}) \cong \mathbb{Z}_p$. On the other hand, by Theorem 2.8 $|G||M(G)|$ divides $|G \otimes G|$, that is, $M(G) = 1$. Figure 1 shows that $G \wedge G \cong G' \cong \mathbb{Z}_p$. Since G^{ab} is finitely generated group that has no element of order two, then by Theorem 2.7

$$G \otimes G \cong \Gamma(G^{\mathrm{ab}}) \times G \wedge G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p.$$

If $G' \cong \mathbb{Z}_{p^2}$, then $(|G'|, |G^{\mathrm{ab}}|) = 1$ and G' is cyclic. Then by Theorem 2.6 it is clear that

$$G \otimes G \cong G' \times (G^{\mathrm{ab}} \otimes_{\mathbb{Z}} G^{\mathrm{ab}}) \cong \mathbb{Z}_{p^2} \times (\mathbb{Z}_q \otimes \mathbb{Z}_q) \cong \mathbb{Z}_{p^2q}.$$

If $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $(|G'|, |G^{\mathrm{ab}}|) = 1$ and $G^{\mathrm{ab}} \cong \mathbb{Z}_q$ is cyclic, so by Schur–Zassenhaus Theorem, G' has a cyclic complement. Therefore, by Theorems 2.2 and 2.4 we have the following computations:

$$M(G) = M(\mathbb{Z}_q) \times M\left((\mathbb{Z}_p \times \mathbb{Z}_p)\right)^{\mathbb{Z}_p} \cong \mathbb{Z}_p,$$

$(G \wedge G)/M(G) \cong G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is abelian and $|G \wedge G| = p^3$, thus $(G \wedge G)' \leq M(G)$, that is, $(G \wedge G)' = 1$ or $\mathbb{Z}_p$. Thus we have the following cases:
Case 1: If $(G \wedge G)' = 1$, then $G \wedge G \cong (\mathbb{Z}_p)^3$.
Case 2: If $(G \wedge G)' \cong \mathbb{Z}_p$, then $G \wedge G \cong < a, b | a^p = b^p = 1, [a, b]^a = [a, b] = [a, b]^b >$ or $< a, b | a^{p^2} = b^p = 1, a^b = a^{p+1} >$.
Finally, by the relation $G \otimes G \cong G \wedge G \times G^{\mathrm{ab}}$, the result follows.

If $G' \cong \mathbb{Z}_q$, then $(|G'|, |G^{\mathrm{ab}}|) = 1$. We have the following cases:
Case 1: If $G^{\mathrm{ab}} \cong \mathbb{Z}_{p^2}$, then by Theorem 2.6

$$G \otimes G \cong \mathbb{Z}_q \times (\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}} \mathbb{Z}_{p^2}) \cong \mathbb{Z}_{p^2q}.$$

Case 2: If $G^{\mathrm{ab}} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then

$$G \otimes G \cong \mathbb{Z}_q \times ((\mathbb{Z}_p \times \mathbb{Z}_p)) \otimes ((\mathbb{Z}_p \times \mathbb{Z}_p)) \cong \mathbb{Z}_q \times (\mathbb{Z}_p)^4.$$

The proof for the case that $G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ can be found in (see [5, Table 1]). □

3.2. Proof of Theorem 1.2.

Proof. Let G be nonabelian group of order p^2q, where p and q are distinct primes:

(i) If $Z(G) = 1$, it is clear that G is capable.
If $p > q$ and G is capable, then for the center of G we have the following cases:
Case 1: $|Z(G)| = p^2$ or pq.
In these cases $G/Z(G)$ is cyclic. Therefore, G is abelian which is a contradiction.
Case 2: $|Z(G)| = p$.
In this case $G/Z(G) = K$, where K is nonabelian of order pq. By Theorem 2.9,

$$|z|\big|\exp((G/ < z >)^{\mathrm{ab}}); \quad \text{for all } z \in Z(G), \tag{3.1}$$

that is, $p|q$, while $(p, q) = 1$.
Case 3: $|Z(G)| = q$.
By Theorem 2.9, $q|p$ or p^2, which is a contradiction.
Thus the only case left is $Z(G) = 1$.

(ii) If $p < q$ and G is capable, then by a similar way we can show that $|Z(G)|$ cannot be p^2, pq or q.
If $|Z(G)| = p$ and $G^{\mathrm{ab}} = \mathbb{Z}_{p^2}$, then by Theorem 2.10, $Z(G) = Z^\wedge(G)$, that is, $Z^\wedge(G) \neq 1$. Therefore, G is not capable.
Thus the only case left is $G^{\mathrm{ab}} = \mathbb{Z}_p \times \mathbb{Z}_p$.

Now, let $G^{\mathrm{ab}} = \mathbb{Z}_p \times \mathbb{Z}_p$. Since G is a nonabelian group of order p^2q and $p < q$, then G isomorphic to exactly one group in the following list (see [8, Section 59]):

1. $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2} =< a, b|a^q = b^{p^2} = 1, bab^{-1} = a^i >$, where $i^p \equiv 1 \pmod q$ and $p \mid q-1$.
2. $< a, b, c|a^q = b^p = c^p = 1, bab^{-1} = a^i, ac = ca, bc = cb >$, where $i^p \equiv 1 \pmod q$ and $p \mid q-1$.
3. $\mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2} =< a, b|a^q = b^{p^2} = 1, bab^{-1} = a^i, i^{p^2} \equiv 1 \pmod q >$, where $p^2 \mid q-1$.

For groups of types 1 and 3, $G^{\mathrm{ab}} = \mathbb{Z}_{p^2}$ and for type 2, $G^{\mathrm{ab}} = \mathbb{Z}_p \times \mathbb{Z}_p$. Our assumption is $G^{\mathrm{ab}} = \mathbb{Z}_p \times \mathbb{Z}_p$, then G is a group of type 2. In the classification for groups of order p^3q (see [22, Section 31]) , we choose $H =< a, b, c, d|a^p = b^p = c^p = d^q = 1, ab = ba, ac = ca, ad = da, bd = db, c^{-1}bc = ab, c^{-1}dc = d^i >$, where $q \equiv 1 \pmod p$ and $i^p \equiv 1 \pmod q$. Thus $Z(H) = \mathbb{Z}_p, |H/Z(H)| = p^2q$ and $(H/Z(H))^{\mathrm{ab}} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. These computations immediately show that $G \cong H/Z(H)$, that is, G is capable. □

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