World Applied Sciences Journal 16 (12): 1688-1698, 2012
ISSN 1818-4952

Semiprime $(\in,\in\vee q_k)$-fuzzy Quasi-ideals in Ordered Semigroups

[1]Asghar Khan, [2]Nor Haniza Sarmin, [2]Faiz Muhammad Khan and [3]Faizullah

[1]Department of Mathematics, COMSATS Institute of IT Abbottabad, Pakistan
[2]Department of Mathematics, Faculty of Science Universiti Teknologi, Malaysia 81310 UTM, Johor Bahru, Johor, Malaysia
[3]College of Physical and Environmental Oceanography, Ocean University of China, Qingdao 266100, PR China

Abstract: In this paper, the concept of an $(\in,\in\vee q_k)$-fuzzy quasi-ideal in an ordered semigroup is introduced, which is a generalization of the concept of an (α,β)-fuzzy quasi-ideal, $\alpha,\beta\in\{\in,q,\in\vee q,\in\wedge q\}$ and (α,β)-fuzzy quasi-ideal is a generalization of fuzzy quasi-ideals of an ordered semigroup. Using this concept some basic results of ordered semigroups are discussed and the supported examples are provided. The upper/lower parts of an $(\in,\in\vee q_k)$-fuzzy quasi-ideal are defined and some classes of ordered semigroups are characterized by using the concept of an $(\in,\in\vee q_k)$-fuzzy quasi-ideal. The concept of a semiprime $(\in,\in\vee q_k)$-fuzzy quasi-ideal is given and the basic results are investigated in terms of semiprime $(\in,\in\vee q_k)$-fuzzy quasi-ideal.

2000 mathematics subject classification: 06F05 . 20M12 . 08A72

Keywords: Regular . left (resp. right) regular and completely regular ordered semigroups . quasi-ideals . fuzzy quasi-ideals . $(\in,\in\vee q_k)$-fuzzy quasi-ideals

INTRODUCTION

A new type of fuzzy subgroup, that is, the (α,β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1, 2] by using the combined notions of "belongingness" and "quasi-coincidence" of a fuzzy point and a fuzzy set. In particular, the concept of an $(\in,\in\vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroup [19]. Davvaz [4] introduced the concept of $(\in,\in\vee q)$-fuzzy sub-near-ring (R-subgroups, ideals) of a near-ring and investigated some of their properties. Jun and Song [6] discussed the general forms of fuzzy interior ideals in semigroups. Davvaz and Khan [3] discussed some characterizations of regular ordered semigroups in terms of (α,β)-fuzzy bi-ideals, where $\alpha,\beta\in\{\in,q,\in,\in\vee q,\in\vee q\}$ and $\alpha\neq\in\wedge q$. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [8] and gave some properties of fuzzy bi-ideals in terms of $(\in,\in\vee q)$-fuzzy bi-ideals. In [21], Shabir *et al*. gave the concept of more generalized forms of (α,β)-fuzzy ideals in semigroups and defined $(\in,\in\vee q_k)$-fuzzy ideals of semigroups, by generalizing the concept of x_tqF and defined x_tq_kF, as $F(x)+t+k>1$, where $k\in[0,1)$.

The topic of these investigations belongs to the theoretical soft computing (fuzzy structure). Indeed, it is well known that semigroups are basic structures in many applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings [16].

This paper is arranged in the following sections:

In section 2, the basic definitions of quasi-ideals and fuzzy quasi-ideals are given and the necessary results are revised. In section 3, the concept of an $(\in,\in\vee q_k)$-fuzzy quasi-ideal is provided and the basic results of ordered semigroups with supporting examples and counter-example are given. In section 4, we define the upper and lower parts of an $(\in,\in\vee q_k)$-fuzzy quasi-ideal and the properties of some classes of ordered semigroups are presented. We also define semiprime $(\in,\in\vee q_k)$-fuzzy quasi-ideals in this section and give some properties of completely regular ordered semigroups. In the last section, some semilattices of left and right simple semigroups are discussed and hence, it is shown that an ordered semigroup S is completely regular if and only if for every $(\in,\in\vee q_k)$-

Corresponding Author: Asghar Khan, Department of Mathematics, COMSATS Institute of IT Abbottabad, Pakistan

fuzzy quasi-ideal F of S, we have $\overline{F}^k(a)=\overline{F}^k(a^2)$ and $\overline{F}^k(ab)=\overline{F}^k(ba)$ for every $a,b\in S$ where $\overline{F}^k$ denotes the lower part of F and $k\in[0,1)$.

BASIC DEFINITIONS AND PRELIMINARIES

By an ordered semigroup (or po-semigroup) we mean a structure $(S,\cdot,\leq)$ in which the following conditions are satisfied:

(OS1) $(S,\cdot)$ is a semigroup,
(OS2) $(S,\leq)$ is a poset,
(OS3) $a\leq b\rightarrow ax\leq bx$ and $a\leq b\rightarrow xa\leq xb$ for all $a,b,x\in S$.

For subsets A,B of an ordered semigroup S, we denote by $AB=\{ab\in S\mid a\in A, b\in B\}$. If $A\subseteq S$ we denote $(A]=\{t\in S\mid t\leq h$ for some $h\in A\}$. If $A=\{a\}$, then we write $(a]$ instead of $(\{a\}]$. If $A,B\subseteq S$, then $A\subseteq(A]$, $(A](B]\subseteq(AB]$ and $((A]]=(A]$.

Let $(S,\cdot,\leq)$ be an ordered semigroup. A non-empty subset A of S is called a subsemigroup of S if $A^2\subseteq A$. A non-empty subset A of S is called left (resp. right) ideal of S if

(i) $(\forall a\in S)(\forall b\in A)\ (a\leq b\rightarrow a\in A)$
(ii) $SA\subseteq A$ (resp. $AS\subseteq A$)

A non-empty subset A of S is called an ideal if it is both a left and a right ideal of S.

A non-empty subset A of an ordered semigroup S is called a bi-ideal of S if

(i) $(\forall a\in S)(\forall b\in A)\ (a\leq b\rightarrow a\in A)$
(ii) $A^2\subseteq A$
(iii) $ASA\subseteq A$

A non-empty subset A of an ordered semigroup S is called a quasi-ideal of S if

(i) $(\forall a\in S)(\forall b\in A)\ (a\leq b\rightarrow a\in A)$
(ii) $(AS]\cap(SA]\subseteq A$

An ordered semigroup S is regular if for every $a\in S$ there exists, $x\in S$ such that $a\leq axa$, or equivalently, we have (i) $a\in(aSa]\ \forall a\in S$ and (ii) $A\subseteq(ASA]\ \forall A\subseteq S$. An ordered semigroup S is called left (resp. right) regular if for every $a\in S$ there exists $x\in S$, such that $a\leq xa^2$ (resp. $a\leq a^2x$), or equivalently, (i) $a\in(Sa^2]$(resp. $a\in(S^2a]$) $\forall a\in S$ and (ii) $A\subseteq(SA^2]$ resp. $A\subseteq(A^2S]$) $\forall a\subseteq S$. An ordered semigroup S is called left (resp. right) simple if for every left (resp. right) ideal A of S we have A = S and S is called simple if it is both left and right simple. An ordered semigroup S is called completely regular, if it is left regular, right regular and regular.

An equivalence relation σ on S is called congruence if $(a,b)\in\sigma$ implies $(ac,bc)\in\sigma$ and $(ca,cb)\in\sigma$ for every $c\in S$. A congruence σ on S is called semilattice congruence [9] if $(a,a^2)\in\sigma$ and $(ab,ba)\in\sigma$. An ordered semigroup S is called a semilattice of left and right simple semigroups if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left and right simple subsemigroup of S for every $x\in S$, or equivalently, there exists a semilattice Y and a family $\{S_i: i\in y\}$ of left and right simple subsemigroups of S such that

$$S_i\cap S_j=?,\forall i,j\in Y, i\neq j$$
$$S=\bigcup_{i\in Y}S_i$$
$$S_iS_j\subseteq S_{ij}\ \forall i,j\in Y$$

A subset T of S is called semiprime [9], if for every $a\in S$ such that $a^2\in T$, we have $a\in T$, or equivalently, for each subset A of S, such that $A^2\subseteq T$, we have $A\subseteq T$.

Lemma [9]: For an ordered semigroup S, the following are equivalent:

(i) $(x)_N$ is a left (resp. right) simple subsemigroup of S, for every $x\in S$,
(ii) Every left (resp. right) ideal of S is a right (resp. left) ideal of S and semiprime.

Lemma [20]: An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for all quasi-ideals A and B of S, we have $(A^2]=A$ and $(B^2]=B$.

Now, we give some fuzzy logic concepts.
A function $F: S\rightarrow[0,1]$ is called a fuzzy subset of S.

The study of fuzzification of algebraic structures has been started in the pioneering paper of Rosenfeld [19] in 1971. Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups in the theory of fuzzy groups. Kuroki [11] studied fuzzy ideals, fuzzy bi-ideals and semiprime fuzzy ideals in semigroups [12, 13].

If F_1 and F_2 are fuzzy subsets of S then F_1 ? F_2 means $F_1(x)\leq F_2(x)$ for all $x\in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets:

$$F_1 \wedge F_2 : S \to [0,1] | x \mapsto (F_1 \wedge F_2)(x) = F_1(x) \wedge F_2(x) = \min\{F_1(x), F_2(x)\}$$

$$F_1 \vee F_2 : S \to [0,1] | x \mapsto (F_1 \vee F_2)(x) = F_1(x) \vee F_2(x) = \max\{F_1(x), F_2(x)\}$$

for all $x \in S$.

A fuzzy subset F of S is called a fuzzy subsemigroup if $F(xy) \geq \min\{F(x), F(y)\}$ for all $x,y \in S$.

A fuzzy subset F of S is called a fuzzy left (resp. right) ideal [9] of S if

(i) $x \leq y \to F(x) \geq F(y)$

(ii) $F(xy) \geq F(y)$ resp. $F(xy) \geq F(x)$ for all $x,y \in S$.

A fuzzy subset F of S is called a fuzzy ideal if it is both a fuzzy left and a fuzzy right ideal of S.

A fuzzy subsemigroup F is called a fuzzy bi-ideal [9] of S if

(i) $x \leq y \to F(x) \geq F(y)$

(ii) $F(xyz) \geq \min\{F(x) \geq F(z)\}$ for all $x,y,z \in S$.

The fuzzy subsets "1" and "0" of an ordered semigroup S are the greatest and least elements of S and are defined as follows:

$$1 : S \to [0,1], \quad x \mapsto 1(x) = 1 \ \forall x \in S$$
$$0 : S \to [0,1], \quad x \mapsto 0(x) = 0 \ \forall x \in S$$

If $a \in S$ and A is a non-empty subset of S. Then, $A_a = \{(y,z) \in S \times S \mid a \leq yz\}$.

If F_1 and F_2 are two fuzzy subsets of S. Then the product $F_1 \circ F_2$ of F_1 and F_2 is defined by:

$$F_1 \circ F_2 : S \to [0,1] | a \mapsto (F_1 \circ F_2)(a) = \begin{cases} \bigvee_{(y,z) \in A_a} (F_1(y) \wedge F_2(z)) & \text{if } A_a \neq ? \\ 0 & \text{if } A_a = ? \end{cases}$$

A fuzzy subset F of S is called a fuzzy quasi-ideal of S [20] if it satisfies the following conditions:

(i) $x \leq y \to F(x) \geq F(y)$

(ii) $(F \circ 1) \wedge (1 \circ F) ? F$

Theorem [20]: A non-empty subset A of an ordered semigroup S is a quasi-ideal of S if and only if

$$\chi_A : S \to [0,1] | x \mapsto \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is a fuzzy quasi-ideal of S.

Let S be an ordered semigroup and F is a fuzzy subset of S. Then, for all $t \in (0,1]$, the set $U(F;t) = \{x \in S \mid F(x) \geq t\}$ is called a level set of F.

Theorem [20]: A fuzzy subset F of an ordered semigroup S is a fuzzy bi-ideal (resp. fuzzy quasi-ideal) of S if and only if $U(F;t)(\neq ?)$ is a bi-ideal (resp. quasi-ideal) of S, for all $t \in (0,1]$.

Example [Shabir-Khan]: Consider the ordered semigroup S = {a,b,c,d,*f*} with the following multiplication table and the order relation:

·	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

$$\leq := \{(a,a),(a,b),(a,c),(a,d)(a,f),(b,b),(c,c),(d,d),(f,f)\}$$

Then {a}, {a,b}, {a,c}, {a,d}, {a,*f*}, {a,b,d}, {a,e,d}, {a,b,*f*}, {a,c,*f*} and {a,b,c,d,*f*} are quasi-ideals of S. Define a fuzzy subset F of S as follows:

$$F : S \to [0,1] | x \mapsto F(x) = \begin{cases} 0.5 \text{ if } x = a \\ 0.4 \text{ if } x = b \\ 0.3 \text{ if } x = d \\ 0.2 \text{ if } x = c,f \end{cases}$$

Then, F is a fuzzy quasi-ideal of S as shown in [20].

It is well known that every quasi-ideal of S is a bi-ideal of S. Similarly every fuzzy quasi-ideal of S is a fuzzy bi-ideal of S but the converse is not true in general, as shown in [20].

Lemma [9]: Let A and B be non-empty subsets of S, then we have the following:

(1) $A \subseteq B$ if and only if $\chi_A ? \chi_B$,

(2) $\chi_A \wedge \chi_B = \chi_{A \cap B}$,

(3) $\chi_A \circ \chi_B = \chi_{(AB]}$.

Let F be a fuzzy subset of S, then the set of the form:

$$F(y) := \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is called a fuzzy point with support x and value t and is denoted by [x;t]. A fuzzy point [x;t] is said to belong to (resp. quasi-coincident with) a fuzzy set F, written as $[x;t]\in F$ (resp. $[x;t]qF$) if F(x)≥t (resp. F(x)+t>1). If $[x;t]\in F$ or $[x;t]qF$, then we write $[x;t]\in\vee qF$. The symbol $\overline{\in\vee q}$ means $\in\vee q_k$ does not hold.

Generalizing the concept of $[x;t]qF$, in semigroups, Shabir *et al*. [21] defined $[x;t]q_kF$, as F(x)+t+k >1, where $k\in[0,1)$.

$(\in,\in\vee q_k)$-FUZZY QUASI-IDEALS

In what follows, let S denote an ordered semigroup unless otherwise specified. In this section, we define a more generalized form of (α,β)-fuzzy quasi-ideals of an ordered semigroups S and introduced $(\in,\in\vee q_k)$-fuzzy quasi-ideals S where k is an arbitrary element of [0,1) unless otherwise stated.

Definition [7]: A fuzzy subset F of S is called an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S if it satisfies the conditions:

(1) $(\forall x,y\in S)(\forall t\in(0,1])(x\leq y,[y;t]\in F\rightarrow[x;t]\in\vee q_kF)$

(2) $(\forall x,y\in S)\left(F(xy)\geq F(x)\wedge F(y)\wedge\frac{1-k}{2}\right)$

(3) $(\forall x,y,z\in S)\left(F(xyz)\geq F(x)\wedge F(z)\wedge\frac{1-k}{2}\right)$

Definition: A fuzzy subset F of S is called an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S if it satisfies the conditions:

(1) $(\forall x,y\in S)(\forall t\in(0,1])(x\leq y,[y;t]\in F\rightarrow[x;t]\in\vee q_kF)$

(2) $(\forall x\in S)\left(F(x)\geq((F\circ 1)\wedge(1\circ F))(x)\wedge\frac{1-k}{2}\right)$

Theorem: A fuzzy subset F of S is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S if and only if $U(F;t)(\neq ?)$ is a quasi-ideal of S for all $t\in(0,\frac{1-k}{2}]$.

Proof: Suppose that F is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S and let x,y∈S with x≤y such that y∈U(F;t) for some $t\in(0,\frac{1-k}{2}]$. Then F(y)≥t and by hypothesis

$$F(x)\geq F(y)\wedge\frac{1-k}{2}\geq t\wedge\frac{1-k}{2}=t$$

Hence x∈U(F;t).

Now let $x\in(U(F;t)S]\cap(SU(F;t)]$ for some $t\in(0,\frac{1-k}{2}]$. Then $x\in(U(F;t)S]$ and $x\in(SU(F;t)]$ and hence $x\leq pq, x\leq p_1q_1$ for some $q,p_1\in S$ and $p,q_1\in U(F;t)$. Then F(p)≥t and $F(q_1)\geq t$. Since $(p,q),(p_1,q_1)\in A_x$, so we have $A_x\neq ?$. Then

$$\begin{aligned}F(x)&\geq\left[(F\circ 1)(x)\wedge(1\circ F)(x)\wedge\frac{1-k}{2}\right]\\&=\left[\bigvee_{(y,z)\in A_x}\{F(y)\wedge 1(z)\}\wedge\bigvee_{(y_1,z_1)\in A_x}\{1(y_1)\wedge F(z_1)\}\wedge\frac{1-k}{2}\right]\\&\geq\left[\{F(q)\wedge 1(p_1)\}\wedge\{1(q)\wedge F(p)\}\wedge\frac{1-k}{2}\right]\\&=\left[\{F(q)\wedge 1\}\wedge\{1\wedge F(p)\}\wedge\frac{1-k}{2}\right]=\left[F(p)\wedge F(q_1)\wedge\frac{1-k}{2}\right]\\&\geq\left[t\wedge t\wedge\frac{1-k}{2}\right]=t\end{aligned}$$

Hence x∈U(F;t) and $(U(F;t)S]\cap(SU(F;t)]\subseteq U(F;t)$, consequently, U(F;t) is a quasi-ideal of S.

Conversely, assume that for every $t\in(0,\frac{1-k}{2}]$, the set $U(F;t)(\neq ?)$ is a quasi-ideal of S. Suppose there exist x,y∈S with x≤y such that $F(x)<F(y)\wedge\frac{1-k}{2}$. Choose $t_0\in(0,\frac{1-k}{2}]$ such that $F(x)<t_0\leq F(y)\wedge\frac{1-k}{2}$. Thus y∈U(F;t) but x∉U(F;t) since x≤y∈U(F;t) and U(F;t) is a quasi-ideal of S, we have x∈U(F;t), a contradiction. Hence $F(x)\geq F(y)\wedge\frac{1-k}{2}$ for all x,y∈S with x≤y. Suppose there exists x∈S such that

$$F(x)<\left\{(F\circ 1)(x)\wedge(1\circ F)(x)\wedge\frac{1-k}{2}\right\}$$

Choose $t_1\in(0,\frac{1-k}{2}]$ such that

$$F(x)<t_1\leq\left\{(F\circ 1)(x)\wedge(1\circ F)(x)\wedge\tfrac{1-k}{2}\right\}$$

Then $(F\circ 1)(x)\geq t_1$ and $(1\circ F)(x)\geq t_1$ but $F(x)<t_1$ implies that x∉U(F;t). Since $(F\circ 1)(x)\geq t_1$ and $(1\circ F)(x)\geq t_1$, we have

$$\bigvee_{(y_1,z_1)\in A_x}\{F(y_1)\wedge 1(z_1)\}\geq t_1$$

and

$$\bigvee_{(y_2,z_2)\in A_x}\{1(y_2)\wedge F(z_2)\}\geq t_1$$

This implies that there exist $p,q,p_1,q_1\in S$ with $(p,q)\in A_x$ and $(p_1,q_1)\in A_x$, such that $F(p)\geq t_1$ and $F(q_1)\geq t_1$. Then $p,q_1\in U(F;t_1)$ and so $pq\in U(F;t_1)S$ and $p_1q_1\in SU(F;t_1)$. Hence $x\in(U(F;t_1)S]$ and $x\in(SU(F;t_1)]$. Thus $x\in(U(F;t_1)S]\cap(SU(F;t_1)]$. Since

$(U(F;t_1)S] \cap (SU(F;t_1)] \subseteq U(F;t_1)$, we have $x \in U(F;t_1)$, a contradiction. Thus $F(x) \geq \{(F \circ 1)(x) \wedge (1 \circ F)(x) \wedge \frac{1-k}{2}\}$ for all $x \in S$. Consequently, F is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.

Example: Consider the ordered semigroup as shown in example 2.5 and define a fuzzy subset F of S as follows:

$$F : S \to [0,1] | x \mapsto F(x) = \begin{cases} 0.5 \text{ if } x = a \\ 0.4 \text{ if } x = b \\ 0.3 \text{ if } x = d \\ 0.2 \text{ if } x = c,f \end{cases}$$

Then

$$U(F;t) = \begin{cases} S & \text{if } 0 < t \leq 0.2 \\ \{a,c,d\} & \text{if } 0.2 < t \leq 0.3 \\ \{a,d\} & \text{if } 0.3 < t \leq 0.4 \\ \{a\} & \text{if } 0.4 < t \leq 0.5 \\ ? & \text{if } 0.5 < t \leq 1 \end{cases}$$

Then $U(F;t) = \{a,c,d\}$ is a quasi-ideal of S for all $t \in (0.2, 0.3]$ and by Theorem 3.3, F is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S for all $t \in (0, \frac{1-k}{2}]$ with $k = 0.4$.

Proposition: If F is a nonzero $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Then the set $F_0 = \{x \in S \mid F(x) > 0\}$ is a quasi-ideal of S.

Proof: Let F be an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Let $x,y \in S$, $x \leq y$ and $y \in F_0$. Then, $F(y) > 0$. Since F is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S, we have

$$F(x) \geq F(y) \wedge \frac{1-k}{2} > 0, \text{ because } F(y) > 0$$

Let $a \in (F_0 S] \cap (SF_0]$, then $a \in (F_0 S]$ and $a \in (SF_0]$ and we have $a \leq y_1 z_1$, $a \leq p_1 q_1$ for some $z,p \in S$, $(y_1, z_1), (p_1, q_1) \in A_a$ and $y,q \in F_0$. Then $F(y)>0$ and $F(q)>0$. Since, $A_a \neq ?$

$$\begin{aligned}(F \circ 1)(a) &= \bigvee_{(y,z) \in A_a} (F(y) \wedge 1(z)) \\ &\geq (F(y_1) \wedge 1(z_1)) \\ &= (F(y_1) \wedge 1) = F(y_1)\end{aligned}$$

In a similar way, we can prove that $(1 \circ F)(a) \geq F(q_1)$. Thus,

$$\begin{aligned}F(a) &\geq \left\{((F \circ 1) \wedge (1 \circ F))(a) \wedge \frac{1-k}{2}\right\} \\ &= \left\{(F \circ 1)(a) \wedge (1 \circ F)(a) \wedge \frac{1-k}{2}\right\} \\ &\geq \left\{F(y_1) \wedge F(q) \wedge \frac{1-k}{2}\right\} > 0\end{aligned}$$

because $F(y_1)>0$ and $F(q_1)>0$

Thus, $a \in F_0$ and $(F_0 S] \cap (SF_0] \subseteq F_0$. Consequently, F_0 is a quasi-ideal of S.

Lemma: A non-empty subset A of S is a quasi-ideal of S if and only if the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.

Proof: Suppose A is a quasi-ideal of S. Let $x,y \in S$ such that $x \leq y$. If $y \notin A$, then $\chi_A(y) = 0$. Hence $\chi_A(x) \geq \min\{\chi_A(y), \frac{1-k}{2}\}$. If $y \in A$, then since A is a quasi-ideal of S with $x \leq y \in A$, we have $x \in A$. Therefore $\chi_A(x) = \frac{1-k}{2} = \chi_A(y)$. Let $x \in S$ and we have to show that $\chi_A(x) \geq \min\{((\chi_A \circ 1) \wedge (1 \circ \chi_A))(x), \frac{1-k}{2}\}$. Since $1 = \chi_S$, then

$$\begin{aligned}((\chi_A \circ 1) \wedge (1 \circ \chi_A))(x) &= ((\chi_A \circ \chi_S) \wedge (\chi_S \circ \chi_A))(x) \\ &= (\chi_{(AS]} \wedge \chi_{(SA]})(x) \wedge \frac{1-k}{2}, \text{ by Lemma 2.6 (3)} \\ &= (\chi_{(AS] \cap (SA]})(x) \wedge \frac{1-k}{2}, \text{ by Lemma 2.6 (2)} \\ &\leq (\chi_A)(x) \wedge \frac{1-k}{2} \leq \chi_A(x).\end{aligned}$$

Conversely, assume that χ_A is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Let $x,y \in S$ such that $x \leq y$ and $y \in A$, then $\chi_A(y) = \frac{1-k}{2}$. Since χ_A is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. We have

$$\begin{aligned}\chi_A(x) &\geq \min\left\{\chi_A(y), \frac{1-k}{2}\right\} \\ &= \min\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\} = \frac{1-k}{2}\end{aligned}$$

Hence $\chi_A(x) = \frac{1-k}{2}$. Therefore $x \in A$. Let $x \in S$, be such that $x \in (AS] \cap (SA]$. Then there exist $y,z \in S$ and $a,b \in A$ such that $x \leq ay$ and $x \leq zb$, hence (a,y), $(z,b) \in A_x$ and $A_x \neq ?$. Thus,

$$\min\left\{((\chi_A \circ 1) \wedge (1 \circ \chi_A))(x), \frac{1-k}{2}\right\} = \min\left\{(\chi_A \circ 1)(x), (1 \circ \chi_A)(x), \frac{1-k}{2}\right\}$$

$$= \min\left[\bigvee_{(c,d)\in A_x} \min\{\chi_A(c), 1(d)\}, \bigvee_{(c,d)\in A_x} \min\{1(c), \chi_A(d)\}, \frac{1-k}{2}\right] \geq \min\left[\min\{\chi_A(a), 1(y)\}, \min\{1(z), \chi_A(b)\}, \frac{1-k}{2}\right]$$

$$= \min\left[\min\{\chi_A(a), 1\}, \min\{1, \chi_A(b)\}, \frac{1-k}{2}\right] = \min\left[\chi_A(a), \chi_A(b), \frac{1-k}{2}\right] = \min\left[1, 1, \frac{1-k}{2}\right] = \frac{1-k}{2}$$

Thus, $\chi_A(x) \geq \min\left\{(\chi_A \circ 1)(x), (1 \circ \chi_A)(x), \frac{1-k}{2}\right\} = \frac{1-k}{2}$ and we have $\chi_A(x) = \frac{1-k}{2}$, $x \in A$. Therefore $(AS] \cap (SA] \subseteq A$.

Lemma: Let $(S,\cdot,\leq)$ be an ordered semigroup. Then every $(\in, \in\vee q_k)$-fuzzy quasi-ideal of S is an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S.

Proof: Let F be an $(\in, \in\vee q_k)$-fuzzy quasi-ideal of S. Let $x,y \in S$. Then $xy = x(y)$ and so $(x,y) \in A_{xy}$. Since $A_{xy} \neq ?$, we have

$$F(xy) \geq \left\{((F \circ 1) \wedge (1 \circ F))(xy) \wedge \frac{1-k}{2}\right\} = \min\left[(F \circ 1)(xy), (1 \circ F)(xy), \frac{1-k}{2}\right]$$

$$= \min\left[\bigvee_{(p,q)\in A_{xy}} \min\{F(p), 1(q)\}, \bigvee_{(p_1,q_1)\in A_{xy}} \min\{1(p_1), F(q_1)\}, \frac{1-k}{2}\right]$$

$$\geq \min\left[\min\{F(x), 1(y)\}, \min\{1(x), F(y)\}, \frac{1-k}{2}\right] = \min\left[\min\{F(x), 1\}, \min\{1, F(y)\}, \frac{1-k}{2}\right] = \min\left[F(x), F(y), \frac{1-k}{2}\right]$$

Let $x,y,z \in S$, then $(xy)z = x(yz)$ and $(x, yz), (xy, z) \in A_{xyz}$. Since $A_{xyz} \neq ?$, we have

$$F(xyz) \geq \left\{((F \circ 1) \wedge (1 \circ F))(xyz) \wedge \frac{1-k}{2}\right\}$$

$$= \min\left[(F \circ 1)(xyz), (1 \circ F)(xyz), \frac{1-k}{2}\right] = \min\left[\bigvee_{(p,q)\in A_{xyz}} \min\{F(p), 1(q)\}, \bigvee_{(p_1,q_1)\in A_{xyz}} \min\{1(p_1), F(q_1)\}, \frac{1-k}{2}\right]$$

$$\geq \min\left[\min\{F(x), 1(yz)\}, \min\{1(xy), F(z)\}, \frac{1-k}{2}\right] = \min\left[\min\{F(x), 1\}, \min\{1, F(z)\}, \frac{1-k}{2}\right] = \min\left[F(x), F(z), \frac{1-k}{2}\right].$$

Thus F is an $(\in, \in\vee q_k)$-fuzzy quasi-ideal of S.

Remark: The converse of the Lemma 3.7 is not true in general. As shown in the following Example.

Example: Consider the ordered semigroup S = {a,b,c,d} with the following multiplication table and ordered relation below:

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}$$

Then {a,d} is bi-ideal of S but not a quasi-ideal of S. Define a fuzzy subset F of S as follows:

$$F : S \to [0,1] \mid x \mapsto F(x) = \begin{cases} 0.5 \text{ if } x = a,d \\ 0.3 \text{ if } x = b,c \end{cases}$$

then

$$U(F,t) = \begin{cases} S \text{ if } t \in (0, 0.3] \\ \{a,d\} \text{ if } t \in (0.3, 0.5] \end{cases}$$

Then U(F, t) = {a,d} is a bi-ideal of S for all $t \in (0.3, 0.5]$ and by Theorem 2.4, F is an $(\in, \in\vee q_k)$-fuzzy bi-ideal. Furthermore, U(F, t) = {a,d} is not a quasi-ideal and so by Theorem 3.3, F is not an $(\in, \in\vee q_k)$-fuzzy quasi-ideal of S.

UPPER AND LOWER PARTS OF $(\in, \in\vee q_k)$-FUZZY QUASI-IDEALS

In this section, we define the upper/lower parts of an $(\in, \in\vee q_k)$-fuzzy bi-ideal and characterize regular and intra-regular ordered semigroups in terms of $(\in,\in\vee q_k)$-fuzzy bi-ideals.

Definition [7]: Let F_1 and F_2 be a fuzzy subsets of S. Then the fuzzy subsets $\overline{F_1}^k$, $(F_1 \wedge^k F_2)^-$, $(F_1 \vee^k F_2)^-$, $(F_1 \circ^k F_2)^-$, $\overset{+}{F_1}^k$, $(F_1 \wedge^k F_2)^+$, $(F_1 \vee^k F_2)^+$ and $(F_1 \circ^k F_2)^+$ of S are defined as follows:

$$\overline{F_1}^k : S \to [0,1] | x \mapsto F_1^k(x) = F_1(x) \wedge \frac{1-k}{2},$$

$$(F_1 \wedge^k F_2)^- : S \to [0,1] | x \mapsto (F_1 \wedge^k F_2)(x) = (F_1 \wedge F_2)(x) \wedge \frac{1-k}{2},$$

$$(F_1 \vee^k F_2)^- : S \to [0,1] | x \mapsto (F_1 \vee^k F_2)(x) = (F_1 \vee F_2)(x) \wedge \frac{1-k}{2},$$

$$(F_1 \circ^k F_2)^- : S \to [0,1] | x \mapsto (F_1 \circ^k F_2)(x) = (F_1 \circ F_2)(x) \wedge \frac{1-k}{2},$$

and

$$\overset{+}{F_1}^k : S \to [0,1] | x \mapsto F_1^k(x) = F_1(x) \vee \frac{1-k}{2},$$

$$(F_1 \wedge^k F_2)^+ : S \to [0,1] | x \mapsto (F_1 \wedge^k F_2)(x) = (F_1 \wedge F_2)(x) \vee \frac{1-k}{2},$$

$$(F_1 \vee^k F_2)^+ : S \to [0,1] | x \mapsto (F_1 \vee^k F_2)(x) = (F_1 \vee F_2)(x) \vee \frac{1-k}{2},$$

$$(F_1 \circ^k F_2)^+ : S \to [0,1] | x \mapsto (F_1 \circ^k F_2)(x) = (F_1 \circ F_2)(x) \vee \frac{1-k}{2},$$

for all $x \in S$.

Lemma [7]: Let F_1 and F_2 be fuzzy subsets of S. Then the following hold:

(i) $\left(F_1 \wedge^k F_2\right)^- = \left(\overline{F}^k \wedge \overline{F_2}^k\right)$

(ii) $\left(F_1 \vee^k F_2\right)^- = \left(\overline{F}^k \vee \overline{F_2}^k\right)$

(iii) $\left(F_1 \circ^k F_2\right)^- = \left(\overline{F}^k \circ \overline{F_2}^k\right)$

Lemma [7]: Let F_1 and F_2 be fuzzy subsets of S. Then the following hold:

(i) $\left(F_1 \wedge^k F_2\right)^+ = \left(\overset{+}{F_1}^k \wedge \overset{+}{F_2}^k\right)$

(ii) $\left(F_1 \vee^k F_2\right)^+ = \left(\overset{+}{F_1}^k \vee \overset{+}{F_2}^k\right)$

(iii) $\left(F_1 \circ^k F_2\right)^+ ? \left(\overset{+}{F_1}^k \circ \overset{+}{F_2}^k\right)$ if $A_x = ?$

and

$$\left(F_1 \circ^k F_2\right)^+ = \left(\overset{+}{F_1}^k \circ \overset{+}{F_2}^k\right) \text{ if } A_x \neq ?.$$

Let A be a non-empty subset of S, then the upper and lower parts of the characteristic function χ_A are defined as follows:

$$\overline{\chi}_A^k : S \to [0,1] | x \mapsto \overline{\chi}_A^k(x) = \begin{cases} \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$\overset{+}{\chi}_A^k : S \to [0,1] | x \mapsto \overset{+}{\chi}_A^k(x) = \begin{cases} 1 & \text{if } x \in A \\ \frac{1-k}{2} & \text{otherwise} \end{cases}$$

Lemma [7]: Let A and B be non-empty subset of S. Then the following hold:

(1) $(\chi_A \wedge^k \chi_B)^- = \overline{\chi}_{A\cap B}^k$

(2) $(\chi_A \vee^k \chi_B)^- = \overline{\chi}_{A\cup B}^k$

(3) $(\chi_A \circ^k \chi_B)^- = \overline{\chi}_{(AB]}^k$

Lemma: The lower part of the characteristic function χ_A of A

$$\overline{\chi}_A^k : S \to [0,1] | x \mapsto \overline{\chi}_A^k(x) = \begin{cases} \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S if and only if A is a quasi-ideal of S.

Proof: The proof follows from Lemma 3.6.

The proof of the following Proposition is straightforward.

Proposition: If F is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S, then $\overline{F}^k$ is a fuzzy quasi-ideal of S.

In [20], left and right regular and completely regular ordered semigroups are characterized by the properties of their fuzzy quasi-ideals. Here we further extend our studies from fuzzy quasi-ideals to $(\in,\in\vee q_k)$-fuzzy quasi-ideals in ordered semigroups and discuss left and right regular and completely regular ordered semigroups in terms of $(\in,\in\vee q_k)$-fuzzy quasi-ideals. We also define semiprime $(\in,\in\vee q_k)$-fuzzy quasi-ideals and give a characterization of completely regular ordered semigroups in terms of semiprime $(\in,\in\vee q_k)$-fuzzy quasi-ideals.

Lemma [20]: If $(S,\cdot,\leq)$ is an ordered semigroup and $? \neq A \subseteq S$, then the set $(A \cup (AS \cap SA)]$ is the quasi-ideal of S generated by A. If A = {x} ($x \in S$), we write $(x \cup (xS \cap Sx)]$ instead of $(\{x\} \cup (\{x\}S \cap S\{x\})]$.

Lemma [9]: An ordered semigroup S is completely reqular if and only if for every $A\subseteq S$, we have, $A\subseteq (A^2SA^2]$, or equivalently, $a\in(a^2Sa^2]$ for every $a\in S$.

Theorem: An ordered semigroup S is completely regular if and only if for every $(\in,\in\vee q_k)$-fuzzy quasi-ideal F of S, we have $\overline{F}^k(a)=\overline{F}^k(a^2)$ for every $a\in S$.

Proof: Let $a\in S$. Since S is left and right regular, so $a\in(Sa^2]$ and $a\in(a^2S]$. Then there exist $x,y\in S$, such that $a\le xa^2$ and $a\le a^2y$ and hence $(a^2,y)\in A_a$ and $(x,a^2)\in A_a$. Since $A_a\ne ?$, we have

$$\begin{aligned}
F(a) &\ge \left\{((F\circ 1)\wedge(1\circ F))(a)\wedge\frac{1-k}{2}\right\}\\
&=\min\left[(F\circ 1)(a),(1\circ F)(a),\frac{1-k}{2}\right]\\
&=\min\left[\bigvee_{(p,q)\in A_a}\min\{F(p),1(q)\},\bigvee_{(p_1,q_1)\in A_a}\{1(p_1),F(q_1)\},\frac{1-k}{2}\right]\\
&\ge\min\left[\min\{F(a^2),1(y)\},\{1(x),F(a^2)\},\frac{1-k}{2}\right]\\
&=\min\left[\min\{F(a^2),1\},\{1,F(a^2)\},\frac{1-k}{2}\right]=\min\left[F(a^2),\frac{1-k}{2}\right]\\
&\ge\min\left[\min\left\{F(a),F(a),\frac{1-k}{2}\right\},\frac{1-k}{2}\right]\\
&=\min\left[F(a),\frac{1-k}{2}\right]
\end{aligned}$$

Thus

$$\overline{F}^k(a)=F(a)\wedge\tfrac{1-k}{2}\ge F(a^2)\wedge\tfrac{1-k}{2}=\overline{F}^k(a^2)\ge F(a)\wedge\tfrac{1-k}{2}=\overline{F}^k(a)$$

and it follows that $\overline{F}^k(a)=\overline{F}^k(a^2)$ for every $a\in S$.

Conversely, let $a\in S$ and we consider the quasi-ideal $A(a^2)=(a^2\cup(a^2S\cap Sa^2)]$ of S generated by $a^2(a\in S)$. Then by Lemma 4.5,

$$\overline{\chi}^k_{A(a^2)}:S\to[0,1]\,|\,x\mapsto\overline{\chi}^k_{A(a^2)}(x)=\begin{cases}\frac{1-k}{2} & \text{if } x\in A(a^2)\\ 0 & \text{if } x\notin A(a^2)\end{cases}$$

is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S. By hypothesis, $\overline{\chi}^k_{A(a^2)}(a)=\overline{\chi}^k_{A(a^2)}(a^2)$. Since $a^2\in A(a^2)$, we have

$$\overline{\chi}^k_{A(a^2)}(a^2)=\chi_{A(a^2)}(a^2)\wedge\tfrac{1-k}{2}=1\wedge\tfrac{1-k}{2}=\tfrac{1-k}{2}$$

and hence, $\overline{\chi}^k_{A(a^2)}(a)=\frac{1-k}{2}$, so $a^2\in A(a^2)$ and we have, $a\le a^2$ or $a\le a^2x$ and $a\le ya^2$ for some $x,y\in S$. If $a\le a^2$, Then, $a\le a^2=aa\le a^2a^2=aaa^2\le a^2aa^2\in a^2Sa^2$ and $a\in(a^2Sa^2]$. If $a\le a^2x$ and $a\le ya^2$, then $a\le a^2(xy)a^2=a^2ta^2$, where $t=xy\in S$ and so $a\in(a^2Sa^2]$. Therefore, S is completely regular.

Definition: Let $(S,\cdot,\le)$ be an ordered semigroup and F an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S. Then F is called semiprime $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S if

$$F(a)\ge F(a^2)\wedge\frac{1-k}{2}$$

for all $a\in S$

Theorem: An ordered semigroup S is completely regular if and only if every $(\in,\in\vee q_k)$-fuzzy quasi-ideal F of S is semiprime.

Proof: Suppose that S is completely regular ordered semigroup. Let $a\in S$. Since S is left and right regular, there exist $x,y\in S$ such that $a\le xa^2$ and $a\le a^2y$, then $(a^2,y)\in A_a$ and $(x,a^2)\in A_a$. Since $A_a\ne\phi$, we have

$$\begin{aligned}
F(a) &\ge\left\{((F\circ 1)\wedge(1\circ F))(a)\wedge\frac{1-k}{2}\right\}\\
&=\min\left[(F\circ 1)(a),(1\circ F)(a),\frac{1-k}{2}\right]\\
&=\min\left[\bigvee_{(p,q)\in A_a}\min\{F(p),1(q)\},\bigvee_{(p_1,q_1)\in A_a}\min\{1(p_1),F(q_1)\},\frac{1-k}{2}\right]\\
&\ge\min\left[\min\{F(a^2),1(y)\},\min\{1(x),F(a^2)\},\frac{1-k}{2}\right]\\
&=\min\left[\min\{F(a^2),1\},\min\{1,F(a^2)\},\frac{1-k}{2}\right]\\
&=\min\left[F(a^2),F(a^2),\frac{1-k}{2}\right]\\
&=\min\left[F(a^2),\frac{1-k}{2}\right]=F(a^2)\wedge\frac{1-k}{2}.
\end{aligned}$$

Conversely, assume that F is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S such that $F(a)\ge F(a^2)\wedge\frac{1-k}{2}$ for all $a\in S$. We consider the quasi-ideal $A(a^2)$ of S generated by a^2. Then by Lemma 4.5, $\chi^{-k}_{A(a^2)}$ is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S. By hypothesis, $\chi^{-k}_{A(a^2)}(a)\ge\chi^{-k}_{A(a^2)}(a^2)$. Since $a^2\in A(a^2)$, we have

$$\chi^{-k}_{A(a^2)}(a^2)=\chi_{A(a^2)}(a^2)\wedge\tfrac{1-k}{2}=1\wedge\tfrac{1-k}{2}=\tfrac{1-k}{2}$$

Hence $\chi^{-k}_{A(a^2)}(a^2)=\frac{1-k}{2}$ and we have $a\in A(a^2)$. Then $a\le a^2$ or $a\le a^2x$ and $a\le za^2$ for some $x,z\in S$. If $a\le a^2$, then

$a \le aa \le a^2a^2 = aaa^2 \le a^2a\,a^2 \in a^2Sa^2$ and $a \in (a^2Sa^2]$. If $a \le a^2x$ and $a \le za^2$, then $a \le (a^2x)(za^2) = a^2(xz)a^2 = a^2ya^2$ for y = xz∈S and so $a^2ya^2 \in a^2Sa^2$. Thus $a \in (a^2Sa^2]$. Consequently, S is completely regular.

Some semilattices of ordered semigroups in terms of $(\in,\in\vee q_k)$-fuzzy quasi-ideals

Theorem: An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for every $(\in,\in\vee q_k)$-fuzzy quasi-ideal F of S, we have

$$\overline{F}^k(a) = \overline{F}^k(a^2) \text{ and } \overline{F}^k(ab) = \overline{F}^k(ba) \text{ for all } a,b \in S$$

Proof: Suppose that F is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal and by hypothesis, there exist a semilattice Y and a family $\{S_i: i\in Y\}$ of left and right simple subsemigroups of S such that:

$$S_i \cap S_j = ?, \forall i,j \in Y, i \ne j, S = \bigcup_{i\in Y} S_i, S_iS_j \subseteq S_{ij} \forall\, i,j \in Y$$

(i) To prove that $\overline{F}^k(a) = \overline{F}^k(a^2)$ for every a∈S. By Theorem 4.9 and Lemma 4.8, it is enough to prove that $a \in (a^2Sa^2]$, for every a∈S. Let a∈S then there exists a semilattice y such that $a\in S_i$. Since each S_i is a left and right simple, we have $(S_ia] = S_i$ and $(aS_i] = S_i$ and so $a \in (aS_i] = (a(S_ia]] = (aS_ia]$. Since $a \in (aS_ia]$, there exists $x\in S_i$ such that $a \le axa$. Since $x \in (aS_ia]$, there exists $y\in S_i$ such that $x \le aya$. Thus, $a \le axa \le a(aya)a = a^2ya^2$. Since $y\in S_i$, we have $a^2ya^2 \in a^2S_ia^2 \subseteq a^2Sa^2$ and $a \in (a^2Sa^2]$. Since F is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S, by Theorem 4.9, it follows that $\overline{F}^k(a) = \overline{F}^k(a^2)$ for every a∈S.

(ii) Let a,b∈S. By (i), we have $\overline{F}^k(ab) = \overline{F}^k((ab)^2) = \overline{F}^k((ab)^4)$.

Also

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \subseteq (Q(aba)Q(babab)] \\
&= (Q(babab)Q(aba)] \text{ (by Lemma 2.2)} \\
&= ((babab \cup (bababS \cap Sbabab)](aba \cup (abaS \cap Saba)]] \\
&\subseteq (((babab \cup (bababS \cap Sbabab))(aba \cup (abaS \cap Saba))]] \text{ (as } (A](B] \subseteq (AB]) \\
&\subseteq ((babab \cup bababS)(aba \cup Saba)] \subseteq ((baS)(Sba)] \subseteq ((baS](Sba]] \ (\text{as } A \subseteq (A]) \\
&= ((baS] \cap (Sba]]. \ (\text{Since S is regular})
\end{aligned}$$

Then $(ab)^4 \le (ba)x$ and $(ab)^4 \le y(ba)$ for some x,y∈S. Thus, $(ba,x) \in A_{(ab)^4}$ and $(y,ba) \in A_{(ab)^4}$. Since $A_{(ab)^4} \ne \phi$, we have

$$\begin{aligned}
F((ab)^4) &\ge \left\{((F\circ 1)\wedge(1\circ F))((ab)^4) \wedge \frac{1-k}{2}\right\} = \min\left[(F\circ 1)((ab)^4), (1\circ F)((ab)^4), \frac{1-k}{2}\right] \\
&= \min\left[\bigvee_{(p,q)\in A_{(ab)^4}} \min\{F(p),1(q)\}, \bigvee_{(p_1,q_1)\in A_{(ab)^4}} \min\{1(p_1),F(q_1)\}, \frac{1-k}{2}\right] \ge \min\left[\min\{F(ba),1(x)\}, \min\{1(y),F(ba)\}, \frac{1-k}{2}\right] \\
&= \min\left[\min\{F(ba),1\}, \min\{1,F(ba)\}, \frac{1-k}{2}\right] = \min\left[F(ba),F(ba),\frac{1-k}{2}\right] = \min\left[F(ba),\frac{1-k}{2}\right]
\end{aligned}$$

Thus, $F((ab)^4) \ge F(ba) \wedge \frac{1-k}{2}$ and so

$$F^{-k}((ab)^4) = F((ab)^4) \wedge \frac{1-k}{2} \ge \left(F(ba) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} = F(ba) \wedge \frac{1-k}{2} = F^{-k}(ba)$$

Since $F^{-k}((ab)^4) = F^{-k}(ab)$, we have $F^{-k}(ab) \ge F^{-k}(ba)$. In a similar way, we can prove that $F^{-k}(ba) \ge F^{-k}(ab)$. Thus $F^{-k}(ab) = F^{-k}(ba)$.

Conversely, since N is a semilattice congruence on S, by Lemma 2.1, it is enough to prove that every one sided ideal of S is a two-sided ideal and semiprime. Let R be a right ideal of S and so a quasi-ideal of S. Let a∈R and t∈S. Since R is a quasi-ideal of S, by Lemma 4.5, χ_R^{-k} is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S. By hypothesis, $\chi_R^{-k}(at)=\chi_R^{-k}(ta)$. Since $at\in RS\subseteq R$, we have $\chi_R^{-k}(at)=\chi_R(at)\wedge\frac{1-k}{2}=1\wedge\frac{1-k}{2}=\frac{1-k}{2}$ and so $\chi_R^{-k}(ta)=\frac{1-k}{2}$, hence ta∈R. Thus SR⊆R. This shows that R is a left ideal of S and is an ideal of S. Let x∈S such that $x^2\in R$. Since R is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S, by Lemma 4.5, χ_R^{-k} is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S. By hypothesis, $\chi_R^{-k}(x^2)=\chi_R^{-k}(x)$. Since $x^2\in R$, we have $\chi_R^{-k}(x^2)=\frac{1-k}{2}$ and so $\chi_R^{-k}(x)=\frac{1-k}{2}$. It follows that x∈R and hence R is semiprime. In a similar way we can prove that every left ideal of S is an ideal and semiprime.

CONCLUDING REMARKS

In this paper, we have presented the generalization of (α,β)-fuzzy quasi-ideals in ordered semigroups. We have also provided different characterizations theorems in terms of this notion. Some important classes of ordered semigroups have been characterized by lower part of $(\in,\in\vee q_k)$-fuzzy quasi-ideal. In particular, if J = {t|t∈(0,1] and U(F; t) is an empty set or a quasi-ideal of S}, we have answered the following question.

If J=(0, $\frac{1-k}{2}$], what kind of fuzzy quasi-ideal of S will F be?

In our future work, we want to define prime (α,β)-fuzzy quasi-ideal and study those ordered semigroups for which each generalized fuzzy quasi-ideal is idempotent.

Hopefully, our results presented in this paper will constitute a platform for further development of ordered semigroups and their applications in other branches of algebra.

ACKNOWLEDGEMENT

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

REFERENCES

1. Bhakat, S.K. and P. Das, 1996. $(\in,\in\vee q_k)$-fuzzy subgroups. Fuzzy Sets and Systems, 80: 359-368.
2. Bhakat, S.K. and P. Das, 1996. Fuzzy subrings and ideals redefined. Fuzzy Sets and Systems, 81: 383-393.
3. Davvaz, B. and A. Khan, 2011. Characterizations of regular ordered semigroups in terms of (α,β)-fuzzy generalized bi-ideals. Information Sciences, pp: 12.
4. Davvaz, B., 2006. $(\in,\in\vee q_k)$-fuzzy subnear-rings and ideals. Soft Computing, 10: 206-211.
5. Jun, Y.B., A. Khan and M. Shabir, 2009. Ordered semigroups characterized by their $(\in,\in\vee q_k)$-fuzzy bi-ideals. Bull. Malays. Math. Sci. Soc., (2) 32(3): 391-408.
6. Jun, Y.B. and S. Z. Song, 2006. Generalized fuzzy interior ideals in semigroups. Inform. Sci., 176: 3079-3093.
7. Khan, A., B. Davvaz, N.H. Sarmin and F.M. Khan, Ordered semigroups characterized by $(\in,\in\vee q_k)$-fuzzy generalized bi-ideals, submitted.
8. Kazanci, O. and S. Yamak, 2008. Generalized fuzzy bi-ideals of semigroup. Soft Computing, 12: 1119-1124.
9. Kehayopulu, N. and M. Tsingelis, 2005. Fuzzy bi-ideals in ordered semigroups. Information Sciences, 171: 13-28.
10. Khan, A., Y.B. Jun and S.Z. Abbas, 2010. Characterizations of ordered semigroups in terms of $(\in,\in\vee q_k)$-fuzzy interior ideals. Neural Comp. Appli. DOI 10.1007/s00521-010-0463-8.
11. Kuroki, N., 1981. On fuzzy ideals and fuzzy bi-ideals in semigroups. Fuzzy Sets and Systems, 5: 203-215.
12. Kuroki, N., 1993. Fuzzy semiprime quasi-ideals in semigroups. Information Sciences, 75: 201-211.
13. Kuroki, N., 1991. On fuzzy semigroups. Information Sciences, 53: 203-236.
14. Ma, X., J. Zhan and Y.B. Jun, 2009. On $(\in,\in\vee q_k)$-fuzzy filters of R_0-algebras. Math. Log. Quart., 55: 493-508.
15. Ma, X., J. Zhan and Y.B. Jun, 2008. Interval valued $(\in,\in\vee q_k)$-fuzzy ideals of pseudo-MV algebras. International Journal of Fuzzy Systems, 10 (2): 84-91.
16. Mordeson, J.N., D.S. Malik and N. Kuroki, 2003. Fuzzy Semigroups, Studies in Fuzziness and Soft Computing, Springer-Verlag Berlin, Vol: 131.
17. Murali, V., 2004. Fuzzy points of equivalent fuzzy subsets. Information Science, 158: 277-288.
18. Pu, P.M. and Y.M. Liu, 1980. Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appl., 76: 571-599.

19. Rosenfeld, A., 1971. Fuzzy groups. J. Math. Anal. Appl., 35: 512-517.
20. Shabir. M. and A. Khan, 2011. Fuzzy quasi-ideals of ordered semigroups. Bull. Malays. Math. Sci. Soc., (2) 34 (1): 87-102.
21. Shabir, M., Y.B. Jun and Y. Nawaz, 2010. Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals, Computer and Mathematics with Applications, 60: 1473-1493.
22. Zadeh, L.A., 1965. Fuzzy sets. Information and Control, 8: 338-353.